HOMOGENIZATION AND CORRECTOR RESULTS FOR A NONLINEAR REACTION-DIFFUSION EQUATION IN DOMAINS WITH SMALL HOLES

Joel Souza **Airton Kist**

Abstract

This paper is mainly devoted to the homogenization for a nonlinear reaction-diffusion equation with homogeneous Dirichlet boundary condition in a domain containing tiny holes periodically distributed in each direction of the axis. For holes within a critical size (for example, in the three dimensional case, the obstacles have a size of ε^3 and are located at the nodes of a regular mesh of size ε), the proofs are performed in the abstract framework introduced by D. Cioranescu and F. Murat for the study of homogenization of elliptic problems in domains with tiny holes. This is based on the use of suitable test functions adapted to the geometry of the problem.

1 Introduction

In this paper, we study the homogenization and corrector for a nonlinear reaction-diffusion equation with homogeneous Dirichlet boundary condition in a domain containing "tiny" holes periodically distributed in each direction of the axis.

Let Ω be a fixed bounded domain of $\mathbb{R}^N$ $(N \geq 2)$. Denote by Ω_ε the domain obtained by removing from Ω a set $S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S_\varepsilon^i$ of $N(\varepsilon)$ tiny closed subsets of Ω, namely, $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} S_\varepsilon^i$. Here, $\varepsilon > 0$ denotes a parameter which takes its values in a sequence which tends to zero while $N(\varepsilon)$ tends to infinity. Finally, let $T > 0$ be fixed. We consider the following nonlinear reaction-diffusion equation

$$\begin{cases} u_\varepsilon' - \beta\Delta u_\varepsilon + |u_\varepsilon|^\rho u_\varepsilon + \alpha^2 u_\varepsilon = f_\varepsilon & \text{in} \quad Q_\varepsilon = \Omega_\varepsilon \times (0,T), \\ u_\varepsilon = 0 & \text{on} \quad \Sigma_\varepsilon = \partial\Omega_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{in} \quad \Omega_\varepsilon, \end{cases} \tag{1.1}$$

where β and ρ are positive constants and $\alpha \in \mathbb{R}$.

The study of this equation is motivated by the reaction-diffusion equation that appears in Brèzis [3],

$$\begin{cases} u' - M\Delta u = f(u) \quad \text{in} \quad \Omega \times (0,T) \\ + \text{ Boundary conditions and initial data,} \end{cases}$$

where $u(x,t)$ is a vector with N components, M is a $N \times N$ diagonal square matrix and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear application.

We are considering a decoupled vector equation, thus, it suffices to study the equation with only one component, i.e., in the scalar form. Such reaction-diffusion equations model phenomena that appear in several fields of the science such that: chemistry, biology, neurofisilogy, epidemiology, combustion, genetics of population, etc.

The work that we develop here follows from [9] and is essentially based on [7]. The homogenization of the reaction-diffusion equation is also studied here by using some techniques established since 1977, by Luc Tartar (see [18]).

In the whole paper, the sets Ω_ε satisfy the conditions of the abstract framework introduced by Doina Cioranescu and François Murat [9] (see hypothesis (2.1)). The study of the homogenization of elliptic problems in perforated domains with "tiny" holes, with homogeneous Dirichlet boundary conditions.

The model case to our study is provided by a periodically perforated domain (with a period 2ε in the direction of each coordinative axis) with holes of size $a_{S^i_\varepsilon}$, where $a_{S^i_\varepsilon}$ is asymptotically equal to the "critical size" a_ε. This critical size a_ε is given by

$$a_\varepsilon = \begin{cases} \delta_\varepsilon \exp(-C_0/\varepsilon^2) & \text{for } N = 2, \\ C_0 \varepsilon^{N/(N-2)} & \text{for } N \geq 3, \end{cases} \tag{1.2}$$

where $C_0 > 0$ is fixed and $\varepsilon^2 \log \delta_\varepsilon \to 0$, where $\varepsilon \to 0$ (see [7], Section 2).

This condition is fundamental in the construction of the abstract framework of hypotheses about the holes. The proofs given in this paper are based on the existence of such an abstract framework of hypotheses.

We observe that the main difficulty of our work is that, due to the hypotheses about the data of the problem, we only obtain weak convergence of the term Δu_ε. This motivates us to use the abstract framework (2.1) about the holes.

Thus, it will be possible the passage to the limit in this term in the variational formulation. We still observe that for the treatment of this problem other techniques would be possible, such as compensated compactness or the Bloch-Waves Method (see [10]).

This paper is only concerned with homogeneous Dirichlet boundary data. We observe that the case of homogeneous Neumann boundary conditions gives completely different results, with "critical" size being $a_\varepsilon = \varepsilon$ (c.f. [6], for the homogenization of this problem).

The present paper is organized as follows:

In §2, we recall the abstract framework of [9] on the geometry of the holes. We present the main result of this paper in §3, which gives us the convergence of the homogenization process of the nonlinear reaction-diffusion equation. Lower semicontinuity of the corresponding energy is also demonstrated. In §4, we still present the corrector result for the nonlinear reaction-diffusion equation in a perforated domain. At last, in §5, we present similar results in the case where the size of the holes is smaller than the critical one.

We have the following result.

Theorem 1.1 *(Existence and uniqueness) Assume Ω_ε as above and that the functions f_ε and u_ε^0 satisfy the following hypotheses*

$$\begin{cases} u_\varepsilon^0 \in H_0^2(\Omega_\varepsilon), \;\; f_\varepsilon' \in L^1(0,T;L^2(\Omega_\varepsilon)), \;\; f_\varepsilon(0) \in L^2(\Omega_\varepsilon), \\ \rho \text{ - any real number, when } N=2, \text{ and } \rho \leq \dfrac{2}{N-2}, \text{ when } N \geq 3. \end{cases} \tag{1.3}$$

Then, there exists only one function $u_\varepsilon : Q_\varepsilon \to \mathbb{R}$ such that

$$u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega_\varepsilon)), \tag{1.4}$$

$$u_\varepsilon^{'} \in L^2(0,T;H_0^1(\Omega_\varepsilon)) \cap L^\infty(0,T;L^2(\Omega_\varepsilon)), \tag{1.5}$$

and

$$\begin{cases} u_\varepsilon' - \beta\Delta u_\varepsilon + |u_\varepsilon|^\rho u_\varepsilon + \alpha^2 u_\varepsilon = f_\varepsilon & \text{on} \quad Q_\varepsilon = \Omega_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{on} \quad \Omega_\varepsilon, \\ u_\varepsilon \in C^0([0,T];H_0^1(\Omega_\varepsilon)). \end{cases} \tag{1.6}$$

Proof. Standard (see, for example, [16], pg. 8).

□

Remark 1.1 *Applying the Faedo-Galerkin method, we obtain that the approximate solution $u_{\varepsilon m}$ satisfies*

$$|u'_{\varepsilon m}|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \qquad \textit{and} \qquad \|u'_{\varepsilon m}\|_{L^2(0,T;H_0^1(\Omega_\varepsilon))} \leq C, \tag{1.7}$$

where

$$C = C\left(|f_\varepsilon(0)|_{L^2(\Omega_\varepsilon)},\ \|u_\varepsilon^0\|_{H_0^2(\Omega_\varepsilon)},\ \|f'_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))}\right),$$

is independent of m, for each $\varepsilon > 0$ fixed. Thus, from $(1.3)_1$, *(1.7) and Lemma 1.1 of Temam [19], pg. 250, we obtain*

$$\|u_{\varepsilon m}\|_{C^0([0,T];H_0^1(\Omega_\varepsilon))} \leq C, \tag{1.8}$$

independent of m, for each $\varepsilon > 0$ fixed (see also [16], pg. 8).

2 Preliminary results

2.1 Geometric setting

Instead of making direct geometric assumptions on the holes S_ε^i, we adopt here the abstract framework introduced by D. Cioranescu and F. Murat (c.f.,[9]) where the assumption on the geometry of the holes is made by assuming the existence of a suitable family of test functions. Precisely we will assume that

$$\left\{\begin{array}{ll} \multicolumn{2}{l}{\text{there exists a sequence of functions } (w_\varepsilon,\ \mu_\varepsilon,\ \gamma_\varepsilon) \ \text{ such that}} \\ (i) & w_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega),\ \|w_\varepsilon\|_{L^\infty(\Omega)} \leq M_0, \\ & \quad \text{where } M_0 \text{ is a fixed positive constant.} \\ (ii) & w_\varepsilon = 0 \text{ on } S_\varepsilon, \\ (iii) & w_\varepsilon \rightharpoonup 1, \text{ weakly in } H^1(\Omega), \text{ and a.e. in } \Omega, \\ (iv) & -\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon, \text{ where } \mu_\varepsilon,\ \gamma_\varepsilon \in H^{-1}(\Omega), \\ & \mu_\varepsilon \to \mu, \text{ strongly in } H^{-1}(\Omega), \text{ and} \\ & \langle \gamma_\varepsilon,\ \nu_\varepsilon \rangle_\Omega = 0, \text{ for any } \nu_\varepsilon \in H_0^1(\Omega), \text{ such that } \nu_\varepsilon = 0 \text{ on } S_\varepsilon. \end{array}\right. \tag{2.1}$$

In (2.1) and henceforth, $\langle \cdot\ ,\ \cdot \rangle_\Omega$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, while $\langle \cdot\ ,\ \cdot \rangle_{\Omega_\varepsilon}$ will denote the duality pairing between $H^{-1}(\Omega_\varepsilon)$ and $H_0^1(\Omega_\varepsilon)$.

Remark 2.1 *Examples, where the hypothesis (2.1) is satisfied, are provided in [7], Section 2.1; [9], Example model 2.1; and [14], Chapter 1.*

From the abstract framework of hypothesis (2.1) the following result can be proved

Lemma 2.1 *If (2.1) holds true, the distribution $\mu \in H^{-1}(\Omega)$, which appears in (iv), is given by*

$$\langle \mu, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \lim_{\varepsilon \to 0} \int_\Omega \varphi |\nabla w_\varepsilon|^2 dx, \ \forall \ \varphi \in \mathcal{D}(\Omega). \tag{2.2}$$

Thus, μ is a positive Radon's measure as well as an element of $H^{-1}(\Omega)$; moreover $\mu(\Omega)$ is finite.

Proof. See [9], Chapter 1, and [15], Paragraph 2.

□

We now present a result of 1950s, due to J. Deny and included in [4], that is fundamental for what we desire.

Theorem 2.1 *Let Ω be an open set of $\mathbb{R}^N$ and μ a positive Radon's measure such that $\mu \in H^{-1}(\Omega)$. Let $v \in H_0^1(\Omega)$, then one has that $v \in L^1(\Omega, d\mu)$ and*

$$\langle \mu, v \rangle_\Omega = \int_\Omega v d\mu. \tag{2.3}$$

□

This allows one to define, without ambiguity, the space

$$V = H_0^1(\Omega) \cap L^2(\Omega, d\mu) \tag{2.4}$$

as a Hilbert space with the scalar product

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx \ + \int_\Omega uv d\mu. \tag{2.5}$$

Finally, for any $v \in L^2(\Omega_\varepsilon)$, we define $\widetilde{v}$ as the extension of v, which is zero outside Ω_ε, i.e.,

$$\widetilde{v}(x) = \begin{cases} v(x) & \text{if} \quad x \in \Omega_\varepsilon, \\ 0 & \text{if} \quad x \in S_\varepsilon. \end{cases} \tag{2.6}$$

Based on [9] and [15], we recall the following result about lower semicontinuity of the energy concerning the homogenization of elliptic problems.

Theorem 2.2 *Assume that (2.1) holds true and consider a sequence z_ε such that*

$$\begin{cases} z_\varepsilon \in H_0^1(\Omega), \\ z_\varepsilon = 0 \ \text{ on } S_\varepsilon, \\ z_\varepsilon \rightharpoonup z, \ \text{ weakly in } \ H_0^1(\Omega). \end{cases} \tag{2.7}$$

Then,

$$\begin{cases} z \in V, \\ \liminf_{\varepsilon \to 0} \int_\Omega |\nabla z_\varepsilon|^2 dx \geq \int_\Omega |\nabla z|^2 dx + \int_\Omega |z|^2 d\mu. \end{cases} \tag{2.8}$$

Moreover, when z_ε also satisfies

$$\int_\Omega |\nabla z_\varepsilon|^2 dx \to \int_\Omega |\nabla z|^2 dx + \int_\Omega |z|^2 d\mu, \tag{2.9}$$

one has

$$\begin{cases} z_\varepsilon = w_\varepsilon z + r_\varepsilon, \\ r_\varepsilon \to 0, \ \text{ strongly in } W_0^{1,1}(\Omega). \end{cases} \tag{2.10}$$

Finally, if z belongs to $H_0^1(\Omega) \cap C^0(\overline{\Omega})$, the convergence of r_ε in (2.10) takes place in the strong topology of $H_0^1(\Omega)$.

□

2.2 Compactness results

Now, we recall some compactness results. When X and Y are two reflexive Banach spaces, and $X \subset Y$ with compact and dense embedding, we have the following results:

Proposition 2.1 *Assume that*

$$\begin{cases} v_\varepsilon \rightharpoonup v, & \text{weakly in } L^1(0,T;X), \\ v'_\varepsilon \rightharpoonup v', & \text{weakly in } L^1(0,T;Y). \end{cases} \tag{2.11}$$

Then,

$$v_\varepsilon \longrightarrow v, \ \text{ strongly in } C^0([0,T];Y). \tag{2.12}$$

Proposition 2.2 *Assume that (2.1) holds true and consider a sequence of functions v_ε in $L^\infty(0,T;H_0^1(\Omega_\varepsilon)) \cap W^{1,\infty}(0,T;L^2(\Omega_\varepsilon))$ satisfying*

$$\begin{cases} \widetilde{v_\varepsilon} \overset{*}{\rightharpoonup} v, & \text{weakly-star in } \ L^\infty(0,T;H_0^1(\Omega)), \\ \widetilde{v'_\varepsilon} \overset{*}{\rightharpoonup} v', & \text{weakly-star in } \ L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.13}$$

Then,

$$\langle \theta, \widetilde{v_\varepsilon}(\cdot) \rangle_\Omega \to \langle \theta, v(\cdot) \rangle_\Omega, \text{ strongly in } C^0([0,T]), \tag{2.14}$$

for any $\theta \in H^{-1}(\Omega)$, *and, on the other hand,*

$$v \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)).$$

The proofs of these results are classical and can be found in [7] and [17].

3 The homogenization result

After we have obtained a solution of the nonlinear reaction-diffusion equation in the domain Ω_ε, for each $\varepsilon > 0$, fixed, we will obtain in this section a solution of this equation in the whole domain Ω. For this, we will make $\varepsilon \to 0$. This is what we call the homogenization result, which is presented in the following Theorem.

Theorem 3.1 *Assume that (2.1) holds true and consider a sequence of data which satisfy*

$$\begin{cases} \widetilde{u_\varepsilon^0} \rightharpoonup u^0, \text{ weakly in } H_0^1(\Omega), \\ \widetilde{f_\varepsilon} \to f, \quad \text{strongly in } L^1(0,T;L^2(\Omega)), \\ \text{with } \widetilde{f_\varepsilon'} \text{ and } \widetilde{f_\varepsilon}(0) \text{ uniformly bounded, respectively, in} \\ L^1(0,T;L^2(\Omega)) \text{ and } L^2(\Omega). \end{cases} \tag{3.1}$$

Then, the sequence of solutions u_ε *of (1.1) satisfies*

$$\begin{cases} \widetilde{u_\varepsilon} \to u, & \text{strongly in} & C^0([0,T];L^2(\Omega)), \\ \widetilde{u_\varepsilon} \overset{*}{\rightharpoonup} u, & \text{weakly-star in} & L^\infty(0,T;H_0^1(\Omega)), \\ \widetilde{u_\varepsilon'} \overset{*}{\rightharpoonup} u', & \text{weakly-star in} & L^\infty(0,T;L^2(\Omega)), \\ \widetilde{u_\varepsilon'} \rightharpoonup u', & \text{weakly in} & L^2(0,T;H_0^1(\Omega)), \end{cases} \tag{3.2}$$

where $u = u(x,t)$ *is the unique solution of the homogenized nonlinear reaction-diffusion equation*

$$\begin{cases} u' - \beta \Delta u + \beta \mu u + |u|^\rho u + \alpha^2 u = f & \text{in } \; Q = \Omega \times (0,T), \\ u = 0 & \text{in } \; \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x) & \text{in } \; \Omega, \\ u \in C^0([0,T];V), \end{cases} \tag{3.3}$$

where $V = H_0^1(\Omega) \cap L^2(\Omega, d\mu)$ *and* μ *is a Radon measure given by Lemma (2.1).*

Remark 3.1 *In view of definition (2.5) of the scalar product $a(\cdot,\cdot)$ in V, the variational formulation of the reaction-diffusion equation $(3.3)_1$ is*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u(x,t)v(x)dx + \beta a(u(t),v) + \int_\Omega |u(x,t)|^\rho u(x,t)v(x)dx \\ \quad + \alpha^2 \displaystyle\int_\Omega u(x,t)v(x)dx = \int_\Omega f(x,t)v(x)dx, \ \ in\ \mathcal{D}'(0,T), \quad \forall\, v \in V \\ u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)). \end{cases}$$

Note that, according to Theorem 2.2, the function u^0 (which is the weak limit in $H_0^1(\Omega)$ of functions $\widetilde{u}_\varepsilon^0$ vanishing on the holes S_ε) belongs to V, so there is no contradiction between the two assertions $u(x,0) = u^0$ and $u \in C^0([0,T];V)$.

Proof. We proceed in four steps.

First step - A priori estimates

From (1.7) and (1.8), making use of Banach-Steinhaus' Theorem, it follows that there exists a subsequence, still denoted by the same symbol, such that

$$\begin{cases} \widetilde{u}_\varepsilon \overset{*}{\rightharpoonup} u, & \text{weakly-star in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ \widetilde{u'_\varepsilon} \overset{*}{\rightharpoonup} u', & \text{weakly-star in} \quad L^\infty(0,T;L^2(\Omega)), \\ \widetilde{u'_\varepsilon} \rightharpoonup u', & \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)). \end{cases} \tag{3.4}$$

Now, from (3.4) and Proposition 2.1, considering $X = H_0^1(\Omega)$, and $Y = L^2(\Omega)$, we have

$$\widetilde{u}_\varepsilon \to u, \quad \text{strongly in} \quad C^0([0,T];L^2(\Omega)), \tag{3.5}$$

and from Lions' Lemma, making use of the limitation of $|\widetilde{u}_\varepsilon|^\rho \widetilde{u}_\varepsilon$ in $L^2(Q)$, we have that

$$|\widetilde{u}_\varepsilon|^\rho \widetilde{u}_\varepsilon \rightharpoonup |u|^\rho u, \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.6}$$

On the other hand, in view of Proposition 2.2, we have

$$u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;H_0^1(\Omega)). \tag{3.7}$$

Second step - Passage to the limit in the equation $(1.1)_1$

Using the test function $\psi(t)w_\varepsilon(x)\varphi(x)$, with $\psi \in \mathcal{D}(0,T)$, $\varphi \in \mathcal{D}(\Omega)$ and w_ε of the abstract framework (2.1), we obtain

$$\begin{aligned}\langle u_\varepsilon', \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon} + \langle -\beta\Delta u_\varepsilon, \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon} + \langle |u_\varepsilon|^\rho u_\varepsilon, \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon} \\ +\langle \alpha^2 u_\varepsilon, \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon} = \langle f_\varepsilon, \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon},\end{aligned} \tag{3.8}$$

where $\langle \cdot\,,\ \cdot\rangle_{Q_\varepsilon}$ will henceforth denote the duality pairing $L^1(0,T;H^{-1}(\Omega_\varepsilon))$ and $L^\infty(0,T;H_0^1(\Omega_\varepsilon))$.

Integrating by parts the second term of (3.8), with respect to the variable x, we obtain

$$\langle -\Delta u_\varepsilon, \psi w_\varepsilon\varphi\rangle_{Q_\varepsilon} = \int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi \nabla w_\varepsilon \varphi dxdt + \int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi w_\varepsilon \nabla\varphi dxdt. \tag{3.9}$$

On the other hand, we also have

$$\int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi \nabla w_\varepsilon \varphi dxdt = \langle -\Delta w_\varepsilon, \psi u_\varepsilon\varphi\rangle_{Q_\varepsilon} - \int_{Q_\varepsilon} \nabla w_\varepsilon \cdot \psi u_\varepsilon \nabla\varphi dxdt. \tag{3.10}$$

Thus, combining (3.8)-(3.10), we arrive at

$$\begin{aligned}&\langle u_\varepsilon', \psi w_\varepsilon\varphi\rangle_{Q_\varepsilon} + \langle -\beta\Delta w_\varepsilon, \psi u_\varepsilon\varphi\rangle_{Q_\varepsilon} - \beta\int_{Q_\varepsilon} \nabla w_\varepsilon \cdot \psi u_\varepsilon \nabla\varphi dxdt \\ &\quad +\beta\int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi w_\varepsilon \nabla\varphi dxdt + \langle |u_\varepsilon|^\rho u_\varepsilon, \psi(t)w_\varepsilon(x)\varphi(x)\rangle_{Q_\varepsilon} \\ &\quad +\langle \alpha^2 u_\varepsilon, \psi w_\varepsilon\varphi\rangle_{Q_\varepsilon} = \langle f_\varepsilon, \psi w_\varepsilon\varphi\rangle_{Q_\varepsilon}.\end{aligned} \tag{3.11}$$

Next, we are going to analyze only the convergence of the 2nd, 4th and 5th terms of the equation (3.11), since the convergence of the others terms don't offer any difficulty.

2nd term. Now, consider the function $\mathcal{U}_\varepsilon \in H_0^1(\Omega)$ defined by $\mathcal{U}_\varepsilon = \displaystyle\int_0^T \psi\widetilde{u}_\varepsilon dt$. The convergence $(3.4)_1$ implies that the sequence $\mathcal{U}_\varepsilon$ satisfies

$$\begin{cases} \mathcal{U}_\varepsilon(x) \rightharpoonup \displaystyle\int_0^T \psi u dt, & \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega) \\ \mathcal{U}_\varepsilon(x) = 0, & \text{on } T_\varepsilon. \end{cases} \tag{3.12}$$

In view of the hypothesis (2.1)(iv), $-\Delta w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon$, and applying Fubini's Theorem, we have:

$$\begin{aligned}\langle -\Delta w_\varepsilon, \psi u_\varepsilon \varphi\rangle_{Q_\varepsilon} &= \langle (\mu_\varepsilon - \gamma_\varepsilon), \psi u_\varepsilon \varphi\rangle_{Q_\varepsilon} \\ &= \langle (\mu_\varepsilon - \gamma_\varepsilon), \left(\int_0^T \psi \widetilde{u}_\varepsilon dt\right) \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle \mu_\varepsilon, \mathcal{U}_\varepsilon \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},\end{aligned} \tag{3.13}$$

since $\langle \gamma_\varepsilon, \mathcal{U}_\varepsilon\varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0$.

Thus, from (3.12) and hypothesis (2.1)(iv), we obtain the following convergence in (3.13)

$$\langle -\Delta w_\varepsilon, \psi u_\varepsilon \varphi\rangle_{Q_\varepsilon} \to \langle \mu, \left(\int_0^T \psi u dt\right) \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{3.14}$$

4th term. Applying Fubini's Theorem, we obtain

$$\int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi w_\varepsilon \nabla\varphi dxdt = \int_\Omega w_\varepsilon \nabla\varphi \cdot \nabla \left(\int_0^T \psi \widetilde{u}_\varepsilon dt\right) dx. \tag{3.15}$$

From $(3.12)_1$, for a subsequence still denoted by the same symbol, we see that

$$\nabla \left(\int_0^T \psi \widetilde{u}_\varepsilon dt\right) \rightharpoonup \nabla \left(\int_0^T \psi u dt\right), \text{ weakly in } [L^2(\Omega)]^N. \tag{3.16}$$

But, from the hypothesis (2.1)(iii), and Relich-Kondrachoff's Theorem, for a subsequence, still denoted by the same symbol, we have

$$w_\varepsilon \to 1, \text{ strongly in } L^2(\Omega). \tag{3.17}$$

Thus, from (3.16) and (3.17), we obtain the following convergence in (3.15)

$$\int_{Q_\varepsilon} \nabla u_\varepsilon \cdot \psi w_\varepsilon \nabla\varphi dxdt \to \int_\Omega \nabla\varphi \cdot \nabla \left(\int_0^T \psi u dt\right) dx. \tag{3.18}$$

5th term. Analogously applying Fubini's Theorem, we obtain

$$\langle |u_\varepsilon|^\rho u_\varepsilon, \psi w_\varepsilon \varphi\rangle_{Q_\varepsilon} = \int_\Omega w_\varepsilon \varphi \left(\int_0^T \psi |\widetilde{u}_\varepsilon|^\rho \widetilde{u}_\varepsilon dt\right) dx. \tag{3.19}$$

From the convergence (3.6), for a subsequence still denoted by the same symbol, we see that

$$\int_0^T \psi |\widetilde{u}_\varepsilon|^\rho \widetilde{u}_\varepsilon dt \rightharpoonup \int_0^T \psi |u|^\rho u dt, \text{ weakly in } L^2(\Omega). \tag{3.20}$$

Thus, from (3.20) and (3.17), we obtain the following convergence in (3.19)

$$\langle |u_\varepsilon|^\rho u_\varepsilon, \psi w_\varepsilon \varphi\rangle_{Q_\varepsilon} \to \int_\Omega \varphi \left(\int_0^T \psi |u|^\rho u dt \right) dx. \tag{3.21}$$

It is now easy to pass to the limit in each term of (3.11). Using Fubini's Theorem and Deny's Theorems (see Theorem (2.1)), we obtain

$$\begin{aligned} &\langle \int_\Omega u' \varphi \, dx, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)} + \langle \beta \int_\Omega u\varphi \, d\mu, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)} \\ &\quad + \langle \beta \int_\Omega \nabla u \cdot \nabla \varphi \, dx, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)} + \langle \int |u|^\rho u \varphi \, dx, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)} \\ &\quad + \langle \alpha^2 \int_\Omega u\varphi \, dx, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)} = \langle \int_\Omega f\varphi \, dx, \psi\rangle_{\mathcal{D}',\mathcal{D}(0,T)}. \end{aligned} \tag{3.22}$$

Since $\psi \in \mathcal{D}(0,T)$ is arbitrary, we have that

$$\begin{aligned} &\int_\Omega u' \varphi \, dx + \beta \int_\Omega u\varphi \, d\mu + \beta \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega |u|^\rho u\varphi \, dx \\ &\quad + \alpha^2 \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx, \end{aligned} \tag{3.23}$$

for any $\varphi \in \mathcal{D}(\Omega)$.

In view of (3.7), and the density of $\mathcal{D}(\Omega)$ in V (see [7], Appendix) we have that (3.23) allows one to extend to every function $v \in V$. Thus, we have proved $(3.3)_1$ in the weak sense.

Third step - Passing to the limit in the initial data

From Proposition 2.2 we deduce that

$$\langle \varphi, \widetilde{u_\varepsilon}(\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \to \langle \varphi, u(\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \text{ strongly in } C^0([0.T]), \tag{3.24}$$

for any $\varphi \in H^{-1}(\Omega)$. Since $\widetilde{u_\varepsilon}(0) = \widetilde{u_\varepsilon^0}$ tends to u^0, weakly in $H_0^1(\Omega)$, from uniqueness of the limit, we obtain

$$u(x,0) = u^0.$$

Fourth step - End of the proof

In conclusion, we proved that, up to the extraction of a subsequence still denoted by $\{u_\varepsilon\}$, the subsequence $\{u_\varepsilon\}$ satisfies (3.2), where the limit

$$u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;H_0^1(\Omega))$$

and satisfies $(3.3)_1 - (3.3)_3$ in the weak sense.

The uniqueness of the solution of (3.3) allows us to deduce that the whole sequence satisfies $(3.3)_4$. This completes the proof of Theorem 3.1.

□

Now, we will make some observations.

Remark 3.2 *We can rewrite the equation* $(1.1)_1$ *in the form*

$$u_\varepsilon' - \beta\Delta u_\varepsilon + \alpha^2 u_\varepsilon = g_\varepsilon, \tag{3.25}$$

where $g_\varepsilon = f_\varepsilon - |u_\varepsilon|^\rho u_\varepsilon$ *is bounded in* $L^\infty(0,T;L^2(\Omega_\varepsilon))$*, since* $|u_\varepsilon|^\rho u_\varepsilon$ *is bounded in* $L^\infty(0,T;L^2(\Omega_\varepsilon))$ *and* f_ε *is bounded in* $L^\infty(0,T;L^2(\Omega_\varepsilon))$ *due to the assumption* $(3.1)_3$.

We also have that $g_\varepsilon' = f_\varepsilon' - (\rho+1)|u_\varepsilon|^\rho u_\varepsilon'$. *Thus, from the Sobolev's embedding, using that* $\frac{2}{q} + \frac{2}{N} = 1$*, by the Hölder's inequality, we obtain*

$$\int_{\Omega_\varepsilon} ||u_\varepsilon|^\rho u_\varepsilon'|^2\, dx = \int_{\Omega_\varepsilon} |u_\varepsilon|^{2\rho}|u_\varepsilon'|^2 dx \le \left(\int_{\Omega_\varepsilon} |u_\varepsilon|^{\rho N} dx\right)^{\frac{2}{N}} \left(\int_{\Omega_\varepsilon} |u_\varepsilon'|^q dx\right)^{\frac{2}{q}} < \infty, \tag{3.26}$$

because $\rho N \le q$ *what implies* $H_0^1(\Omega_\varepsilon) \hookrightarrow L^q(\Omega_\varepsilon) \hookrightarrow L^{\rho N}(\Omega_\varepsilon)$. *So, from (3.26), (1.4) and (1.5), we conclude* $|u_\varepsilon|^\rho u_\varepsilon' \in L^2(0,T;L^2(\Omega_\varepsilon))$. *Therefore,* $g_\varepsilon' \in L^1(0,T;L^2(\Omega_\varepsilon))$.

In the same way, the homogenized equation takes the following form

$$u' - \beta\Delta u + \beta\mu u + \alpha^2 u = g, \tag{3.27}$$

where $g = f - |u|^\rho u$*, with* $g \in L^1(0,T;L^2(\Omega))$ *and* $g' \in L^1(0,T;L^2(\Omega))$.

4 Energy and corrector results

This section is dedicated to the establishment of the corrector result for the nonlinear reaction-diffusion equation, using the article of D. Cioranescu et al., [7]. An important result in this section is the strong convergence of the energy in $C^0[0,T])$. However, before we state the strong convergence of the energy we establish the pointwise convergence in time and lower semi-continuity property of the energy.

Proposition 4.1 *Assume that the hypotheses of Theorem 3.1 are satisfied. Then, for all fixed $t \in [0, T]$, we have*

$$\widetilde{u_\varepsilon}(t) \rightharpoonup u(t), \text{ weakly in } H_0^1(\Omega), \tag{4.1}$$

$$\int_\Omega |u(x,t)|^2 dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |u_\varepsilon(x,t)|^2 dx, \tag{4.2}$$

$$\begin{aligned} \int_0^t \int_\Omega |\nabla u(x,s)|^2 dx ds + \int_0^t \int_\Omega |u(x,s)|^2 d\mu(x) ds \\ \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x,s)|^2 dx ds, \end{aligned} \tag{4.3}$$

$$E(t) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(t), \tag{4.4}$$

where the energy $E_\varepsilon(\cdot)$, associated to the equation (3.25)), is defined by

$$E_\varepsilon(t) = \frac{1}{2}|u_\varepsilon(t)|^2_{L^2(\Omega_\varepsilon)} + \beta \int_0^t |\nabla u_\varepsilon(s)|^2_{[L^2(\Omega_\varepsilon)]^N} ds + \alpha^2 \int_0^t |u_\varepsilon(s)|^2_{L^2(\Omega_\varepsilon)} ds,$$

and the energy $E(\cdot)$, associated to the equation (3.27)), is defined by

$$\begin{aligned} E(t) = \ & \frac{1}{2}|u(t)|^2_{L^2(\Omega)} + \beta \int_0^t |\nabla u(s)|^2_{[L^2(\Omega)]^N} ds \\ & + \beta \int_0^t \|u(s)\|^2_{L^2(\Omega, d\mu)} ds + \alpha^2 \int_0^t |u(s)|^2_{L^2(\Omega)} ds. \end{aligned}$$

Proof. The convergence (4.1) has been proved in the third step of the proof of Theorem 3.1 (see (3.24)). Thus, since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we obtain (4.2). Now, in order to obtain (4.3), it is sufficient to apply Theorem 2.2, since $\widetilde{u_\varepsilon}(t)$, for any $t \in [0, T]$, satisfies the hypotheses of this theorem. Then, we integrate on $(0, t)$, employing Fatou's Lemma. From (4.2) and (4.3), we immediately obtain (4.4).

□

Now, we state the following proposition about the strong convergence of the energy. For this, due to Remark 3.2, we have that $\{\widetilde{g_\varepsilon}\}_\varepsilon$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$.

Proposition 4.2 *Assume that the hypotheses of Theorem 3.1 are satisfied, then*

$$E_\varepsilon(t) \to E(t), \quad \text{strongly in} \quad C^0([0, T]). \tag{4.5}$$

Proof. We have the following energy identities

$$E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} g_\varepsilon(x,s) u_\varepsilon(x,s)\, dx ds, \tag{4.6}$$

$$E(t) = E(0) + \int_0^t \int_{\Omega} g(x,s) u(x,s)\, dx ds, \tag{4.7}$$

with

$$E_\varepsilon(0) = \frac{1}{2}|u_\varepsilon^0|^2_{L^2(\Omega_\varepsilon)} \quad \text{and} \quad E(0) = \frac{1}{2}|u^0|^2_{L^2(\Omega)}.$$

In view of (3.5) and from the fact that, in particular, $\widetilde{g}_\varepsilon \rightharpoonup g$, in $L^1(0,T;L^2(\Omega))$, we have

$$\int_0^t \int_{\Omega_\varepsilon} g_\varepsilon(x,s) u_\varepsilon(x,s)\, dx ds \to \int_0^t \int_{\Omega} g(x,s) u(x,s)\, dx ds, \tag{4.8}$$

for any $t \in [0,T]$. On the other hand, (4.2) implies that

$$E_\varepsilon(0) \to E(0) \tag{4.9}$$

Therefore,

$$E_\varepsilon(t) \to E(t), \quad \text{for any } t \in [0,T], \tag{4.10}$$

which is a pointwise convergence in time.

In order to obtain the uniform convergence of the energy, we apply the Ascoli-Arzela's Theorem. Thus, we need to show that the family of energies $\{E_\varepsilon(t)\}_{\varepsilon>0}$ is equicontinuous. In fact, given any $t \in [0,T]$, and $h > 0$ small enough, we have

$$\begin{aligned} |E_\varepsilon(t+h) - E_\varepsilon(t)| &= \left| \int_t^{t+h} \int_\Omega \widetilde{g}_\varepsilon(x,s) \widetilde{u}_\varepsilon(x,s)\, dx ds \right| \\ &\le \|\widetilde{g}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \int_t^{t+h} \|\widetilde{u}_\varepsilon(s)\|_{L^2(\Omega)} ds. \end{aligned}$$

Since $\widetilde{g}_\varepsilon$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and since, in particular, $\widetilde{u}_\varepsilon$ converges strongly in $L^1(0,T;L^2(\Omega))$, this implies that

$$|E_\varepsilon(t+h) - E_\varepsilon(t)| \to 0, \quad \text{as} \quad h \to 0, \tag{4.11}$$

uniformly on ε, which proves that the family of functions $\{E_\varepsilon(t)\}_{\varepsilon>0}$ is equicontinuous.

Thus, from (4.10), (4.11) and Ascoli-Arzela's Theorem, it follows the uniform convergence of the energy (4.5). $\square$

Now we define,

$$e_\varepsilon(v)(t) = \frac{1}{2}|v(t)|^2_{L^2(\Omega_\varepsilon)} + \beta\int_0^t |\nabla v(s)|^2_{[L^2(\Omega_\varepsilon)]^N} ds + \alpha^2 \int_0^t |v(s)|^2_{L^2(\Omega_\varepsilon)} ds, \tag{4.12}$$

for $v \in C^0([0,T]; H^1_0(\Omega_\varepsilon))$, and

$$\begin{aligned} e(v)(t) = \ & \frac{1}{2}|v(t)|^2_{L^2(\Omega)} + \beta\int_0^t |\nabla v(s)|^2_{[L^2(\Omega)]^N} ds \\ & +\beta\int_0^t \langle \mu v(s), v(s)\rangle_\Omega ds + \alpha^2\int_0^t |v(s)|^2_{L^2(\Omega)} ds, \end{aligned} \tag{4.13}$$

for $v \in C^0([0,T]; V)$.

From these definitions, we present the following result

Proposition 4.3 *Assume that the hypotheses of Proposition 4.2 are satisfied. Then,*

$$e_\varepsilon(u_\varepsilon - w_\varepsilon\varphi)(\cdot) \longrightarrow e(u-\varphi)(\cdot), \quad \textit{strongly in} \quad C^0([0,T]) \tag{4.14}$$

for every $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega))$.

Proof. We have

$$\begin{aligned} e_\varepsilon(u_\varepsilon - w_\varepsilon\varphi)(t) = \ & e_\varepsilon(u_\varepsilon)(t) + e_\varepsilon(w_\varepsilon\varphi)(t) - \int_\Omega \widetilde{u}_\varepsilon(x,t)w_\varepsilon(x)\varphi(x,t)dx \\ & -2\beta\int_0^t\int_\Omega \nabla\widetilde{u}_\varepsilon(x,s)\cdot\nabla(w_\varepsilon(x)\varphi(x,s))dxds \\ & -2\alpha^2\int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)w_\varepsilon(x)\varphi(x,s)dxds. \end{aligned} \tag{4.15}$$

We will pass successively to the limit in each term on the right hand side of (4.15).

First term. Since $e_\varepsilon(u_\varepsilon)(t) = E_\varepsilon(t)$, we have from Proposition 4.2 that

$$e_\varepsilon(u_\varepsilon)(\cdot) \to e(u)(\cdot), \quad \text{strongly in} \quad C^0([0,T]). \tag{4.16}$$

Second term. Differentiating in time, one shows that the function $|w_\varepsilon\varphi(\cdot)|^2_{L^2(\Omega_e)}$ is bounded in $W^{1,\infty}(0,T) \hookrightarrow C^0([0,T])$ with compact embedding, from Rellich-Kondrachov's Theorem. Thus, using (2.1)(iii), we obtain

$$|w_\varepsilon\varphi(\cdot)|^2_{L^2(\Omega_e)} = |w_\varepsilon\varphi(\cdot)|^2_{L^2(\Omega)} \to |\varphi|^2_{L^2(\Omega)}, \quad \text{strongly in} \quad C^0([0,T]), \tag{4.17}$$

and also that

$$\int_0^t |w_\varepsilon \varphi(s)|^2_{L^2(\Omega_e)} ds \to \int_0^t |\varphi(s)|^2_{L^2(\Omega)} ds, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.18}$$

On the other hand,

$$\begin{aligned}\int_0^t |\nabla(w_\varepsilon \varphi(s))|^2_{[L^2(\Omega_\varepsilon)]^N} ds &= \int_0^t |\nabla(w_\varepsilon \varphi(s))|^2_{[L^2(\Omega)]^N} ds \\ &= -\int_0^t \langle \Delta w_\varepsilon \varphi(s), w_\varepsilon \varphi(s)\rangle_\Omega ds - 2\int_0^t \int_\Omega \nabla w_\varepsilon \cdot \nabla\varphi(s) w_\varepsilon \varphi(s) dx ds \\ &\quad - \int_0^t \int_\Omega \Delta\varphi(s)|w_\varepsilon|^2 \varphi(s) dx ds\end{aligned}$$

Still, in view of (2.1)(iii) and (iv), we can pass to the limit in each term on the right hand side and observe that each term is bounded in $W^{1,\infty}(0,T)$. As a result, we obtain

$$\int_0^t \int_\Omega \Delta\varphi(s)|w_\varepsilon|^2 \varphi(s) dx ds \to \int_0^t \int_\Omega \Delta\varphi(s)\varphi(s) dx ds, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.19}$$

and

$$2\int_0^t \int_\Omega \nabla w_\varepsilon \cdot \nabla\varphi(s) w_\varepsilon \varphi(s) dx ds \to 0, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.20}$$

Using (2.1)(iv), we have that

$$\langle -\Delta w_\varepsilon \varphi(s), w_\varepsilon \varphi(s)\rangle_\Omega = \langle \mu_\varepsilon, w_\varepsilon \varphi^2(s)\rangle_\Omega \to \langle \mu, \varphi^2(s)\rangle_\Omega, \quad \text{strongly in} \quad C^0([0,T]).$$

Therefore,

$$\int_0^t \langle \Delta w_\varepsilon \varphi(s), w_\varepsilon \varphi(s)\rangle_\Omega ds \to \int_0^t \langle \mu, \varphi^2(s)\rangle_\Omega ds, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.21}$$

Now, combining (4.17)-(4.21), we obtain that

$$e_\varepsilon(w_\varepsilon \varphi(\cdot)) \to e(\varphi)(\cdot), \quad \text{strongly in} \quad C^0([0,T]). \tag{4.22}$$

Third term. Since $\widetilde{u}_\varepsilon$ is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$ (see (3.2)), the function

$$t \mapsto \int_\Omega \widetilde{u}_\varepsilon(x,t) w_\varepsilon(x) \varphi(x,t) dx$$

is bounded in $W^{1,\infty}(0,T)$, thus relatively compact in $C^0([0,T])$. This implies that

$$\int_\Omega \widetilde{u}_\varepsilon(x,t)w_\varepsilon(x)\varphi(x,t)dx \to \int_\Omega u(x,t)\varphi(x,t)dx, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.23}$$

Fourth term. We have

$$\begin{aligned}\int_0^t \int_\Omega \nabla\widetilde{u}_\varepsilon(x,s)\cdot\nabla(w_\varepsilon(x)\varphi(x,s))dxds &= \int_0^t \langle -\Delta w_\varepsilon(x)\varphi(x,s), \widetilde{u}_\varepsilon(x,s)\rangle_\Omega ds \\ -2\int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)\nabla w_\varepsilon(x)\cdot\nabla\varphi(x,s)dxds &- \int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)w_\varepsilon(x)\Delta\varphi(x,s)dxds.\end{aligned} \tag{4.24}$$

Now we consider the funtion

$$t \longmapsto -2\int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)\nabla w_\varepsilon(x)\cdot\nabla\varphi(x,s)dxds - \int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)w_\varepsilon\Delta\varphi(x,s)dxds.$$

From (3.2), $\widetilde{u}_\varepsilon$ is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$. Thus, this family of functions is bounded in $W^{1,\infty}(0,T)$ and therefore is relatively compact in $C^0([0,T])$, due to the compact embedding of $W^{1,\infty}(0,T)$ in $C^0([0,T])$. This implies that

$$\begin{aligned}-2\int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)\nabla w_\varepsilon(x)\cdot\nabla\varphi(x,s)dxds - \int_0^t\int_\Omega \widetilde{u}_\varepsilon(x,s)w_\varepsilon\Delta\varphi(x,s)dxds \\ \longrightarrow -\int_0^t\int_\Omega u(x,s)\Delta\varphi(x,s)dxds = \int_0^t\int_\Omega \nabla u(x,s)\cdot\nabla\varphi(x,s)dxds,\end{aligned} \tag{4.25}$$

strongly in $C^0([0,T])$.

Consider now the remaining term $\langle -\Delta w_\varepsilon, \widetilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega$. We have from (2.1)$(iv)$, that

$$\langle -\Delta w_\varepsilon, \widetilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega = \langle \mu_\varepsilon, \widetilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega,$$

since $\widetilde{u}_\varepsilon\varphi(t) = 0$ on S_ε.

On the other hand, using the fact that $W^{1,2}(0,T) \hookrightarrow C^0([0,T])$ with compact embedding, we have that

$$\langle -\Delta w_\varepsilon, \widetilde{u}_\varepsilon(t)\varphi(t)\rangle_\Omega \to \langle \mu, u(t)\varphi(t)\rangle_\Omega, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.26}$$

Therefore,

$$\int_0^t \langle -\Delta w_\varepsilon, \widetilde{u}_\varepsilon(s)\varphi(s)\rangle_\Omega ds \to \int_0^t \langle \mu\varphi(s), u(s)\rangle_\Omega ds, \quad \text{strongly in} \quad C^0([0,T]). \tag{4.27}$$

Fifth term. Considering the last term in (4.15), from (4.23), it immediately follows that

$$\int_0^t \int_\Omega \widetilde{u}_\varepsilon(x,s) w_\varepsilon(x) \varphi(x,s) dx ds \to \int_0^t \int_\Omega u(x,s) \varphi(x,s) dx ds, \tag{4.28}$$

strongly in $C^0([0,T])$.

Furthermore, putting together, (4.15), (4.16), (4.22), (4.23),(4.25), (4.27) and (4.28), we obtain (4.14). The proof of Proposition 4.3 is now complete.

□

We now present the corrector result.

Theorem 4.1 *Assume that the hypotheses of Proposition 4.2 are satisfied. If u denotes the unique solution of the homogenized problem (3.3), then the sequence of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ of the problem (1.1) satisfies*

$$\widetilde{u}_\varepsilon \to u, \quad in \quad C^0([0,T]; L^2(\Omega)), \tag{4.29}$$

$$\widetilde{u}_\varepsilon = w_\varepsilon u + r_\varepsilon, \quad with \tag{4.30}$$

$$r_\varepsilon \to 0, \quad strongly\ in \quad L^2(0,T; W_0^{1,1}(\Omega)). \tag{4.31}$$

Moreover, if $u \in C^0([0,T]; C^1(\overline{\Omega}))$, then

$$r_\varepsilon \to 0, \quad strongly\ in \quad L^2(0,T; H_0^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)). \tag{4.32}$$

Proof. We now observe that due to (3.4) and the Proposition 2.1, we easily obtain that

$$\widetilde{u}_\varepsilon \to u, \quad \text{strongly in} \quad C^0([0,T]; L^2(\Omega)),$$

and therefore (4.29) is shown. From Theorem 3.1, we know that, in particular, $u \in C^0([0,T]; H_0^1(\Omega))$. Then, let us consider a sequence φ_k in $\mathcal{D}(Q)$ such that

$$\varphi_k \to u, \quad \text{strongly in} \quad C^0([0,T]; H_0^1(\Omega)), \quad \text{as} \quad k \to \infty. \tag{4.33}$$

From Proposition 4.3 we have

$$\limsup_{\varepsilon \to 0} \left\{ \|\widetilde{u}_\varepsilon - w_\varepsilon \varphi_k\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\beta \int_0^T |\nabla(\widetilde{u}_\varepsilon(s) - w_\varepsilon \varphi_k(s))|^2_{L^2(\Omega)} ds \right.$$
$$\left. + 2\alpha^2 \int_0^T |\widetilde{u}_\varepsilon(s) - w_\varepsilon \varphi_k(s)|^2_{L^2(\Omega)} ds \right\} \leq 3\|e(u-\varphi_k)\|_{L^\infty(0,T)},$$

and thus,

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \left\{ \|\widetilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\beta \int_0^T |\nabla(\widetilde{u}_\varepsilon(s) - w_\varepsilon\varphi_k(s))|^2_{L^2(\Omega)} ds \right.$$
$$\left. +2\alpha^2 \int_0^T |\widetilde{u}_\varepsilon(s) - w_\varepsilon\varphi_k(s)|^2_{L^2(\Omega)} ds \right\} = 0. \tag{4.34}$$

On the other hand, we have

$$\begin{aligned} &\|\nabla(\widetilde{u}_\varepsilon - w_\varepsilon u)\|_{L^2(0,T;L^1(\Omega))} \\ &\quad \le C\|\nabla(\widetilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^2(0,T;L^2(\Omega))} + \|\nabla w_\varepsilon(\varphi_k - u)\|_{L^2(0,T;L^1(\Omega))} \\ &\qquad + \|w_\varepsilon \nabla(\varphi_k - u)\|_{L^2(0,T;L^1(\Omega))} \\ &\quad \le C\|\nabla(\widetilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^2(0,T;L^2(\Omega))} + C\|\nabla w_\varepsilon\|_{[L^2(\Omega)]^N} \cdot \|\varphi_k - u\|_{C^0([0,T];L^2(\Omega))} \\ &\qquad + C\|w_\varepsilon\|_{L^2(\Omega)} \cdot \|\varphi_k - u\|_{C^0([0,T];H^1_0(\Omega))}. \end{aligned} \tag{4.35}$$

By (2.1), (4.33), (4.34) and (4.35) we conclude that

$$\nabla r_\varepsilon = \nabla(\widetilde{u}_\varepsilon - w_\varepsilon u) \to 0, \quad \text{strongly in} \quad L^2(0,T;L^1(\Omega)).$$

Thus, (4.31) is proved.

Now, if $u \in C^0([0,T];C^1(\overline{\Omega}))$, then

$$r_\varepsilon \to 0, \quad \text{strongly in} \quad L^2(0,T;H^1_0(\Omega))$$

In fact, is such case, an approximating sequence φ_k may be also chosen to satisfy the hypothesis

$$\varphi_k \to u, \quad \text{strongly in} \quad C^0([0,T];C^1(\overline{\Omega})).$$

In this case, we can estimate $\nabla(\widetilde{u}_\varepsilon - w_\varepsilon u)$ in $L^2(0,T;L^2(\Omega))$ and not only in $L^2(0,T;L^1(\Omega))$, as it was previously done in (4.35). In fact, we have

$$\begin{aligned} &\|\nabla(w_\varepsilon(\varphi_k - u))\|_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le \|\nabla w_\varepsilon(\varphi_k - u)\|_{L^2(0,T;L^2(\Omega))} + \|w_\varepsilon\nabla(\varphi_k - u)\|_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le C\|\nabla w_\varepsilon\|_{[L^2(\Omega)]^N} \cdot \|\varphi_k - u\|_{C^0([0,T];C^0(\overline{\Omega}))} \\ &\qquad\quad + C\|w_\varepsilon\|_{L^2(\Omega)} \cdot \|\varphi_k - u\|_{C^0([0,T];C^1(\overline{\Omega}))}, \end{aligned}$$

which is sufficient to obtain the result (4.32) above.

We also observe that we can similarly show the following convergence $r_\varepsilon \to 0, \quad \text{strongly in} \quad C^0([0,T];L^2(\Omega)).$

□

5 The case of holes smaller than the critical

In this section we will consider the particular case where the holes are smaller than the critical size. In this case, we assume that the function w_ε of the abstract framework (2.1) strongly converges in $H^1(\Omega)$, which implies that $\mu = 0$. And thus, all the results of the Sections 3 and 4 hold true.

We assume that the holes S_ε are such that

$$\begin{cases} \text{there exists a sequence of test functions } w_\varepsilon \text{ satisfying} \\ \quad (i) \quad w_\varepsilon \in H^1(\Omega), \ \|w_\varepsilon\|_{L^\infty(\Omega)} \le M_0, \\ \quad (ii) \quad w_\varepsilon = 0 \text{ on } S_\varepsilon, \\ \quad (iii) \quad w_\varepsilon \to 1, \text{ strongly in } H^1(\Omega). \end{cases} \tag{5.1}$$

Remark 5.1. The assumption (5.1) means that the size of the holes $a_{S_\varepsilon^i}$ is smaller than the critical one given by (1.2), i.e. that

$$\begin{cases} \varepsilon^2 \log a_{S_\varepsilon^i} \to -\infty, & \text{if } N = 2, \\ \dfrac{\varepsilon^{N/(N-2)}}{a_{S_\varepsilon^i}} \to +\infty, & \text{if } N \ge 3. \end{cases} \tag{5.2}$$

The main differences between the hypotheses in (2.1) and (5.1) is that, in (5.1)(iii), we assume the strong convergence of w_ε. In this case, (2.1)(iv) is obviously satisfied with $\gamma_\varepsilon = 0$, $\mu_\varepsilon = -\Delta w_\varepsilon$ and $\mu = 0$.

Examples where the assumption (5.1) is satisfied can be found in [7].

Under the hypothesis (5.1), all the results of Sections 3 and 4 obviously hold true, but the strong convergence of the data now implies strong convergence of the solutions.

Theorem 5.1 *Assume that (5.1) holds true and consider a sequence of data that satisfy (3.1). Then the sequence u_ε of solutions of (1.1) satisfies*

$$\begin{cases} \widetilde{u}_\varepsilon \overset{*}{\rightharpoonup} u, & \textit{weakly-star in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ \widetilde{u'_\varepsilon} \overset{*}{\rightharpoonup} u', & \textit{weakly-star in} \quad L^\infty(0,T;L^2(\Omega)), \\ \widetilde{u'_\varepsilon} \rightharpoonup u', & \textit{weakly in} \quad L^2(0,T;H_0^1(\Omega)), \end{cases} \tag{5.3}$$

and

$$\widetilde{u}_\varepsilon(t) \rightharpoonup u(t), \quad \textit{weakly in } H_0^1(\Omega),$$

for all $t \in [0,T]$, where the limit u is the unique solution of

$$\begin{cases} u' - \beta\Delta u + |u|^\rho u + \alpha^2 u = f & \textit{in} \quad Q = \Omega \times (0,T), \\ u = 0 & \textit{on} \quad \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x) & \textit{in} \quad \Omega, \\ u \in C^0([0,T];H_0^1(\Omega)). \end{cases} \tag{5.4}$$

Moreover,

$$\begin{cases} \widetilde{u}_\varepsilon \to u & \textit{strongly in} \quad C^0([0,T];L^2(\Omega)), \\ \widetilde{u}_\varepsilon \to u & \textit{strongly in} \quad L^2(0,T;H_0^1(\Omega)). \end{cases} \tag{5.5}$$

Proof. The first part of the theorem involves passing to the limit in (1.1)), which is a mere rewriting of Theorems 3.1 and 3.2.

To prove $(5.5)_2$ we proceed as in the proof of Theorem 4.1. Thus, let $\varphi_k \in \mathcal{D}(Q)$ be a sequence satisfying (4.33). We have

$$\begin{aligned} \|\nabla(\widetilde{u}_\varepsilon - u)\|_{L^2(0,T;[L^2(\Omega)]^N)} &\leq \|\nabla(\widetilde{u}_\varepsilon - w_\varepsilon\varphi_k)\|_{L^2(0,T;L^2(\Omega))} \\ &+\|\nabla((1-w_\varepsilon)\varphi_k)\|_{L^2(0,T;L^2(\Omega))} + \|\nabla(\varphi_k - u)\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{5.6}$$

Arguing as in the proof of Theorem 4.1, and using the strong convergence of w_ε to 1, in $H^1(\Omega)$, in the second term on the right side of inequality (5.6), we obtain $(5.5)_2$.

$\square$

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J. Souza
Departamento de Matemática
Univ. Federal de Santa Catarina
Caixa Postal 476
88040-900 Florianópolis, Brasil
jsouza@mtm.ufsc.br

A. Kist
Departamento de Matemática
Univ. Estadual de Ponta Grossa
Uvaranas
84030-000 Ponta Grossa, Brasil
kist@uepg.br