EXPONENTIAL DECAY FOR A SYSTEM OF ELASTIC WAVES WITH A NONLINEAR LOCALIZED DAMPING

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Abstract

We show that the total energy of the solutions of a system in Elasticity Theory perturbed with a dissipative localized nonlinear term, but with a linear behavior, decay exponentially to zero, that is, denoting by $E(t)$ the first order total energy associated to the system, then there exist positive constants C and γ satisfying:

$$E(t) \leq CE(0)e^{-\gamma t}.$$

1 Introduction

In this work we study decay and properties of the solutions for the following initial boundary value problem related with the system of elastic waves with a localized damping given by a nonlinear term, but with a linear behavior

$$u_{tt} - b^2\Delta u - (a^2 - b^2)\nabla\, div\, u + \rho(x, u_t) = 0, \qquad \text{in } \Omega \times \mathbb{R} \qquad (1)$$

$$u(x, 0) = u_o(x), \quad u_t(x, 0) = u_1(x) \qquad \text{in } \Omega \qquad (2)$$

$$u(x, t) = 0 \qquad \text{in } \Gamma \times \mathbb{R} \qquad (3)$$

where the medium Ω is a bounded domain in $\mathbb{R}^3$ with C^1 boundary Γ. The function $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ is the vector displacement, $\Delta\, u = (\Delta\, u^1(x,t), \Delta\, u^2(x,t), \Delta\, u^3(x,t))$ is the Laplacian operator, $div\, u$ is the usual divergent of u and ∇ is the gradient operator. The coefficients a and b are

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related with Lamé coefficients of Elasticity Theory, $a^2 > b^2 > 0$, (see [3]). This system has applications in geophysics and seismic waves propagation.

The vector function ρ, which appear in (1), represents a localized dissipative term. Therefore, our interest in this work is to show the exponential decay of total energy for the system (1)-(3) using suitable hypotheses on the dissipation ρ which is localized in a neighborhood of part of the boundary of Ω.

To prove this result we use some energy identities for the solution of the system (1)- (3) which are obtained choosing localized multipliers and using ideas of Control Theory (see J.-L. Lions [12], V. Komornik [10] and A. Haraux [7]).

In the case that $a^2 = b^2$ we have a vector wave equation. Many results related to wave equation can be generalized for the system (1). It is important to observe that the solutions of the system (1) are more complicated than the solutions of the wave equation. In fact, we observe that the solutions of the free system of elastic waves are a superposition of two waves which propagate with different phase velocities a and b (see [5]).

The stabilization of the system of elastic waves in a unbounded domain was studied by B. Kapitonov [8] and R. C. Charão [4]. For bounded domains the stabilization of the system (1) with linear localized unbounded damping was studied by M. A. Astaburuaga and R. C. Charão [6] and only algebraic decay rate was obtained for the total energy. To show the result, they use some estimates and multipliers which are not equal with that ones we used here. Moreover, the manner to estimate the dissipative term is different. In the case of wave equation, stabilization results can be found in M. Nakao [15], [16], E. Zuazua [18], P. Martinez [13], [14] and L. R. T. Tébou [17] and the references therein.

The system (1) with $\rho \equiv 0$ damped by a linear boundary feedback was studied by F. Alabau and V.Komornik [1] and they proved uniform stabilization. In [2] F. Alabau and Komornik considered an anisotropic system of elasticity and established uniform decay rates when feedback control is acting via natural and physically implementable boundary conditions. Their results require even more stringent geometric conditions. In fact, they must assume that the domain is a sphere.

Next, we give the hypotheses on the dissipative term and we prove some technical lemmas in order to obtain the main result.

2 Hypotheses

Let us consider the following hypotheses on the nonlinear function $\rho(x, s)$

$$\rho \colon \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^3, \tag{4}$$

$$\rho(x, s) \cdot s \geq 0, \quad \forall\, x \in \overline{\Omega}, \quad \forall\, s \in \mathbb{R}^3 \tag{5}$$

$$a(x)|s|^2 \leq \rho(x, s) \cdot s, \quad \forall\, x \in \overline{\Omega}, \quad \forall\, s \in \mathbb{R}^3 \tag{6}$$

$$|\rho(x, s)| \leq C a(x)|s|, \quad \forall\, x \in \overline{\Omega}, \quad \forall\, s \in \mathbb{R}^3 \tag{7}$$

where C is a positive constant and $a(x)$ is a bounded function such that

$$a \colon \overline{\Omega} \to \mathbb{R} \tag{8}$$

$$a(x) \geq 0 \quad \text{in} \quad \overline{\Omega} \tag{9}$$

$$a(x) \geq a_o > 0 \quad \text{in} \quad \omega \subset \overline{\Omega} \tag{10}$$

with a_o constant and ω is a neighborhood of part of the boundary of Ω. Thus, the nonlinearity $\rho(x, u_t)$ is effective only on a part of Ω. The dot " $\cdot$ " represents the usual inner product in $\mathbb{R}^3$.

3 Existence of Solutions and Exponential Decay

About the existence of solutions for the system (1)-(3) we can prove in a standard way (Theory of Semigroups) the following Theorem.

Theorem 1 (Existence of Solutions) *Let Ω be a C^1 bounded open set in $\mathbb{R}^3$ and ρ a function with the conditions* (4)-(7)*. Then, for each initial data*

$u_o \in (H_o^1(\Omega))^3$, $u_1 \in (L^2(\Omega))^3$, *there exists a unique solution* $u = u(x,t)$ *of the system* (1)-(3) *in the class*

$$u \in C(\mathbb{R}; (H_o^1(\Omega))^3) \cap C^1(\mathbb{R}; (L^2(\Omega))^3) \tag{11}$$

Our goal in this work is to prove the uniform stabilization of the total energy $E(t)$,

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + b^2|\nabla u|^2 + (a^2 - b^2)(div\, u)^2 \, dx \tag{12}$$

where $|\nabla u|^2 = \sum_{i=1}^{3} |\nabla u^i|^2$ and $u = u(x,t)$ is the solution of (1)-(3).

We note that the the following identity holds for the solution of (1)-(3)

$$E(S) - E(T) = \int_S^T \int_\Omega \rho(x, u_t) \cdot u_t \, dx \, ds, \tag{13}$$

for all T and S such that $0 \leq S < T < \infty$.

The identity (13) is obtained taking inner product between equation (1)-(3) and u_t and integrating over $[S,T] \times \Omega$. So, the energy is a nonincreasing function of t because $\rho(x, u_t) \cdot u_t \geq 0$ always. Then, it is possible that the energy decay in some rate. In fact, we have the following result.

Theorem 2 (Exponential Decay) *Under the hypotheses on function* $\rho(x,s)$ *and Theorem 1, the total energy for the solution* $u = u(x,t)$ *of the problem* (1)-(3) *has the following asymptotic behavior in time*

$$E(t) = E(u(x,t)) \leq C\, E(0)\, \mathbf{e}^{-\sigma t}, \qquad t \geq 0 \tag{14}$$

where C *and* σ *are positive constants.*

In order to prove this theorem we need some Lemmas and special estimates about the solution $u(x,t)$.

4 Technical Lemmas

Lemma 1 *Let* $E\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function. If there exists a real constant* $\sigma > 0$ *such that*

$$\int_S^T E(t)\, dt \leq \frac{1}{\sigma} E(S) \tag{15}$$

for all S and T such that $0 \leq S < T < +\infty$, then

$$E(t) \leq E(0)\mathbf{e}^{-\sigma t}, \quad t \geq 0 \tag{16}$$

Proof. See V. Komornik [9], [10] (See also P. Martinez [13]).

Lemma 2 (Energy Identities) *Let $h\colon \overline{\Omega} \to \mathbb{R}^3$ be a vector field of C^1-class, $\xi \in C_o^\infty(\mathbb{R}^3)$ and u the solution of (1) − (3). Then, for $0 \leq S < T < +\infty$ we have the identities*

$$\begin{aligned}&\int_\Omega 2\, u_t \cdot u\, dx\Big|_S^T - 2\int_S^T\int_\Omega |u_t|^2\, dx\, dt \\ &+ 2\int_S^T\int_\Omega \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dx\, dt + \int_S^T\int_\Omega 2\rho(x,u_t)\cdot u\, dx\, dt = 0\end{aligned} \tag{17}$$

$$\begin{aligned}&\int_\Omega (2h(x) * \nabla u)\cdot u_t\, dx\Big|_S^T + \int_S^T\int_\Omega div\, h\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] dx\, dt \\ &+ 2\int_S^T\int_\Omega \sum_{i,j,k=1}^3 \left[b^2\frac{\partial h^k}{\partial x_j}\frac{\partial u^i}{\partial x_j}\frac{\partial u^i}{\partial x_k} + (a^2-b^2)\frac{\partial h^k}{\partial x_i}\frac{\partial u^j}{\partial x_j}\frac{\partial u^i}{\partial x_k}\right] dx\, dt \\ &+ \int_S^T\int_\Omega (2h(x)*\nabla u)\cdot\rho(x,u_t)\, dx\, dt \\ =& \int_S^T\int_\Gamma (h\cdot\eta)\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] d\Gamma\, dt \\ &+ \int_S^T\int_\Gamma \left[b^2(2h*\nabla u)\cdot\frac{\partial u}{\partial\eta} + (a^2-b^2)(2h*\nabla u)\cdot\eta\, div\, u\right] d\Gamma\, dt\end{aligned} \tag{18}$$

$$\begin{aligned}&\int_\Omega u_t\cdot\xi u\, dx\Big|_S^T + \int_S^T\int_\Omega \xi\left[-|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dx\, dt \\ &- \int_S^T\int_\Omega \left[\frac{b^2}{2}|u|^2\Delta\xi - (a^2-b^2)\nabla\xi\cdot u\, div\, u\right] dx\, dt \\ &+ \int_S^T\int_\Omega \rho(x,u_t)\cdot\xi u\, dx\, dt = 0\end{aligned} \tag{19}$$

where $\eta = \eta(x)$ is the outward unit normal on the boundary Γ and $\frac{\partial u}{\partial\eta} = \left(\frac{\partial u^1}{\partial\eta}, \frac{\partial u^2}{\partial\eta}, \frac{\partial u^3}{\partial\eta}\right)$ with $\frac{\partial u^i}{\partial\eta}$ the normal derivative of u^i , $i = 1,2,3$.

Proof. The identity (17) is obtained taking inner product of the equation (1) with the multiplier $M(u) = 2u$ and integrating in $\Omega \times [S,T]$. The identity (18) is obtained in the same way with the vector multiplier $M(u) = 2h * \nabla u \equiv 2\,(h\cdot\nabla u^1, h\cdot\nabla u^2, h\cdot\nabla u^3) \in \mathbb{R}^3$. Finally, the identity (19) is obtained with the multiplier $M(u) = \xi\, u$.

Now we want to estimate the integrals on the boundary in the identity (18) which appears in Lemma 2. To do this, we consider Ω_o a subset of Ω, $\Omega_o \subset \Omega$, $x_o \in \mathbb{R}^3$ fixed and

$$\Gamma_o = \Gamma(x_0) = \{x \in \Gamma\colon (x - x_o)\cdot\eta(x) > 0\}. \tag{20}$$

We assume that exists $\varepsilon > 0$ such that

$$\Omega \cap V_\varepsilon \subset \omega \subset \Omega \tag{21}$$

where $V_\varepsilon = V_\varepsilon[\Gamma_o \cup (\Omega \setminus \Omega_o)]$ is a neighborhood of $\Gamma_o \cup (\Omega \setminus \Omega_o)$ with radius $\varepsilon > 0$. The set ω is given in (8)-(10)

We consider numbers $\varepsilon_2 > \varepsilon_1 > \varepsilon_o > 0$, such that $\varepsilon_2 < \varepsilon$ and we define the sets

$$\mathcal{O}_i = V_{\varepsilon_i} = V_{\varepsilon_i}[\Gamma_o \cup (\Omega \setminus \Omega_o)] \quad i = 0,1,2. \tag{22}$$

We note that $(\overline{\Omega}_o \setminus \mathcal{O}_1) \cap \overline{\mathcal{O}}_o$ is an empty set, therefore we can construct a function $\psi \in C_o^\infty(\mathbb{R}^3)$ such that

$$0 \le \psi \le 1, \tag{23}$$

$$\psi = 1 \qquad \text{in} \qquad \overline{\Omega}_o \setminus \mathcal{O}_1, \tag{24}$$

$$\psi = 0 \qquad \text{in} \qquad \mathcal{O}_o \tag{25}$$

Now, with the above considerations we have

Lemma 3 *Let u be the solution of system* (1)-(3). *Then*

$$\begin{aligned}
&\int_S^T \int_{\partial\Omega_0} \psi(x)(x-x_o)\cdot\eta\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)\,(div\, u)^2\right] d\Gamma\, dt \\
&+ \int_S^T \int_{\partial\Omega_0} 2b^2(\psi(x)(x-x_o) * \nabla u)\cdot\frac{\partial u}{\partial\eta} d\Gamma\, dt \\
&+ \int_S^T \int_{\partial\Omega_0} 2(a^2-b^2)(\psi(x)(x-x_o) * \nabla u)\cdot\eta\, div\, u d\Gamma\, dt \le 0
\end{aligned}$$

where $\eta = \eta(x)$ is the normal in $x \in \Gamma$.

Proof. From the construction of the function ψ we can see that $\psi\Big|_{\partial\Omega_o} = 0$ outside of $(\partial\Omega_o \setminus \Gamma_o) \cap \partial\Omega$. In fact, if $x \notin (\partial\Omega_o \setminus \Gamma_o) \cap \partial\Omega$ and $x \in \partial\Omega_o$ then $x \in \mathcal{O}_o \cap \partial\Omega_o \subset \mathcal{O}_o$. Thus, we have

$$
\begin{aligned}
I(\partial\Omega_o) = & \int_S^T \int_{\partial\Omega_0} \psi(x)(x-x_o)\cdot\eta\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] d\Gamma\, dt \\
& + \int_S^T \int_{\partial\Omega_0} \Big[2b^2(\psi(x)(x-x_o)*\nabla u)\cdot\frac{\partial u}{\partial\eta} \\
& \qquad + 2(a^2-b^2)(\psi(x)(x-x_o)*\nabla u)\cdot\eta\, div\, u\Big] d\Gamma\, dt \\
= & \int_S^T \int_{[\partial\Omega_o\setminus\Gamma_o]\cap\partial\Omega} \psi(x)(x-x_o)\cdot\eta\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] d\Gamma\, dt \\
& + \int_S^T \int_{[\partial\Omega_o\setminus\Gamma_o]\cap\partial\Omega} \Big[2b^2(\psi(x)(x-x_o)*\nabla u)\cdot\frac{\partial u}{\partial\eta} \\
& \qquad + 2(a^2-b^2)(\psi(x)(x-x_o)*\nabla u)\cdot\eta\, div\, u\Big] d\Gamma\, dt
\end{aligned} \tag{26}
$$

In this point we observe that $\nabla u^i = \dfrac{\partial u^i}{\partial\eta}\eta$ on $\partial\Omega$ because $u = 0$ on $\partial\Omega$ (see Lions [12]). Then we have

$$
\begin{aligned}
(2\Psi(x)(x-x_o)*\nabla u)\cdot\frac{\partial u}{\partial\eta} &= \sum_{i=1}^{3}(2\Psi(x)(x-x_o)\cdot\nabla u^i)\frac{\partial u^i}{\partial\eta} \\
&= \sum_{i=1}^{3}\left(2\Psi(x)(x-x_o)\cdot\frac{\partial u^i}{\partial\eta}\eta\right)\frac{\partial u^i}{\partial\eta} \\
&= 2\Psi(x)(x-x_o)\cdot\eta\left|\frac{\partial u}{\partial\eta}\right|^2
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
(2\Psi(x)(x-x_o)*\nabla u)\cdot\eta\; div\,u &= \sum_{i=1}^{3}(2\Psi(x)(x-x_o)\cdot\nabla u^i)\eta^i\; div\,u \\
&= \sum_{i=1}^{3} 2(\Psi(x)(x-x_o)\cdot\frac{\partial u^i}{\partial\eta}\eta)\eta^i\; div\,u \\
&= 2\Psi(x)(x-x_o)\cdot\eta\; div\,u\sum_{i=1}^{3}\eta^i\frac{\partial u^i}{\partial\eta} \\
&= 2\Psi(x)(x-x_o)\cdot\eta\;(div\,u)^2 \qquad (28)
\end{aligned}
$$

where $\eta=(\eta^1,\eta^2,\eta^3)$ is the normal on $\Gamma=\partial\Omega$.

Substituting (27) and (28) in (26) and observing that $\dfrac{\partial u}{\partial t}=0$ on $\partial\Omega$ (because $u(x,t)=0$ on $\partial\Omega\times\mathbb{R}$) we obtain that

$$
\begin{aligned}
I(\partial\Omega_o) &= \int_S^T\int_{(\partial\Omega_0\setminus\Gamma_o)\cap\partial\Omega}\psi(x)(x-x_o)\cdot\eta\left[-b^2\,|\nabla u|^2-(a^2-b^2)\,(div\,u)^2\right]d\Gamma\,dt \\
&+\int_S^T\int_{(\partial\Omega_0\setminus\Gamma_o)\cap\partial\Omega}2(\psi(x)(x-x_o)\cdot\eta)\left\{b^2\left|\frac{\partial u}{\partial\eta}\right|^2+(a^2-b^2)(div\,u)^2\right\}d\Gamma\,dt \\
&=\int_S^T\int_{[\partial\Omega_o\setminus\Gamma_o]\cap\partial\Omega}\psi(x)(x-x_o)\cdot\eta\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2+(a^2-b^2)\,(div\,u)^2\right]d\Gamma\,dt\le 0
\end{aligned}
\tag{29}
$$

because $(x-x_o)\cdot\eta(x)\le 0$ on $\partial\Omega\setminus\Gamma(x_o)$. We have used the fact that $\left|\dfrac{\partial u}{\partial\eta}\right|^2=|\nabla u|^2$ on $\partial\Omega$.

Here we fixed a real arbitrary number $\lambda>0$ and we need the following estimates.

Lemma 4 *Let u be the solution of* (1)-(3). *Then there exists a positive constant $C>0$ such that*

$$
\left|\int_\Omega M(u)\cdot u_t\,dx\right|\le C\,E(S),\quad \forall\,t\ge S \tag{30}
$$

$$
\left|\int_S^T\int_\Omega M(u)\cdot\rho(x,u_t)\,dxdt\right|\le\frac{\lambda}{2}\int_S^T E(t)dt+\frac{C}{\lambda}\int_S^T\int_\Omega|\rho(x,u_t)|^2\,dxdt \tag{31}
$$

where $M(u)=2\Psi(x)(x-x_o)*\nabla u+2u$ and $\Psi(x)$ is the function defined in (23)-(25).

Proof. The proof of Lemma 4 is a consequence of Hölder and Poincare's inequalities and the fact that $E(t)$ is a nonincreasing function.

$\square$

Lemma 5 *Let u be the solution of (1)-(3). Then there exists a positive constant C such that*

$$\left|\int_S^T \int_{\Omega\cap\mathcal{O}_1} \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt\right|$$
$$\leq C\, E(S) + C(2+\frac{1}{\delta})\int_S^T \int_{\Omega\cap\mathcal{O}_2} |u|^2 dxdt + C\int_S^T \int_{\Omega\cap\mathcal{O}_2} |\rho(x,u_t)|^2 dxdt$$
$$+\int_S^T \int_{\Omega\cap\mathcal{O}_2} |u_t|^2 dxdt + \frac{\delta}{2}\int_S^T E(t)dt$$

with $\delta > 0$ a real number to be chosen later.

Proof. Since $\overline{\mathbb{R}^3 \setminus \mathcal{O}_2} \cap \overline{\mathcal{O}}_1 = \emptyset$ we can construct a function $\xi \in C_o^\infty(\mathbb{R})$ such that

$$0 \leq \xi \leq 1 \tag{32}$$
$$\xi = 1, \quad \text{in} \quad \mathcal{O}_1 \tag{33}$$
$$\xi = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \mathcal{O}_2 \tag{34}$$

With this function we obtain from identity (19) in Lemma 2:

$$\begin{aligned}
&\int_S^T \int_{\Omega\cap\mathcal{O}_1} \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt \\
&\leq \int_S^T \int_\Omega \xi(x)\left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt \\
&= -\int_S^T \int_\Omega \xi u\cdot\rho(x,u_t)dxdt - \int_\Omega \xi u\cdot u_t\, dx\Big|_S^T + \int_S^T \int_\Omega \xi |u_t|^2 dxdt \\
&\quad + \int_S^T \int_\Omega \left[\frac{b^2}{2}|u|^2\Delta\xi - (a^2-b^2)\nabla\xi\cdot u\, div\, u\right] dxdt
\end{aligned} \tag{35}$$

Using the fact that $\xi = 0$ outside the set $\mathcal{O}_2$ we have

$$\begin{aligned}\left|\int_S^T \int_\Omega \xi(x) u \cdot \rho(x,u_t) dx dt\right| &= \left|\int_S^T \int_{\Omega\cap\mathcal{O}_2} \xi(x) u \cdot \rho(x,u_t) dx dt\right| \\ &\le \int_S^T \int_{\Omega\cap\mathcal{O}_2} |u||\rho(x,u_t)| dx dt \\ &\le \frac{1}{2}\int_S^T \int_{\Omega\cap\mathcal{O}_2} \left[|u|^2 + \rho(x,u_t)^2\right] dx dt \qquad (36)\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega \xi(x) u \cdot u_t \, dx\Big|_S^T\right| &\le \int_\Omega \left(\frac{|u|^2}{2} + \frac{|u_t|^2}{2}\right) dx\Big|_S^T \\ &\le \int_\Omega \left(C(\Omega)\frac{|\nabla u|^2}{2} + \frac{|u_t|^2}{2}\right) dx\Big|_S^T \\ &\le C\, E(t)\Big|_S^T \le C\, E(T) \\ &\le C\, E(S), \quad 0 \le S \le T. \qquad (37)\end{aligned}$$

because $E(t)$ is non increasing and we have used Poincare's inequality.

Because $\xi = 0$ outside $\mathcal{O}_2$, we note that

$$\int_S^T \int_\Omega \xi(x) |u_t|^2 dx dt \le \int_S^T \int_{\Omega\cap\mathcal{O}_2} |u_t|^2 dx dt \qquad (38)$$

Finally, we estimate

$$\begin{aligned}&\int_S^T \int_\Omega \Big[\frac{b^2}{2}|u|^2 \Delta\xi - (a^2 - b^2)\nabla\xi \cdot u\, div\, u\Big] dx dt \\ &= \int_S^T \int_{\Omega\cap\mathcal{O}_2} \Big[\frac{b^2}{2}|u|^2 \Delta\xi - (a^2 - b^2)\nabla\xi \cdot u\, div\, u\Big] dx dt \\ &\le C \int_S^T \int_{\Omega\cap\mathcal{O}_2} \Big[|u|^2 + |u||div\, u|\Big] dx dt \\ &\le C(1+\frac{1}{\delta}) \int_S^T \int_{\Omega\cap\mathcal{O}_2} |u|^2 dx dt + \frac{\delta}{4}(a^2-b^2) \int_S^T \int_\Omega (div\, u)^2\, dx dt \\ &\le C(1+\frac{1}{\delta}) \int_S^T \int_{\Omega\cap\mathcal{O}_2} |u|^2 dx dt + \frac{\delta}{2}\int_S^T E(t) dt \qquad (39)\end{aligned}$$

Substituting (36)-(39) in (35) we obtain Lemma 5.

Lemma 6 *We consider u the solution of problem (1)-(3). Then there exists a positive constant C such that*

$$\int_S^T \int_{\Omega\cap\mathcal{O}_2} |u|^2 dx dt \leq C\, E(S) + \frac{\lambda}{2}\int_S^T E(t)\, dt$$
$$+ C\int_S^T \int_\Omega |\rho(x,u_t)|^2 dx dt + \frac{C}{\lambda}\int_S^T \int_\omega |u_t|^2 dx dt$$

where $\lambda > 0$ is the same number in Lemma 4.

Proof. We use the fact that $\overline{\mathbb{R}^3 \setminus \omega} \cap \overline{\mathcal{O}}_2 = \emptyset$ to construct a function $\varphi \in C_o^\infty(\mathbb{R}^3)$ such that

$$0 \leq \varphi \leq 1 \tag{40}$$
$$\varphi = 1, \quad \text{in} \quad \mathcal{O}_2 \tag{41}$$
$$\varphi = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \omega \tag{42}$$

For fixed t, we consider v the solution of the vector elliptic problem

$$b^2\Delta v + (a^2 - b^2)\nabla div\ v = \varphi(x)\, u \quad \text{in } \Omega \tag{43}$$
$$v = 0, \quad \text{on} \quad \Gamma = \partial\Omega \tag{44}$$

Taking the inner product between the equation (43) and v, integrating on Ω, using (44) and Poincare's inequality, we can conclude that

$$\int_\Omega |v|^2 dx \leq C\int_\Omega |u|^2 dx \tag{45}$$

On the other hand we observe that v_t is the solution of the problem

$$b^2\Delta v_t + (a^2 - b^2)\nabla div\ v_t = \varphi(x)\, u_t \quad \text{in } \Omega \tag{46}$$
$$v_t = 0, \quad \text{on} \quad \Gamma = \partial\Omega \tag{47}$$

In the same way we have obtained (45) we conclude that

$$\int_\Omega |v_t|^2 dx \leq C\int_\Omega \varphi(x)|u_t|^2 dx \tag{48}$$

Now, taking the inner product of equation (1) with the vector function v and integrating in $\Omega \times [S,T]$, we obtain

$$
\begin{aligned}
0 = & \int_\Omega v \cdot u_t \, dx \Big|_S^T \\
& + \int_S^T \int_\Omega \left[-v_t \cdot u_t - b^2 u \cdot \Delta v - (a^2 - b^2) u \cdot \nabla div\, v + v \cdot \rho(x, u_t) \right] dxdt
\end{aligned}
$$

From this identity and the fact that v is the solution of (43)-(44) and using the definition of function $\varphi(x)$, we have

$$
\begin{aligned}
\int_S^T \int_{\Omega \cap \mathcal{O}_2} |u|^2 dxdt \leq & \int_S^T \int_\Omega \varphi(x) |u|^2 dxdt \\
= & \int_\Omega v \cdot u_t \, dx \Big|_S^T - \int_S^T \int_\Omega v_t \cdot u_t \, dxdt \\
& + \int_S^T \int_\Omega v \cdot \rho(x, u_t) \, dxdt
\end{aligned} \tag{49}
$$

We are going to estimate the terms of (49). We have

$$
\left| \int_\Omega v \cdot u_t \, dx \Big|_S^T \right| \leq C\, E(S) \tag{50}
$$

$$
\left| \int_S^T \int_\Omega v_t \cdot u_t dxdt \right| \leq \frac{C}{\lambda} \int_S^T \int_\omega |u_t|^2 dxdt + \frac{\lambda}{4} \int_S^T E(t)\, dt \tag{51}
$$

$$
\left| \int_\Omega v \cdot \rho(x, u_t) dxdt \right| \leq \frac{C}{\lambda} \int_S^T \int_\Omega |\rho(x, u_t)|^2 dxdt + \frac{\lambda}{4} \int_S^T E(t)\, dt \tag{52}
$$

where we have used the estimates (45), (48), Poincaré's inequality and the definition of the function φ.

Using (50)-(52) in (49) the Lemma 6 follows.

□

Lemma 7 *We consider u the solution of system* (1)-(3) *with our previous hypotheses. Then there exists a positive constant C such that*

$$
\begin{aligned}
2 \int_S^T E(t)\, dt + \int_\Omega M(u) \cdot u_t \, dx \Big|_S^T + \int_S^T \int_\Omega M(u) \cdot \rho(x, u_t) dxdt \\
\leq C \int_S^T \int_{\Omega \cap \mathcal{O}_1} \left[|u_t|^2 + b^2 |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 \right] dxdt
\end{aligned}
$$

Proof. The identity (18) in the Lemma 2 holds with Ω_o instead of Ω and $h = \psi(x)(x-x_o)$, where the function $\psi(x)$ is given in (23)-(25). Thus, using Lemma 3 we have

$$\int_{\Omega_o} (2\psi(x)(x-x_o) * \nabla u)\cdot u_t\,dx\Big|_S^T \tag{53}$$
$$+\int_S^T\int_{\Omega_o} div(\psi(x)(x-x_o))\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\,u)^2\right]dxdt$$
$$+2\int_S^T\int_{\Omega_o}\sum_{i,j,k=1}^3\left[b^2\frac{\partial}{\partial x^j}(\psi(x)(x-x_o))^k\frac{\partial u^i}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$
$$+2\int_S^T\int_{\Omega_o}\sum_{i,j,k=1}^3\left[(a^2-b^2)\frac{\partial}{\partial x_i}(\psi(x)(x-x_o))^k\frac{\partial u^j}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$
$$+\int_S^T\int_{\Omega_o}(2\psi(x)(x-x_o)*\nabla u)\cdot\rho(x,u_t)\,dxdt \le 0$$

where $(\psi(x)(x-x_o))^k$ means the $k-$coordinate of the vector $\psi(x)(x-x_o)$.

Now, we observe that

$$\frac{\partial}{\partial x^i}(\psi(x)(x-x_o))^k = \frac{\partial\psi}{\partial x^i}(x)(x^k-x_o^k)+\psi(x)\delta_{ik}$$

where $\delta_{ik}=1$ if $i=k$, $\delta_{ik}=0$ if $i\neq k$ and $(x^k-x_o^k)$ is the $k-$ coordinate of the vector $(x-x_o)\in\mathbb{R}^3$.

Then, because $\psi=1$ in $\overline{\Omega}_o\setminus\mathcal{O}_1$ (see (24)) it is easy to see that

$$\int_S^T\int_{\Omega_o}\sum_{i,j,k=1}^3\left[b^2\frac{\partial}{\partial x^j}(\psi(x)(x-x_o))^k\frac{\partial u^i}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$
$$+\int_S^T\int_{\Omega_o}\sum_{i,j,k=1}^3\left[(a^2-b^2)\frac{\partial}{\partial x^i}(\psi(x)(x-x_o))^k\frac{\partial u^j}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$
$$=\int_S^T\int_{\Omega_o}\left[b^2\psi(x)\sum_{j=1}^3|\nabla u^j|^2+(a^2-b^2)\psi(x)(div\,u)^2\right]dxdt$$
$$+\int_S^T\int_{\Omega_o\cap\mathcal{O}_1}\sum_{i,j,k=1}^3\left[b^2(x^k-x_o^k)\frac{\partial}{\partial x^j}\psi(x)\frac{\partial u^i}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$
$$+\int_S^T\int_{\Omega_o\cap\mathcal{O}_1}\sum_{i,j,k=1}^3\left[(a^2-b^2)(x^k-x_o^k)\frac{\partial}{\partial x_i}\psi(x)\frac{\partial u^j}{\partial x^j}\frac{\partial u^i}{\partial x^k}\right]dxdt$$

Using this fact in (53) we obtain

$$
\begin{aligned}
&\int_{\Omega_o} (2\psi(x)(x-x_o) * \nabla u) \cdot u_t \, dx \Big|_S^T \\
&\quad + \int_S^T \int_{\Omega_o} div(\psi(x)(x-x_o)) \left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2 \right] dxdt \\
&\quad + 2\int_S^T \int_{\Omega_o} (b^2\psi(x)|\nabla u|^2 + (a^2-b^2)\psi(x)(div\, u)^2)\, dxdt \qquad (54) \\
&\quad + \int_S^T \int_{\Omega_o} (2\psi(x)(x-x_o) * \nabla u) \cdot \rho(x, u_t)\, dxdt \\
&\leq -2\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \sum_{i,j,k=1}^{3} \left[b^2(x^k - x_o^k) \frac{\partial}{\partial x^j}\psi(x) \frac{\partial u^i}{\partial x^j} \frac{\partial u^i}{\partial x^k} \right] dxdt \\
&\quad - 2\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \sum_{i,j,k=1}^{3} \left[(a^2-b^2)(x^k - x_o^k) \frac{\partial}{\partial x_i}\psi(x) \frac{\partial u^j}{\partial x^j} \frac{\partial u^i}{\partial x^k} \right] dxdt.
\end{aligned}
$$

From above inequality we have

$$
\begin{aligned}
&\int_{\Omega_o} (2\psi(x)(x-x_o) * \nabla u) \cdot u_t \, dx \Big|_S^T \qquad (55) \\
&\quad + \int_S^T \int_{\Omega_o \setminus \mathcal{O}_1} div(\psi(x)(x-x_o)) \left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2 \right] dxdt \\
&\quad + 2\int_S^T \int_{\Omega_o \setminus \mathcal{O}_1} \left[b^2\psi(x)|\nabla u|^2 + (a^2-b^2)\psi(x)(div\, u)^2 \right] dxdt \\
&\quad + \int_S^T \int_{\Omega_o} (2\psi(x)(x-x_o) * \nabla u) \cdot \rho(x, u_t)\, dxdt \\
&\leq -\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} div(\psi(x)(x-x_o)) \left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2 \right] dxdt \\
&\quad - 2\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \left[b^2\psi(x)|\nabla u|^2 + (a^2-b^2)\psi(x)(div\, u)^2 \right] dxdt \\
&\quad - 2\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \sum_{i,j,k=1}^{3} \left[b^2(x^k - x_o^k) \frac{\partial}{\partial x^j}\psi(x) \frac{\partial u^i}{\partial x^j} \frac{\partial u^i}{\partial x^k} \right] dxdt \\
&\quad - 2\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \sum_{i,j,k=1}^{3} \left[(a^2-b^2)(x^k - x_o^k) \frac{\partial}{\partial x_i}\psi(x) \frac{\partial u^j}{\partial x^j} \frac{\partial u^i}{\partial x^k} \right] dxdt \\
&\leq C\int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2 \right] dxdt
\end{aligned}
$$

where we have used the fact that $0 \leq \psi \leq 1$ and $\psi = 1$ in $\Omega_o \setminus \mathcal{O}_1$ and the fact that

$$\int_{\Omega_o} = \int_{\Omega_o \setminus \mathcal{O}_1} + \int_{\Omega_o \cap \mathcal{O}_1}.$$

We observe that the positive constant C in (55) depends on

$$\sup_{\Omega_o \cap \mathcal{O}_1} |div(\psi(x)(x - x_o))|$$

and

$$\max_{1 \leq j,k \leq 3} \sup_{\Omega_o \cap \mathcal{O}_1} |\frac{\partial \psi(x)}{\partial x_j}(x^k - x_o^k)|.$$

Using again the fact that $\psi(x) = 1$ in $\overline{\Omega}_o \setminus \mathcal{O}_1$ and $\Omega = (\Omega \cap \mathcal{O}_1) \cup (\Omega_o \setminus \mathcal{O}_1)$, the inequality (55) implies that

$$\begin{aligned}
&\int_{\Omega_o} (2\psi(x)(x - x_o) * \nabla u) \cdot u_t \, dx \Big|_S^T \qquad (56)\\
&\quad + \int_S^T \int_{\Omega_o \setminus \mathcal{O}_1} div(\psi(x)(x - x_o)) \left[|u_t|^2 - b^2|\nabla u|^2 - (a^2 - b^2)(div\, u)^2\right] dxdt\\
&\quad + 2\int_S^T \int_{\Omega} \left[b^2|\nabla u|^2 + (a^2 - b^2)(div\, u)^2\right] dxdt\\
&\quad + \int_S^T \int_{\Omega_o} (2\psi(x)(x - x_o) * \nabla u) \cdot \rho(x, u_t) \, dxdt\\
&\leq C \int_S^T \int_{\Omega_o \cap \mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2 - b^2)(div\, u)^2\right] dxdt\\
&\quad + 2\int_S^T \int_{\Omega \cap \mathcal{O}_1} \left[b^2|\nabla u|^2 + (a^2 - b^2)(div\, u)^2\right] dxdt\\
&\leq (2 + C) \int_S^T \int_{\Omega \cap \mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2 - b^2)(div\, u)^2\right] dxdt.
\end{aligned}$$

Now we note that $\psi = 0$ in $\Omega \setminus \Omega_o$ and $div\,(\psi(x)(x-x_o)) = 3$ in $\Omega_o \setminus \mathcal{O}_1$(see definition of function ψ). Then, we obtain

$$
\begin{aligned}
&\int_\Omega (2\psi(x)(x-x_o) * \nabla u)\cdot u_t\, dx\Big|_S^T \\
&\quad + \int_S^T \int_{\Omega_o \setminus \mathcal{O}_1} 3\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] dxdt \\
&\quad + 2\int_S^T \int_\Omega \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt \\
&\quad + \int_S^T \int_\Omega (2\psi(x)(x-x_o) * \nabla u)\cdot \rho(x,u_t)\, dxdt \\
&\le C\int_S^T \int_{\Omega\cap\mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt.
\end{aligned}
\tag{57}
$$

Adding the identity (17) in Lemma 2 and the inequality (57) it results

$$
\begin{aligned}
&\int_\Omega M(u)\cdot u_t\, dx\Big|_S^T + \int_S^T \int_\Omega M(u)\cdot \rho(x,u_t)\, dxdt \\
&\quad + \int_S^T \int_{\Omega_o \setminus \mathcal{O}_1} 3\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2\right] dxdt \\
&\quad + 4\int_S^T \int_\Omega \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt - 2\int_S^T \int_\Omega |u_t|^2 dxdt \\
&\le C\int_S^T \int_{\Omega\cap\mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt
\end{aligned}
$$

Because $\int_\Omega = \int_{\Omega_o\setminus\mathcal{O}_1} + \int_{\Omega\cap\mathcal{O}_1}$ we have

$$
\begin{aligned}
&\int_\Omega M(u)\cdot u_t\, dx\Big|_S^T + \int_S^T \int_\Omega M(u)\cdot \rho(x,u_t) dxdt + \\
&\quad + \int_S^T \int_\Omega \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt + \int_S^T \int_\Omega |u_t|^2\, dxdt \\
&\le C\int_S^T \int_{\Omega\cap\mathcal{O}_1} \left[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dxdt
\end{aligned}
\tag{58}
$$

Thus

$$\int_S^T 2E(t)dt + \int_\Omega M(u)\cdot u_t\,dx\Big|_S^T + \int_S^T\int_\Omega M(u)\cdot\rho(x,u_t)dxdt \tag{59}$$
$$\leq C\int_S^T\int_{\Omega\cap\mathcal{O}_1}\left[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\,u)^2\right]dxdt$$

where we have indicated by C different positive constants which are independent of the solution u. The proof of Lemma 7 is complete.

□

5 Proof of Theorem 2

Combining estimates obtained in Lemma 4 with the inequality in Lemma 7 we conclude that

$$2\int_S^T E(t)dt \leq CE(S) + C\int_S^T\int_{\Omega\cap\mathcal{O}_1}\left[b^2|\nabla u|^2 + (a^2-b^2)(div\,u)^2\right]dxdt$$
$$+ C\int_S^T\int_{\Omega\cap\mathcal{O}_1}|u_t|^2\,dxdt + \frac{C}{\lambda}\int_S^T\int_\Omega|\rho(x,u_t)|^2dxdt + \frac{\lambda}{2}\int_S^T E(t)dt$$

Now we use the estimate in Lemma 5 to obtain

$$2\int_S^T E(t)dt \leq CE(S) + C\int_S^T\int_{\Omega\cap\mathcal{O}_1}|u_t|^2\,dxdt + C(2+1/\delta)\int_S^T\int_{\Omega\cap\mathcal{O}_2}|u|^2dxdt$$
$$+ C\int_S^T\int_{\Omega\cap\mathcal{O}_2}|\rho(x,u_t)|^2dxdt + C\frac{\delta}{2}\int_S^T E(t)dt$$
$$+ C\int_S^T\int_{\Omega\cap\mathcal{O}_2}|u_t|^2\,dxdt + \frac{\lambda}{2}\int_S^T E(t)dt + \frac{C}{\lambda}\int_S^T\int_\Omega|\rho(x,u_t)|^2dxdt$$

Then,

$$2\int_S^T E(t)dt \leq CE(S) + C\int_S^T\int_{\Omega\cap\mathcal{O}_2}|u_t|^2\,dxdt + C(2+1/\delta)\int_S^T\int_{\Omega\cap\mathcal{O}_2}|u|^2dxdt$$
$$+ C(1+1/\lambda)\int_S^T\int_\Omega|\rho(x,u_t)|^2dxdt + (C\frac{\delta}{2}+\frac{\lambda}{2})\int_S^T E(t)dt$$

Now we choose $\delta > 0$ such that $\dfrac{C\,\delta}{2} = 1$ and we apply the result of Lemma 6 to obtain

$$\begin{aligned}
\int_S^T E(t)dt &\leq C\,E(S) + C\int_S^T\int_{\Omega\cap\mathcal{O}_2} |u_t|^2\,dxdt + C\lambda\int_S^T E(t)dt \\
&\quad + C(2+1/\lambda)\int_S^T\int_\Omega |\rho(x,u_t)|^2\,dxdt + \frac{C}{\lambda}\int_S^T\int_\omega |u_t|^2 dxdt \\
&\leq C\,E(S) + C\int_S^T\int_\omega |u_t|^2\,dxdt + C(2+1/\lambda)\int_S^T\int_\Omega |\rho(x,u_t)|^2\,dxdt \\
&\quad + C\lambda\int_S^T E(t)dt
\end{aligned}$$

because $\Omega\cap\mathcal{O}_2 \subset \omega$ due to $0 < \varepsilon_2 < \varepsilon$.

Taking $\lambda > 0$ such that $C\lambda < 1/2$ we obtain the following estimate for the total energy

$$\int_S^T E(t)dt \leq C\left[E(S) + \int_S^T\int_\omega |u_t|^2 dxdt + \int_S^T\int_\Omega |\rho(x,u_t)|^2 dxdt\right] \tag{60}$$

for $0 \leq S < T < +\infty$.

Finally, using the hypotheses

$$\begin{aligned}
a(x)|s|^2 &\leq \rho(x,s)\cdot s \quad \forall\, x\in\overline{\Omega}, \quad \forall\, s\in\mathbb{R}^3 \\
|\rho(x,s)| &\leq C\,a(x)|s|, \quad \forall\, x\in\overline{\Omega}, \quad \forall\, s\in\mathbb{R}^3 \\
\rho(x,s)\cdot s &\geq 0, \quad \forall\, x\in\overline{\Omega}, \quad \forall\, s\in\mathbb{R}^3
\end{aligned}$$

on the function $\rho(x,s)$ and the condition $a(x) \geq a_o > 0$ in ω, we have from estimate (60)

$$\begin{aligned}
\int_S^T E(t)dt &\leq C\,E(S) + \frac{C}{a_o}\int_S^T\int_\Omega a(x)|u_t|^2\,dxdt + C\int_S^T\int_\Omega |\rho(x,u_t)|^2\,dxdt \\
&\leq C\,E(S) + C\int_S^T\int_\Omega a(x)|u_t|^2\,dxdt \\
&\leq C\left[E(S) + \int_S^T\int_\Omega \rho(x,u_t)\cdot u_t\,dxdt\right]
\end{aligned}$$

for $0 \leq S < T < +\infty$, where C depends on $\|a\|_\infty$ and a_o. In fact, the hypotheses on function ρ say that

$$a(x)|u_t|^2 \leq \rho(x,u_t)\cdot u_t$$

and

$$|\rho(x,u_t)|^2 \le C^2 \|a\|_\infty a(x) |u_t|^2$$

Using identity (13) we conclude that

$$\int_S^T E(t)dt \le C\, E(S)$$

for all $0 \le S < T < +\infty$.

Then, Lemma 2 implies Theorem 2.

$\square$

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