Part I

Logics for Approximate Reasoning

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What this talk is NOT about...

- Fuzzy Logics
- Probabilistic Logics
- Multivalued Logics
- Intuitionistic, Relevant, Linear Logics.
General Contents

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1. Motivation

- Approximations deal with hard problems.
  - Classical Propositional Satisfiability and Theorem Proving are hard (NP- and coNP-complete)

- Idealised agents are logically omniscient.
  - Real agents are limited.
  - Each step in an approximation models a limited agent.

- Approximations implicitly define notions of relevance.
1.1 History of Logics for Approximations

- Schaerf & Cadoli [1995]: Families of Logics S1 and S3, clausal form.
- Uses of S1, S3: diagnosis, belief revision.
- Other approaches for approximation: Horn Clause Approximations
  - Linear approximation for an exponential problem.
- Our work follows the paradigm of Schaerf & Cadoli.
2. Intuitions of Approximations

- A family of Logics: $L_1, L_2, \ldots, L_n$
- A target Logic $L$ to approximate.
- The mathematical intuition:

$$\lim_{n \to \infty} |L - L_n| = \emptyset$$

(I know this expression has no formal meaning!)
2.1 Clarifying the Intuitions

- Think of $L$ as $Th(L)$ or $|=L$.
- $|L - L_n| = (L - L_n) \cup (L_n - L)$.
- The notion of approximation can be expressed as:

$$|L - L_1| \supset |L - L_2| \supset \cdots |L - L_n| \supset \cdots \supset \emptyset$$

- Theorem Proving: approximations “from below”, $L_n \subseteq L$

$$L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots \subseteq L$$

- Theorem DisProving, SAT: approximations “from above”, $L_n \supseteq L$

$$L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots \supseteq L$$
3. Schaerf & Cadoli’s Proposal

- Restricted to Clausal Form: $\bigwedge (l_1 \lor \cdots \lor l_m)$. (Later in NNF)
- Based on a context set $S$.
- If $p \in S$, $p$ behaves classically
  
  $$\nu(p) = 1 \quad \text{iff} \quad \nu(\neg p) = 0$$

- If $p \notin S$, $p$ has a special behaviour:

  $\nu(p) = 0 \quad \text{and} \quad \nu(\neg p) = 1 \quad S_3(S)$
  $\nu(p) = 1 \quad \text{and} \quad \nu(\neg p) = 0 \quad S_1(S)$
  $\nu(p) = 1 \quad \text{and} \quad \nu(\neg p) = 1 \quad S_3(S)$
  $\nu(p) = 0 \quad \text{and} \quad \nu(\neg p) = 0 \quad S_1(S)$
3.1 Approximate Entailment

Logics $S_3$ are useful to approximate Theorem Proving:

$$B \models^3_S \alpha \implies B \models \alpha$$

Logics $S_1$ are useful to approximate “Theorem Disproving” or SAT:

$$B \not\models^1_S \alpha \implies B \not\models \alpha$$

When $S = \varnothing$, $S_1(S) = S_3(S) = CL$.

Theorem 1 There exists algorithms for deciding if $B \models^3_S \alpha$ and deciding $B \models^1_S \alpha$ which runs in $O(|B| \cdot |\alpha| \cdot 2^{|S|})$ time.

For a fixed $S$ these algorithms are linear!
4. Theorem proving in $S_3$

Example (due to [SC 95]).

Check whether $B \models \alpha$, where $\alpha = \neg \text{cow} \lor \text{molar-teeth}$ and

\[
B = \{ \neg \text{cow} \lor \text{grass-eater}, \neg \text{dog} \lor \text{carnivore},
\neg \text{grass-eater} \lor \neg \text{canine-teeth}, \neg \text{carnivore} \lor \text{mammal},
\neg \text{mammal} \lor \text{canine-teeth} \lor \text{molar-teeth},
\neg \text{grass-eater} \lor \text{mammal}, \neg \text{mammal} \lor \text{vertebrate},
\neg \text{vertebrate} \lor \text{animal} \}\.
\]

For $S = \{ \text{grass-eater}, \text{mammal}, \text{canine-teeth} \}$
4.1 \( S_3 \) simplification

To decide whether \( B \models^{3}_{S} \alpha \):

- Delete from \( B \) all clauses which contain an atom \( p \not\in S \) that does not occur in \( \alpha \).
- Obtain \( B' \subseteq B \).
- Apply classical theorem proving to the resulting \( B' \models \alpha \).
- \( B' \models \alpha \) iff \( B \models^{3}_{S} \alpha \).
4.2 $S_3$ Example (cont.)

Check whether $B \models \alpha$, where $\alpha = \neg\text{cow} \lor \text{molar-teeth}$ and

$B = \{ \neg\text{cow} \lor \text{grass-eater}, \neg\text{dog} \lor \text{carnivore},$
\[
\neg\text{grass-eater} \lor \neg\text{canine-teeth}, \neg\text{carnivore} \lor \text{mammal},$
\[
\neg\text{mammal} \lor \text{canine-teeth} \lor \text{molar-teeth},$
\[
\neg\text{grass-eater} \lor \text{mammal}, \neg\text{mammal} \lor \text{vertebrate},$
\[
\neg\text{vertebrate} \lor \text{animal} \}.\]

For $S = \{\text{grass-eater}, \text{mammal}, \text{canine-teeth}\}$

We have that $B \models^3_S \alpha$, hence $B \models \alpha$. 
5. Refutation in $S_1$

Check whether $B \not\models \beta$, where $\beta = \neg \text{child} \lor \text{pensioner}$ and

$$B = \{ \neg \text{person} \lor \text{child} \lor \text{youngster} \lor \text{adult} \lor \text{senior},$$
$$\neg \text{adult} \lor \text{student} \lor \text{worker} \lor \text{unemployed},$$
$$\neg \text{pensioner} \lor \text{senior}, \quad \neg \text{youngster} \lor \text{student} \lor \text{worker},$$
$$\neg \text{senior} \lor \text{pensioner} \lor \text{worker}, \quad \neg \text{pensioner} \lor \neg \text{student},$$
$$\neg \text{student} \lor \text{child} \lor \text{youngster} \lor \text{adult},$$
$$\neg \text{pensioner} \lor \neg \text{worker} \}.$$ 

For $S = \{ \text{child}, \text{worker}, \text{pensioner} \}$. 
5.1 \( S_1 \) simplification

- To decide whether \( B \models_1^S \alpha \):
  - If \( p \notin S \), make \( p, \neg p \) false in \( B \)
  - Obtain \( B' \subseteq B \).
  - Apply classical SAT techniques to the resulting \( B' \models \alpha \).
  - \( B' \models \alpha \) iff \( B \models_1^S \alpha \), for \( \alpha \in S \).
5.2 $S_1$ Example (cont)

Check whether $B \not\models \beta$, where $\beta = \neg \text{child} \lor \text{pensioner}$ and

\[
B = \{ \neg \text{person} \lor \text{child} \lor \text{youngster} \lor \text{adult} \lor \text{senior},
\neg \text{adult} \lor \text{student} \lor \text{worker} \lor \text{unemployed},
\neg \text{pensioner} \lor \text{senior}, \quad \neg \text{youngster} \lor \text{student} \lor \text{worker},
\neg \text{senior} \lor \text{pensioner} \lor \text{worker}, \quad \neg \text{pensioner} \lor \neg \text{student},
\neg \text{student} \lor \text{child} \lor \text{youngster} \lor \text{adult},
\neg \text{pensioner} \lor \neg \text{worker} \}.
\]

For $S = \{ \text{child, worker, pensioner} \}$.

We have that $B \not\models^1_{S} \beta$, and hence $B \not\models \beta$. 
6. Analysis of Cadoli & Schaerf’s Method

- **Good points of $S_3$:**
  - $S_3$ approximates classical logic from below.
  - Nice, simple simplifications.
  - The set $S$ defines a notion of relevance.
  - $S_3$ is paraconsistent: $p \land \lnot p \not\models^3_S q$ if $p \not\in S$.

- **Problems with $S_3$:**
  - Clausal form only
  - Algorithm for simplification is not incremental.
  - Incremental method proposed, but no strategy to compute $S$ is suggested.
  - No proof theory.
6.1 Problems with $S_1$

- $S_1$ does not approximate classical logic from above for:

\[ \not \models_{S_1} p \lor \neg p, \quad \text{if } p \notin S. \]

($S_1$ is paracomplete)

- $S_1$ cannot be extended to full propositional logic

- No strategy to compute $S$ is suggested.

- $\models_{S_1}$ is not a local entailment:
  - To show that $B \not \models_{S_1} \alpha$, many irrelevant atoms have to be added to $S$, so that $v(B) = 1$.
  - No notion of relevance is given by $S_1$. 
7. Next Topics

Part II: Approximate Theorem Proving:

- $S_3$ extended to the full propositional language.
- An incremental proof method for $S_3(S)$.
- A strategy to compute $S$.

Part III: Approximation of Classical Logic from Above

- The Family of Logics $s_1(s)$.
- $s_1$ 3-valued semantics for full propositional language.
- The notion of $s_1$-relevance.
- $s_1$-simplifications.