A Nonstandard Standardisation Theorem

Eduardo Bonelli
Joint work with Beniamino Accattoli, Delia Kesner and Carlos Lombardi

December 9, 2013
Overview

1. Introduce a calculus of explicit substitutions called the Linear Substitution Calculus $\lambda_{\sim sub}$
2. Introduce the notion of standardisation
3. Say a thing or two about standardisation for $\lambda_{\sim sub}$

Approach:

- Informal, mostly via examples
- Intersperse the use of slides and the whiteboard
Lambda Calculus and Explicit Substitutions

Standardisation in the $\lambda$ calculus

Standardisation for $\lambda_{\sim sub}$
Review of the Lambda Calculus

\[ t ::= x \mid tt \mid \lambda x.t \]

\[ (\lambda x.s)t \mapsto_\beta s\{x := t\} \]
Explicit Substitutions

\[ t ::= x \mid tt \mid \lambda x.t \mid t[x/t] \]

\[ (\lambda x.t)u \mapsto_{\text{beta}} t[x/u] \]

- We add rules describing behaviour of \( t[x/t] \)
- Typical examples

\[
\begin{align*}
(tu)[x/v] & \mapsto_{\text{app}} t[x/v]u[x/v] \\
(\lambda y.t)[x/u] & \mapsto_{\text{abs}} \lambda y.t[x/u] \quad y \notin \text{fv}(u) \\
x[x/u] & \mapsto_{\text{var}} u
\end{align*}
\]
Problem with Traditional Presentations of ES

- Structure of reduction space is not amenable to algebraic treatment
- In particular, no obvious theory of residuals
- For example, the beta redex is lost in this step (non-orthogonality)

\[((\lambda y.t)u)[x/v] \mapsto_{app} (\lambda y.t)[x/v]u[x/v]\]
Recently – ES that act at a distance

- $\lambda_{\text{sub}}$ or the Linear Substitution Calculus
- Arises from work of Milner on the one hand, and that of Accattoli and Kesner on the other
- Has two parts: rewrite rules + equations
- Rewrite rules:
  
  $$(\lambda x.t)Lu \mapsto_{\text{db}} t[x/u]L$$
  $$C[x][x/u] \mapsto_{\text{lS}} C[u][x/u]$$
  $$t[x/u] \mapsto_{\text{gc}} t \quad \text{if } x \notin \text{fv}(t)$$

- $L = [x_1/t_1] \ldots [x_k/t_k]$ ($k$ may be 0)
- $C$ context (term with a hole); in $C[u]$ the free variables of $u$ are not captured by $C$
Rewrite rules

\[(\lambda x. t)Lu \quad \mapsto_{db} \quad t[x/u]L\]
\[C[x][x/u] \quad \mapsto_{ls} \quad C[u][x/u]\]
\[t[x/u] \quad \mapsto_{gc} \quad t \quad \text{if} \ x \notin \text{fv}(t)\]

Equations (generate what we call \textbf{graphical equivalence} \(\sim\))

\[t[x/u][y/v] \quad \approx_{CS} \quad t[y/v][x/u] \quad x \notin \text{fv}(v) \& \ y \notin \text{fv}(u)\]
\[(\lambda y. t)[x/u] \quad \approx_{\sigma_1} \quad \lambda y. t[x/u] \quad y \notin \text{fv}(u)\]
\[(tv)[x/u] \quad \approx_{\sigma_2} \quad t[x/u]v \quad x \notin \text{fv}(v)\]

Sample reduction (on the board): \((\lambda x.x[y/u]v)(\lambda z.z)\)
Lambda Calculus and Explicit Substitutions

Standardisation in the $\lambda$ calculus

Standardisation for $\lambda_{\sim\text{sub}}$
Introduction

- Sorting a list of numbers.
  
  ![Example of sorting a list of numbers]
  
  \[3, 4, 1, 2\]  
  \[\Rightarrow 3, 1, 4, 2\]  
  \[\Rightarrow 3, 1, 2, 4\]  
  \[\Rightarrow 1, 3, 2, 4\]  
  \[\Rightarrow 1, 2, 3, 4\]

- We would like to do a similar thing with derivations: sort the redexes in a derivation.
Sorting Redexes in Derivations

- left-to-right order

\[(lx)(ly) \rightarrow (lx)y\]

\[x(ly) \rightarrow xy\]

- Gets a little tricky due to duplication (below) and erasure

\[(\lambda x.xx)(ly) \rightarrow (\lambda x.xx)y\]

\[(ly)(ly) \rightarrow yy\]

- These can be made into “square” diagrams using a notion of simultaneous rewrite step (not developed in this talk)
Residuals in $\lambda$-calculus

- Needed to formalise notion of sorting
- The idea: follow a redex along a derivation by coloring it or labeling it
- Example of labeling for $\lambda$-calculus:
  - Labeled terms
    \[
    t ::= \ x \mid tt \mid \lambda x.t \mid (\lambda x^\alpha.s)t
    \]
  - Labeled $\beta$
    \[
    (\lambda x^\alpha.s)t \xrightarrow{\beta} s\{x := t\}
    \]
- Example of the residual relation $A/B$ (on the board): the residuals of redex $A$ after performing $B$
Residuals in $\lambda_{\text{sub}}$ (1/2)

- Labeled terms
  \[ t ::= x \mid x^\alpha \mid tt \mid \lambda x.t \mid \lambda x^\alpha.t \mid t[x/t] \mid t[x^\alpha/t] \]

- Labeled rewriting
  \[
  (\lambda x^\alpha.t)Lu \quad \xrightarrow{\alpha_{dB}} \quad t[x/u]L \\
  C[x^\alpha][x/u] \quad \xrightarrow{\alpha_{ls}} \quad C[u][x/u] \\
  t[x^\alpha/u] \quad \xrightarrow{\alpha_{gc}} \quad t \quad \text{if } x \notin \text{fv}(t)
  \]

- Anchor of a labeled redex is the variable containing the label

- Note: there is an additional well-labeled condition required which is omitted here (eg. $\lambda x.x^\alpha$ is not well-labeled)

- What about the graphical equivalence? We can do the same (next slide)
Residuals in $\lambda_{\text{sub}}$ (2/2)

- Labeled rewriting (same as above)
  
  $$(\lambda x^\alpha.t)Lu \xrightarrow{\alpha}_{\text{dB}} t[x/u]L$$
  $$C[[x^\alpha]][x/u] \xrightarrow{\alpha}_{\text{ls}} C[[u]][x/u]$$
  $$t[x^\alpha/u] \xrightarrow{\alpha}_{\text{gc}} t \quad x \notin \text{fv}(t)$$

- Labeled equivalence ($\langle \alpha \rangle$ means $\alpha$ may or may not appear)

  $t[x^{(\alpha)}/u][y^{(\beta)}/v] \approx_{\text{CS}} t[y^{(\beta)}/v][x^{(\alpha)}/u] \quad x \notin \text{fv}(v) \& y \notin \text{fv}(u)$
  $$(\lambda y^{(\beta)}.t)[x^{(\alpha)}/u] \approx_{\sigma_1} \lambda y^{(\beta)}.t[x^{(\alpha)}/u] \quad y \notin \text{fv}(u)$$
  $$(tv)[x^{(\alpha)}/u] \approx_{\sigma_2} t[x^{(\alpha)}/u]v \quad x \notin \text{fv}(v)$$

- Note: it can be shown that $s \sim t$ determines a bijective relation between the redexes of $s$ and $t$

- Examples (on the board)
Standardisation via Inversion (for total orders)

- **≺-inversion diagram** (
  \[ s \xrightarrow{B} t \quad A \prec B \]
  \[ A \downarrow \quad A/B \downarrow \]
  \[ s' \xrightarrow{B/A} t' \]
)

- **≺-inversion step** \( \Rightarrow \prec \) in a derivation:
  \[ \sigma_1; B; A/B; \sigma_2 \Rightarrow \prec \sigma_1; A; B/A; \sigma_2 \]

- **Definition**: A derivation in which no \( \Rightarrow \prec \) steps are applicable is said to be \( \prec \)-standard

**Theorem**

If \( \sigma : t \rightarrow_{\beta} u \) then there exists a unique \( \prec \)-left-standard \( \beta \)-derivation \( \rho : t \rightarrow_{\beta} u \) s.t. \( \sigma \Rightarrow^* \rho \).

**Proof**: \( \Rightarrow \prec \) SN+CR (Klop)
Standardisation via Inversion (for partial orders)

- $\prec$-inversion diagram ($\prec$ partial ordering on redexes)
  - Same as previous slide
- $\prec$-square diagram ($\prec$ partial ordering on redexes)

\[
\begin{array}{c}
s \xrightarrow{B} t \\
A \downarrow \quad A/B \downarrow \\
s' \xrightarrow{B/A} t'
\end{array}
\]

- $\prec$-square step $\diamondsuit_{\prec}$ (symmetric)
- $\prec$-inversion step $\Rightarrow_{\prec}$ in a derivation: apply $\Rightarrow_{\prec}$ modulo $\diamondsuit_{\prec}$
- Examples (on the board)
Definition: A derivation in which no \( \Rightarrow \bowtie \) steps are applicable is said to be \( \bowtie \)-standard

Theorem

If \( \sigma : t \rightarrow_{\beta} u \) then there exists a unique \( \bowtie \)\_left\)-standard \( \beta \)-derivation \( \rho : t \rightarrow_{\beta} u \) s.t. \( \sigma \Rightarrow^{*} \rho \). Note: uniqueness here means modulo \( \bowtie \). 

Proof1: Repeatedly extract external redex in \( \rho \) (Huet, Lévy, Melliès)

Proof2: \( \Rightarrow \bowtie \) SN+CR (TERESE)
Lambda Calculus and Explicit Substitutions

Standardisation in the $\lambda$ calculus

Standardisation for $\lambda_{\sim_{\text{sub}}}$
The requirement for the order on $\lambda_{\text{sub}}$ redexes

It must preserve the graphical equivalence

$\sim$ is a strong bisimulation between $\lambda_{\text{sub}}$ and itself that reduces the “same” redexes

\[
\begin{align*}
A & \quad \downarrow \quad A' \\
A_1 & \quad \downarrow \quad A'_1 \\
A_2 & \quad \downarrow \quad A'_2 \\
A_n & \quad \downarrow \quad A'_n \\
A_{n+1} & \quad \downarrow \quad A'_{n+1}
\end{align*}
\]

Thus standardisation should be “preserved” via the equations

$A_1; \ldots; A_n$ standard iff $A'_1; \ldots; A'_n$ standard
An example

\[
\begin{align*}
t[x^\alpha/u][y^\beta/v] & \sim t[y^\beta/v][x^\alpha/u] \\
A & \quad A' \\
\downarrow & \quad \downarrow \\
t[y^\beta/v] & \sim t[y^\beta/v] \\
B & \quad B' \\
\downarrow & \quad \downarrow \\
t & \sim t
\end{align*}
\]

- Note \( t[x^\alpha/u][y^\beta/v] \sim_{cs} t[y^\beta/v][x^\alpha/u] \), assuming \( y \notin \text{fv}(u) \)
- \( A; B \) standard iff \( A'; B' \) standard
- The left-to-right order does not make sense due to the graphical equivalence
Action Principle as Guideline

For devising appropriate partial order on redexes in $\lambda_{\text{sub}}$

\[
\begin{align*}
C[x][x/s] & \rightarrow C[x][x/s'] \\
\downarrow & \quad \downarrow \\
C[s][x/s] & \rightarrow C[s'][x/s']
\end{align*}
\]

*Standard* should be down-below since the $ls$-redex acts on (i.e. *nests*) the redexes in $s$
Action Principle as Guideline

\[ t[x/s] \rightarrow t[x/s'] \]

\[ t \]

*Standard* should be down since the erasing redex *acts on* the redexes in *s*
Action Principle as Guideline

\[ x[x/y][y/z] \rightarrow x[x/z][y/z] \]
\[ \downarrow \quad \downarrow \]
\[ y[x/y][y/z] \rightarrow z[x/z][y/z] \]

1s-redex on \( x \) must nest the 1s-redex on \( y \)

- Note that duplicated 1s-redex on \( y \) is not syntactically contained in the acting 1s-redex on \( x \)
- The same diagram applies to terms like \((x[x/y]yz)[y/z]\), where \([x/y]\) and \([y/z]\) are no longer next to each other.
This is the version at a distance of the erasing diagram, requiring the same notion of nesting at a distance.
Definition of the partial “box” order

- A immediately boxes $B$, noted $A \prec^1_B B$ if the anchor of $B$ (i.e. the variable possibly carrying a label) is in the box of $A$
  - i.e. if the pattern of $A$ is any of $(\lambda x.t)Lu$, $C[x][x/u]$ or $t[x/u]$, then the anchor of $B$ appears in $u$.
- A boxes $B$, noted $A \prec_B B$ if $A(\prec^1_B)^+ B$
- A and $B$ are disjoint, noted $A \parallel B$, if $A \not\prec_B B$ and $B \not\prec_B A$.
- Key property: box order is stable by the equivalence $\sim$
Some Results

Theorem (Existence of Standard Derivations for $\lambda_{\text{sub}}$)
If $t \rightarrow_{\lambda_{\text{sub}}} u$ then there is a $\prec_B$-standard $\lambda_{\text{sub}}$-derivation from $t$ to $u$.

Proof uses axiomatics of Melliès

Theorem (Uniqueness Modulo for $\lambda_{\text{sub}}$)
If $t \rightarrow_{\lambda_{\text{sub}}} u$ then there exists a $\prec_B$-standard $\lambda_{\text{sub}}$-derivation from $t$ to $u$ that is unique modulo $\diamondsuit$.

Proof uses

1. Existence of Standard Derivations for $\lambda_{\text{sub}}$;
2. Uniqueness of standardisation for $\lambda_{\text{sub}}$ w.r.t. the left-to-right order; and
3. A simple argument showing that $\prec_L$-inversions of a $\prec_B$-standard derivation swaps only disjoint (w.r.t. $\prec_B$) redexes
Conclusions

- Quick overview of $\lambda_\text{sub}$
- Quick overview of standardisation
- Standardisation for $\lambda_\text{sub}$
- General context of this work: $\lambda_\text{sub}$ as a vehicle to study the metatheory of the $\lambda$-calculus

Further reading: Standardisation (Ch.8: TERESE), This work (POPL 2014)