A NOTE ON THE FIRST EIGENVALUE OF SPHERICALLY SYMMETRIC MANIFOLDS

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Abstract

We give lower and upper bounds for the first eigenvalue of geodesic balls in spherically symmetric manifolds. These lower and upper bounds are $C^0$-dependent on the metric coefficients. It gives better lower bounds for the first eigenvalue of spherical caps than those from Betz-Camera-Gzyl.

1 Introduction

Let $B_{N^\kappa}(r)$ be a geodesic ball of radius $r > 0$ in the simply connected $n$-dimensional space form $N^\kappa$ of constant sectional curvature $\kappa$ and let $\lambda_1(B_{N^\kappa}(r))$ be its first Laplacian eigenvalue, i.e. the smallest real number $\lambda = \lambda_1(B_{N^\kappa}(r))$ for which there exists a function, called a first eigenfunction, $u \in C^2(B_{N^\kappa}(r)) \cap C^0(B_{N^\kappa}(r)) \setminus \{0\}$, satisfying $\Delta u + \lambda u = 0$ in $B_{N^\kappa}(r)$ with $u|\partial B_{N^\kappa}(r) = 0$. In the case $\kappa = 0$, it is well known that $\lambda_1(B_{R^n}(r)) = (c(n)/r)^2$, where $c(n)$ is the first zero of the Bessel function $J_{n/2-1}$. In the case $\kappa = -1$, there are fairly good lower and upper bounds for $\lambda_1(B_{H^n}(r))$. For instance, one has that

$$\sqrt{\lambda_1(B_{H^n}(r))} \leq (n-1)(\coth(r/2) - 1)/2 + [(n-1)^2/4 +$$

$$+ 4\pi^2/r^2 + (n-1)^2(\coth(r/2) - 1)^2/4]^{1/2},$$

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see [6] page 49. For sharper upper bounds for $\lambda_1(B_{B^n}(r))$, see [9]. On the other hand, one has the well known lower bound,

$$\sqrt{\lambda_1(B_{B^n}(r))} \geq \frac{(n - 1) \coth(r)}{2},$$

proved by McKean, [10], [13]. This lower bound was improved by Bessa and Montenegro in [4] to

$$\sqrt{\lambda_1(B_{B^n}(r))} \geq \max \left\{ \frac{n}{2r}, \frac{(n - 1) \coth(r)}{2} \right\}. \quad (1)$$

The case $c = 1$ is more delicate. Although the sphere is a very well studied manifold, the values of the first Laplacian eigenvalue $\lambda_1(B_{S^n}(r))$, (Dirichlet boundary data if $r < \pi$) are pretty much unknown, with the exceptions $\lambda_1(B_{S^n}(\pi/2)) = n$, $\lambda_1(B_{S^n}(\pi)) = 0$. In dimension two and three there are good lower bounds due to Barbosa-DoCarmo [1], Pinsky [11], Sato [12] and Friedland-Hayman [8]. In higher dimension, the only lower bounds known (to the best of our knowledge) are the following lower bounds due to Betz, Camera and Gzyl obtained in [5] via probabilistic methods.

$$\left( \frac{c(n)}{r} \right)^2 > \lambda_1(B_{S^n}(r)) \geq \int_0^r \left[ \frac{1}{\sin^{n-1}(\sigma)} \cdot \int_0^\sigma \sin^{n-1}(s) ds \right] d\sigma. \quad (2)$$

The upper bound is due to Cheng’s eigenvalue comparison theorem [7] since the Ricci curvature of the sphere is positive (in fact, it needed only to be non-negative).

In order to state our result, recall the definition of a spherically symmetric manifold. Let $M$ be a Riemannian manifold and a point $p \in M$. For each vector $\xi \in T_pM$, let $\gamma_\xi$ be the unique geodesic satisfying $\gamma_\xi(0) = p$, $\gamma_\xi'(0) = \xi$ and $d(\xi) = \sup\{ t > 0 : \text{dist}_M(p, \gamma_\xi(t)) = t \}$. Let $D_p = \{ t \xi \in T_pM : 0 \leq t < d(\xi), |\xi| = 1 \}$ be the largest open subset of $T_pM$ such that for any $\xi \in D_p$ the geodesic $\gamma_\xi(t) = \exp_p(t \xi)$ minimizes the distance from $p$ to $\gamma_\xi(t)$ for all $t \in [0, d(\xi)]$. The cut locus of $p$ is the set $\text{Cut}(p) = \{ \exp_p(d(\xi) \xi), \xi \in T_pM, |\xi| = 1 \}$ and $M = \exp_p(D_p) \cup \text{Cut}(p)$. 
The exponential map \( \exp_p : D_p \to \exp_p(D_p) \) is a diffeomorphism and is called the geodesic coordinates of \( M \setminus \text{Cut}(p) \). Fix a vector \( \xi \in T_p M, |\xi| = 1 \) and denote by \( \xi^\perp \) the orthogonal complement of \( \{\mathbb{R}\xi\} \) in \( T_p M \) and let \( \tau_t : T_p M \to T_{\exp_p(t\xi)} M \) be the parallel translation along \( \gamma_\xi \). Define the path of linear transformations

\[ \mathcal{A}(t, \xi) : \xi^\perp \to \xi^\perp \]

by

\[ \mathcal{A}(t, \xi)\eta = (\tau_t)^{-1}Y(t) \]

where \( Y(t) \) is the Jacobi field along \( \gamma_\xi \) determined by the initial data \( Y(0) = 0, (\nabla_{\gamma_\xi} Y)(0) = \eta \). Define the map

\[ \mathcal{R}(t) : \xi^\perp \to \xi^\perp \]

by

\[ \mathcal{R}(t)\eta = (\tau_t)^{-1}R(\gamma_\xi(t), \tau_t\eta)\gamma_\xi(t), \]

where \( R \) is the Riemann curvature tensor of \( M \). It turns out that the map \( \mathcal{R}(t) \) is a self adjoint map and the path of linear transformations \( \mathcal{A}(t, \xi) \) satisfies the Jacobi equation \( \mathcal{A}'' + \mathcal{R}\mathcal{A} = 0 \) with initial conditions \( \mathcal{A}(0, \xi) = 0, \mathcal{A}'(0, \xi) = I \).

On the set \( \exp_p(D_p) \) the Riemannian metric of \( M \) can be expressed by

\[ ds^2(\exp_p(t\xi)) = dt^2 + |\mathcal{A}(t, \xi)d\xi|^2. \quad (3) \]

**Definition 1.1** A manifold \( M \) is said to be spherically symmetric if the matrix

\[ \mathcal{A}(t, \xi) = f(t)I, \text{ for a function } f \in C^2([0, R]), \quad R \in (0, \infty) \text{ with } f(0) = 0, f'(0) = 1, f|(0, R) > 0. \]

The class of spherically symmetric manifolds includes the canonical space forms \( \mathbb{R}^n, S^n(1) \) and \( H^n(-1) \). The \( n \)-volume \( V(r) \) of a geodesic ball \( B_M(r) \) of radius \( r \) in a spherically symmetric manifold is given by \( V(r) = w_n \int_0^r f^{n-1}(s)ds \), whereas the \((n-1)\)-volume \( S(r) \) of the boundary \( \partial B_M(r) \) is given by \( S(r) = w_n f^{n-1}(r) \). Here \( w_n \) denotes the \((n-1)\)-volume of the sphere \( S^{n-1}(1) \subset \mathbb{R}^n \). The authors \[2\] obtained using fixed point methods the following lower bound for the first
eigenvalue $\lambda_1(B_M(r))$ of geodesic balls $B_M(r)$ with radius $r$ in a spherically symmetric manifold $M$,

$$\lambda_1(B_M(r)) \geq \frac{1}{\int_0^r \frac{V(\sigma)}{S(\sigma)} d\sigma}. \quad (4)$$

It is worth mentioning that this lower bound (4) is Betz-Camera-Gzyl’s lower bound when $M = \mathbb{S}^n$. The purpose of this note is give upper and better lower bounds for $\lambda_1(B_M(r))$. We prove the following theorem.

**Theorem 1.2** Let $B_M(r) \subset M$ be a ball in a spherically symmetric Riemannian manifold with metric $dt^2 + f^2(t)d\sigma^2$, where $f \in C^2([0, R])$ with $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for all $t \in (0, R]$. For every non-negative function $u \in C^0([0, r])$ set

$$h(t, u) = \left[ u(t) \int_t^r \int_t^s \left( \frac{f(s)}{f(\sigma)} \right)^{n-1} u(s) d\sigma ds \right].$$

Then

$$\sup_t h(t, u) \geq \lambda_1(B_M(r)) \geq \inf_t h(t, u) \quad (5)$$

Equality holds in (5) if and only if $u$ is a first positive eigenfunction of $B_M(r)$ and $\lambda_1(B_M(r)) = h(t, u)$.

We should remark that taking $u \equiv 1$ in (5) we obtain (4). In the following table we compare our estimates for $\lambda_1(r) = \lambda_1(B_{\mathbb{S}^n}(r))$ for $n = 2, 3$, $r = \pi/8, \pi/4, 3\pi/8, \pi/2, 5\pi/8$ taking $u(t) = \cos(t\pi/2r)$ with the estimates obtained by Betz-Camera-Gzyl.

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<th>$r$</th>
<th>$\pi/8$</th>
<th>$\pi/4$</th>
<th>$3\pi/8$</th>
<th>$\pi/2$</th>
<th>$5\pi/8$</th>
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<tr>
<td>BCG</td>
<td>$\lambda_1(r)$</td>
<td>$\geq 25.77$</td>
<td>$\geq 6.31$</td>
<td>$\geq 2.70$</td>
<td>$\geq 1.44$</td>
<td>$\geq 0.85$</td>
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<tr>
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<td>$\geq 8.78$</td>
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<td>$= 2$</td>
<td>$\geq 1.01$</td>
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<th>$\pi/4$</th>
<th>$3\pi/8$</th>
<th>$\pi/2$</th>
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<tr>
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<td>$= 3$</td>
<td>$\geq 1.27$</td>
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2 Proof of Theorem 1.2

We start recalling the following theorem due to J. Barta.

**Theorem 2.1 (Barta, [3])** Let $\Omega \subset M$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ in a Riemannian manifold. For any $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $f|\Omega > 0$ and $f|\partial \Omega = 0$ one has that

$$\sup_M (-\triangle f/f) \geq \lambda_1(\Omega) \geq \inf_\Omega (-\triangle f/f). \quad (6)$$

Equality in (6) holds if and only if $f$ is a first eigenfunction of $\Omega$. The lower bound inequality needs only that $f|\Omega > 0$.

Let $u \in C^0([0,r])$, $u \geq 0$. Define a function $T(u) \in C^1([0,r])$ by $T(u)(t) = \int_t^r \int_0^\sigma (f(s)/f(\sigma))^{n-1} u(s)d\sigma ds$. Extend $u$ and $Tu$ radially to $B_M(r)$ by $\tilde{u}(\exp_p(t \eta)) = u(t)$ and $\tilde{T}(u)(\exp_p(t \eta)) = T(u)(t)$, for $\eta \in S^{n-1}$. Observe that $\tilde{T}(u)(\exp_p(t \eta)) \geq 0$, with $\tilde{T}(u)(\exp_p(t \eta)) = 0$ if and only if $t = r$. We claim that

$$\triangle \tilde{T} u(\exp_p(t \eta)) = -\tilde{u}(\exp_p(t \eta))$$

as the following straightforward computation shows.

**Proof:** The expression of a spherically symmetric metric in geodesic coordinates is given by $ds^2 = dt^2 + f^2(t)d\theta^2$. The Laplacian in these coordinates is given by

$$\triangle = \frac{\partial^2}{\partial t^2} + (n-1) \cdot \frac{f'(t)}{f(t)} \cdot \frac{\partial}{\partial t} + \frac{1}{f^2(t)} \triangle_{S^{n-1}}$$

Observe that a geodesic ball $B_M(r)$ are covered by one geodesic chart. Since $\tilde{T}u(\exp_p(t \eta)) = T(u)(t)$, we have that

$$\frac{\partial^2}{\partial t^2} T(u)(t) = -u(t) + (n-1) \frac{f'(t)}{f^n(t)} \int_0^t f^{(n-1)}(s)u(s)ds$$

and

$$\frac{\partial}{\partial t} T(u)(t) = -\frac{1}{f^{(n-1)}(t)} \int_0^t f^{(n-1)}(s)u(s)ds.$$ 

Therefore we have that

$$\triangle \tilde{T}u(\exp_p(t \eta)) = -u(t) = \tilde{u}(\exp_p(t \eta)).$$
Applying Barta’s Theorem we obtain that

\[ \sup_t \frac{u}{T(u)}(t) \geq \lambda_1(B(r)) \geq \inf_t \frac{u}{T(u)}(t). \]

Barta’s Theorem says that equality in the above inequality holds if and only if \( \tilde{T}(u) \) is a first eigenfunction. Thus we need only to show that \( \tilde{T}(u) \) is a first eigenfunction if and only if \( u \) is a first eigenfunction. Suppose that we have equality in (7) then \( \tilde{T}(u) \) is an eigenfunction, this is

\[ 0 = \Delta \tilde{T}u + \lambda_1(B_M(r))\tilde{T}u = -\tilde{u} + \lambda_1(B_M(r))\tilde{T}u \tag{8} \]

Applying the Laplacian in both side of the equation (8) we obtain by equation (7) that

\[ 0 = -\Delta \tilde{u} + \lambda_1(B_M(r))\Delta \tilde{T}u = -(\Delta \tilde{u} + \lambda_1(B_M(r))\tilde{u}) \tag{9} \]

Therefore \( u \) is a first eigenfunction with \( \lambda_1(B_M(r)) = \frac{u}{T(u)}. \)

References


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