Lectures Notes on Convex Analysis

Leandro Cioletti

Departamento de Matemática - Universidade de Brasília
Email address: cioletti@mat.unb.br

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CHAPTER 1

General Topology

"Point set topology is a disease which the human race will soon recover"
—attributed to H. Poincaré.

The aim of this chapter is only to fix some notations and recall important notions of General Topology. The material covered is completely standard and should be presented in most of (under)graduate text books on this subject. The presentation here follows closely the references [**1, 2, 3, 4**].

1.1. Topological Spaces

This chapter introduces what sometimes is called point set topology. It is not our intention to provide motivations or examples for the concepts introduced here and neither background on metric spaces, although they should appear sooner or latter. The goal is to conduct the discussion towards to understand and prove compactness theorems in infinite dimensional vector spaces in some suitable weak topologies. A lot of definitions will be required to achieve this goal and due to lack of time, very occasionally examples are given. So we encourage the reader to take a look at the above cited references for a "conventional" (motivation, examples and so on...) exposition.

Definition 1.1 (Topological Space). A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X (the topology) that obeys three axioms:

(1) $\emptyset, X \in \tau$;
(2) for all $U_1, \ldots, U_n \in \tau$ we have $U_1 \cap \ldots \cap U_n \in \tau$;
(3) if $\{U_\alpha\}_{\alpha \in I}$ is an arbitrary subcollection of τ, then $\bigcup_{\alpha\in I} U_\alpha \in \tau$.

The sets in τ are called *open sets*.

When there is no danger of confusion we denote a topological space (X, τ) simply by its underlying set X. So when we say things like, let X be a topological space the topology will be completely clear from the context or the statement will be valid for a general topology on X.

Proposition 1.2. *Let I be an arbitrary index set. If τ_i, for each $i \in I$, is a topology on X. Then the collection*

$$\tau \equiv \{U \subset X : U \in \tau_i,\ \forall i \in I\}$$

is a topology on X. In other words, the intersection of any family of topologies on X is a topology on X.

Definition 1.3 (Closed Sets). Let (X, τ) be a topological space. We say that a set $A \subset X$ is a *closed set* if $A^c \equiv X \setminus A$ is an open set.

The proof of the following theorem is an obvious application of de Morgan's laws together with the definition of a topology on X, and will be omitted.

THEOREM 1.4. *If $\mathscr{C}$ is the collection of all closed sets in a topological space (X, τ), then*

i) $\emptyset, X \in \mathscr{C}$;
ii) if $A_1, \ldots, A_n \in \mathscr{C}$, *then* $A_1 \cup \ldots \cup A_n \in \mathscr{C}$;
iii) if I is an arbitrary index set and $A_\alpha \in \mathscr{C}$, $\forall \alpha \in I$. *Then* $\bigcap_{\alpha \in I} A_\alpha \in \mathscr{C}$.

Conversely, given a set X and any family $\mathscr{C}$ of subsets of X satisfying i)–iii)*, the collection of complements of members of $\mathscr{C}$ is a topology on X in which the family of closed sets is exactly $\mathscr{C}$.*

This theorem is a result of, and illustrates, the obvious duality between the notions of open and closed set. More formally, any result about the open sets in a topological space becomes a result about closed sets upon replacing "open" by "closed" and interchanging $\bigcup$ by $\bigcap$.

DEFINITION 1.5 (Closure). Let (X, τ) be a topological space and $E \subset X$. The *closure* of E in (X, τ) is the set

$$\overline{E} \equiv \bigcap \{A \subset X : A \text{ is closed and } E \subset A\}.$$

Note that $\overline{E}$ is the "smallest" closed set containing E, in the sense that it is contained in every closed set containing E.

LEMMA 1.6. *Let (X, τ) be a topological space and $A \subset B \subset X$. Then $\overline{A} \subset \overline{B}$.*

THEOREM 1.7. *Let (X, τ) be a topological space and $\mathscr{P}(X)$ the set of parts of X. The operation $\mathscr{P}(X) \ni A \longmapsto \overline{A} \in \mathscr{P}(X)$ has the following properties:*

i) $A \subset \overline{A}$;
ii) $\overline{\overline{A}} = \overline{A}$;
iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
iv) $\overline{\emptyset} = \emptyset$;
v) A is closed if and only if $A = \overline{A}$.

Moreover, if $\Phi : \mathscr{P}(X) \to \mathscr{P}(X)$ is a set-function satisfying i)– v) *and we define the collection $\mathscr{C}$ of the sets satisfying* v)*. We have that the complements of members of $\mathscr{C}$ determines a topology on X, whose the closure operation is just the operation Φ we began with.*

Any operation $\mathscr{P}(X) \ni A \longmapsto \overline{A} \in \mathscr{P}(X)$ in a set X satisfying *i*)– *v*) is called a *Kuratowski* closure operation. Thus every Kuratowski closure operation determines and is determined by some topology.

DEFINITION 1.8 (Interior). Let (X, τ) be a topological space and $E \subset X$. The *interior* of E in (X, τ) is the set

$$\text{Int}(E) \equiv \bigcup \{U \subset X : \ U \text{ is open and } U \subset E\}.$$

Evidently, $\text{Int}(E)$ is an open set. It is the largest open set contained in E, in the sense that it contains any other open set contained in E. We also remark that it might be possible, for some sets to have $\text{Int}(E) = \emptyset$.

The notions of interior and closure are dual to each other, in much the same way that are "open" and "closed". The strictly formal nature of this duality can be brought out by observing that

$$X \setminus \mathrm{Int}(E) = \overline{X \setminus E}$$
$$X \setminus \overline{E} = \mathrm{Int}(X \setminus E).$$

Thus any theorem about closures in a topological space can be translated to a theorem about interiors. The next results are, for example, of this duality.

LEMMA 1.9. *Let (X, τ) be a topological space and $A \subset B \subset X$. Then we have* $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.

THEOREM 1.10. *Let (X, τ) be a topological space and $\mathscr{P}(X)$ the set of parts of X. The operation $\mathscr{P}(X) \ni A \longmapsto \mathrm{Int}(A) \in \mathscr{P}(X)$ has the following properties:*

i) $\mathrm{Int}(A) \subset A$;
ii) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$;
iii) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$;
iv) $\mathrm{Int}(X) = X$;
v) *A is open if and only if* $A = \mathrm{Int}(A)$.

Moreover, if $\Psi : \mathscr{P}(X) \to \mathscr{P}(X)$ is a set-function satisfying i)– iv) and we define τ as the collection of the sets satisfying v), then τ is a topology on X. In this topology the interior of a set A is just $\Psi(A)$.

1.2. Neighborhoods

The characterizations we have so far to describe a topology (open sets, the closure operation and so on) are not the most convenient, and for this reason are rarely used. In this section we present the two most popular ways to describe topologies.

Very often the topology we wish to present is quite "regular", in the sense that the open sets containing one point look no different from the open sets containing any other(this is true for example, in the Euclidean spaces). In such cases one can describe the topology by describing what it is look like "around" one point, or a few points. Considerable saving of effort can result, and topologies will often be presented this way here, so we will present in what follows a detailed discussion of the "local" description of topologies and topological concepts.

DEFINITION 1.11 (Neighborhoods). Let (X, τ) be a topological space and $x \in X$ an arbitrary point. A *neighborhood* of x is a subset $U \subset X$ satisfying:

- $x \in U$;
- there is $V \in \tau$ such that $x \in V$ and $V \subset U$.

Clearly, U is a neighborhood of x if and only if $x \in \mathrm{Int}(U)$.

DEFINITION 1.12 (Neighborhood System at x). Let (X, τ) be a topological space and $x \in X$ an arbitrary point. The collection

$$\mathscr{U}_x \equiv \{U \subset X : U \text{ is a neighborhood of } x\}$$

is called *neighborhood system at x*.

The next result lists some fundamental properties of a neighborhood system at x in a topological space and provides a converse which says whenever neighborhoods has been assigned to each point of a set, satisfying these properties, one has a topology.

THEOREM 1.13. *Let (X, τ) be a topological space, $x \in X$ an arbitrary point, and $\mathscr{U}_x$ a* neighborhood *system at x. Then*

i) if $U \in \mathscr{U}_x$, then $x \in U$;
ii) if $U, V \in \mathscr{U}_x$, then $U \cap V \in \mathscr{U}_x$;
iii) if $U \in \mathscr{U}_x$, then $\exists V \in \mathscr{U}_x$ such that for each $y \in V$, we have $U \in \mathscr{U}_y$;
iv) if $U \in \mathscr{U}_x$ and $U \subset V$, then $V \in \mathscr{U}_x$;
v) A is open, if and only if A contains a neighborhood of each of its points.

Conversely, if in a set X a nonempty collection $\mathscr{U}_x$ of subsets of X is assigned to each $x \in X$ satisfying i) –iv) and v) is used to define a collection τ of subsets of X, then τ is a topology on X, in which the neighborhood system at $x \in X$ is precisely the collection $\mathscr{U}_x$.

DEFINITION 1.14 (Neighborhood base at x). Let (X, τ) be a topological space, $x \in X$ an arbitrary point, and $\mathscr{U}_x$ a *neighborhood* system at x. A subcollection $\mathscr{B}_x$ taken from $\mathscr{U}_x$, having the property that each $U \in \mathscr{U}_x$ contains some $V \in \mathscr{B}_x$ is called a *neighborhood base at x*. That is, $\mathscr{U}_x$ must be determined by $\mathscr{B}_x$ in the following sense

$$\mathscr{U}_x = \{U \subset X : \text{ there exists some } V \in \mathscr{B}_x \text{ such that } V \subset U\}.$$

Once a neighborhood base at x has been chosen (there are many to choose from, all producing the same neighborhood system at x) its elements are called *basic neighborhoods*. Obviously, a neighborhood system at x is itself always a neighborhood base at x. In general, we are interested in smaller basis.

We turn now to the problem of specifying a topology by giving a collection of basic neighborhoods at each point of the space.

The following theorem is used much more often than the corresponding Theorem 1.13 about neighborhood systems.

THEOREM 1.15. *Let (X, τ) be a topological space and for each $x \in X$, let $\mathscr{B}_x$ be a neighborhood base at x. Then*

i) if $V \in \mathscr{B}_x$, then $x \in V$;
ii) if $V_1, V_2 \in \mathscr{B}_x$, then $\exists V_3 \in \mathscr{B}_x$ such that $V_3 \subset V_1 \cap V_2$;
iii) if $V \in \mathscr{B}_x$, there is some $V_0 \in \mathscr{B}_x$ such that for each $y \in V_0$ there is some $W \in \mathscr{B}_y$ with $W \subset V$;
iv) $A \subset X$ is open if and only if A contains a basic neighborhood of each of its points.

Conversely, in a set X if a collection $\mathscr{B}_x$ of subsets of X is assigned to each $x \in X$ satisfying i)– iii) and if we define a collection τ by using v), then τ is a topology on X in which $\mathscr{B}_x$ is a neighborhood base at x, for each $x \in X$.

Since neighborhood bases are important descriptive devices in dealing with topologies, it will be useful to have neighborhood characterizations of all concepts so far introduced for topological spaces.

THEOREM 1.16. *Let (X, τ) be a topological space and suppose that $\mathscr{B}_x$, for each $x \in X$, is a fixed neighborhood base at x. Then*

i) $A \subset X$ *is open if and only if for each* $x \in A$ *there is some* $U \in \mathscr{B}_x$ *such that* $U \subset A$*;*
ii) $A \subset X$ *is closed if and only if for each point* $x \in X \setminus A$ *there is* $U \in \mathscr{B}_x$ *such that* $A \cap U = \emptyset$*;*
iii) $\overline{A} = \{x \in X : \forall U \in \mathscr{B}_x$ *we have* $U \cap A \neq \emptyset\}$*;*
iv) $\mathrm{Int}(A) = \{x \in X : \exists U \in \mathscr{B}_x$ *satisfying* $U \subset A\}$*;*
v) $\partial A = \{x \in X : \forall U \in \mathscr{B}_x$ *we have* $U \cap A \neq \emptyset$ *and* $U \cap (X \setminus A) \neq \emptyset\}$*.*

Roughly speaking the next theorem states that "small neighborhoods make large topologies". This is intuitive since, smaller the neighborhoods in a space are, the easier it is for a set to contain neighborhoods of all its points and then the more open sets there will be.

Theorem 1.17 (Hausdorff Criterion). *For each* $x \in X$ *let* $\mathscr{B}_x^1$ *be a neighborhood base at* x *for a topology* τ_1 *on* X*, and* $\mathscr{B}_x^2$ *be a neighborhood base at* x *for a topology* τ_2 *on* X*. Then* $\tau_1 \subset \tau_2$ *if and only if for each* $x \in X$ *given* $B_1 \in \mathscr{B}_x^1$ *there is some* $B_2 \in \mathscr{B}_x^2$ *such that* $B_2 \subset B_1$*.*

We close this section by introducing a concept that will play a major role in the subsequent sections.

Definition 1.18 (Accumulation point). An *accumulation point* of a set A in a topological space (X, τ) is a point $x \in X$ such that each neighborhood (basic neighborhood, if you prefer) of x contains some point of A, other than x. The set of all accumulation points of A is denoted by A' and sometimes called *derived* set of A.

Theorem 1.19. *Let* (X, τ) *be a topological space. Then for any* $A \subset X$ *we have* $\overline{A} = A \cup A'$*.*

Definition 1.20 (Cluster point). Let (X, τ) be a topological space. A point $x \in X$ is a *cluster point of a sequence* $(x_n)_{n\in\mathbb{N}}$ if for any open set $A \ni x$ and any $N \in \mathbb{N}$ there exists $n \geq N$ so that $x_n \in A$.

Note that the notions of cluster and accumulation points are, in general, different. For example, consider the topological space (X, τ), where $X = \{0\} \cup \{n^{-1} : n \geq 1\}$ and τ is the discrete topology on X. Let $(x_n)_{n\in\mathbb{N}}$ be a constant sequence on X given by $x_n = x \neq 0$, for all $n \in \mathbb{N}$. Then x is a cluster point of $(x_n)_{n\in\mathbb{N}}$, but x is not an accumulation point of $A = (x_n)_{n\in\mathbb{N}}$.

1.3. Bases and subbases

As we observed in last section, we can specify the neighborhood system at a point x of a topological space (X, τ) by giving a somewhat smaller collection of sets, that is, a neighborhood base at x. In much the same way, the idea of *a base for a topology* τ will be a way to specify it without the needing of describe each and every of its open sets.

Definition 1.21 (Base for a topology). Let (X, τ) be a topological space. A collection $\mathscr{B} \subset \tau$ is a base for τ if

$$\tau = \left\{ \bigcup_{U \in \mathscr{C}} U : \mathscr{C} \subset \mathscr{B} \right\}.$$

As we can see if $\mathscr{B}$ is a base for a topology τ on X, then the topology can be recovered from $\mathscr{B}$ by taking all possible unions of subcollections from $\mathscr{B}$.

PROPOSITION 1.22. *A collection $\mathscr{B}$ is a base for (X, τ) if and only if whenever A is a open set and $x \in A$ is an arbitrary point, there is some $U \in \mathscr{B}$ such that $x \in U \subset A$.*

The following theorem is similar to theorems 1.7, 1.10, 1.13, and 1.15. That is, it list a few properties that a bases enjoy and provides the converse assertion: any collection of subsets of X enjoying these properties provides a topology on X.

THEOREM 1.23. *A collection $\mathscr{B}$ of subsets of a set X is a base for a topology $\tau(\mathscr{B}) = \left\{\bigcup_{U\in\mathscr{C}} U : \mathscr{C} \subset \mathscr{B}\right\}$ on X if and only if*

i) $X = \bigcup_{U\in\mathscr{B}} U$;
ii) *for any pair $U_1, U_2 \in \mathscr{B}$ with $x \in U_1 \cap U_2$, there exists $U_3 \in \mathscr{B}$ such that $x \in U_3 \subset U_1 \cap U_2$.*

The reader might well suspected that more than a casual similarity exists between the idea of a neighborhood base at each point of X on the one hand and the notion of a base for the topology of X on the other hand. Indeed, as the next theorem make clear, the only real difference between the two notions is that neighborhoods bases need not consist of open sets.

THEOREM 1.24. *If $\mathscr{B}$ is a collection of open sets in (X, τ), $\mathscr{B}$ is a base for τ if and only if for each $x \in X$, the collection $\mathscr{B}_x \equiv \{U \in \mathscr{B} : x \in U\}$ is a neighborhood base at x.*

We can go one step further in reducing the size of the collection we must specify to describe a topology. Recall that the reduction from topology to a base was accomplished essentially by dropping the requirement that any union of elements of τ belongs to τ. The further reduction to subbase is accomplished essentially by dropping the requirement that any finite intersection of elements of τ belongs to τ.

DEFINITION 1.25 (Subbase). If (X, τ) is a topological space, a *subbase* for τ is a collection $\mathscr{C} \subset \tau$ such that the collection of all finite intersections of elements from $\mathscr{C}$ forms a base for τ.

THEOREM 1.26. *Any collection $\mathscr{C}$ of subsets of X is a subbase for some topology $\tau(\mathscr{C})$ on X. Moreover,*

$$\tau(\mathscr{C}) \equiv \bigcap_{\substack{\tau\supset\mathscr{C} \\ \tau \text{ is a topology on } X}} \tau,$$

that is, the smallest topology on X containing $\mathscr{C}$.

1.4. Subspaces

A subset of a topological space inherits a topology of its own, in a obvious way. This topology and some of its easily developed properties will be presented here.

DEFINITION 1.27 (Subspace). Let (X, τ) be a topological space and $Y \subset X$. The collection $\tau_Y \equiv \{U \cap Y : U \in \tau\}$ is a topology on Y, called *relative topology* for Y. The fact that a subset of X is being given this topology is signified by referring to it as a subspace of X.

Note that any subspace of a discrete topological space is discrete and any subspace of a trivial space is trivial. A subspace of a subspace is a subspace. That is, if $Z \subset Y \subset X$, then the relative topology induced on Z by the relative topology of Y is just the relative topology of Z in X.

The open sets in a subspace Y of X are the intersections with Y of the open sets in X. Most, but no all, of the related topological notions are introduced into Y in the same way, by intersection, as the following theorem shows.

THEOREM 1.28. *If (Y, τ_Y) is a subspace of (X, τ) then*

i) *$V \subset Y$ is open in (Y, τ_Y) if and only if $V = U \cap Y$, where U is open in (X, τ);*
ii) *$B \subset Y$ is closed in (Y, τ_Y) if and only if $B = A \cap Y$, where A is closed in (X, τ);*
iii) *If $A \subset Y$, then $\mathrm{Cl}_Y(A) = Y \cap \mathrm{Cl}_X(A)$, where $\mathrm{Cl}_Y(A)$ is the closure of A in (Y, τ_Y).*
iv) *if $x \in Y$, then V is a neighborhood of x in (Y, τ_Y) if and only if $V = U \cap Y$, where U is a neighborhood of x in (X, τ).*
v) *if $x \in Y$, and $\mathscr{B}_x$ is neighborhood base at x in (X, τ), then $\{U \cap Y : U \in \mathscr{B}_x\}$, is a neighborhood base at x in (Y, τ_Y).*
vi) *if $\mathscr{B}$ is a base for τ, then $\{U \cap Y : U \in \mathscr{B}\}$ is a base for τ_Y.*

1.5. Continuous Functions

It is the aim this section to define continuous functions on a topological space and establish their elementary properties.

DEFINITION 1.29. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $f(U) \subset V$. We say that f is continuous on X if f is continuous at each $x_0 \in X$.

Is left to the reader as exercise to verify that the concept of continuous is not altered if we replace neighborhoods by basic neighborhoods throughout.

In the sequel we see an alternative, and somewhat surprising set of characterizations of continuity. This theorem, in one or another of its form, is more often used to check "global" continuity than the alternative, that is, check continuity at each point of X individually. The fourth characterization, although not often used as a test for continuity, is interesting. It provide us with description of the continuous functions $f : X \to Y$ as precisely those functions which take the points close to a set E in X close it its image in Y.

THEOREM 1.30. *If (X, τ_X) and (Y, τ_Y) are topological spaces and $f : X \to Y$. Then the following are all equivalent:*

i) *f is continuous;*
ii) *for each open set $V \subset Y$, we have that $f^{-1}(V)$ is open in X;*
iii) *for each closed set $B \subset Y$, we have that $f^{-1}(B)$ is closed in X;*
iv) *for each set $A \subset X$, we have that $f(\overline{A}) \subset \overline{f(A)}$.*

THEOREM 1.31. *Let (X, τ_X) and (Y, τ_Y) be a topological spaces and $f : X \to Y$ a function. If $\mathscr{C}$ is a subbase for the topology τ_Y and $f^{-1}(S)$ is an open set in X, for any $S \in \mathscr{C}$, then f is a continuous function.*

Proof. Let us first note that if the topology of the range space Y is given by a basis $\mathscr{B}$, then to prove continuity of f it suffices to show that the inverse image of every basis element is open. Indeed, an arbitrary open set $V \subset Y$ can be written as a union of basis elements

$$V = \bigcup_{i \in I} W_i.$$

Therefore

$$f^{-1}(V) = f^{-1}\Big(\bigcup_{i \in I} W_i\Big) = \bigcup_{i \in I} f^{-1}(W_i),$$

and so f is continuous.

Since $\mathscr{C}$ is a subbase of τ_Y we have that and arbitrary base element $W \in \tau_Y$ can be written as a finite intersection $W = S_1 \cap \ldots \cap S_n$ of subbase elements. From the elementary properties of functions we have

$$f^{-1}(W) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n).$$

Thus proving that the inverse image of any base element of τ_Y is open set in X. From the observation at the beginning of this proof, we get that f is continuous. ■

The following result is intuitive, easily proved and surprisingly important.

Theorem 1.32. *If X, Y and Z are topological spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is continuous.*

Definition 1.33. Let $f : X \to Y$ be a function and $A \subset X$. The restriction of f to A is a function from A to Y, denoted by $f|_A$, given by $f|_A(a) = f(a)$ for all $a \in A$.

Theorem 1.34. *Let (X, τ) be a topological space, $A \subset X$ be a generic subset endowed with the relative topology τ_A and $f : X \to Y$ is a continuous function. Then $f|_A : A \to Y$ is a continuous function.*

Proof. If V is a open set in Y, then $f|_A^{-1}(V) = A \cap f^{-1}(V)$, and the latter is open in the relative topology on A. ■

Proposition 1.35. *If (A, τ_A) is a subspace of a topological space (X, τ), then the inclusion function $j : A \to X$ is a continuous function.*

Lemma 1.36 (Local Formulation of Continuity). *Let (X, τ_X) and (Y, τ_Y) be general topological spaces. Suppose that $X = \cup_{\alpha \in \Gamma} A_\alpha$, where $A_\alpha \in \tau_X$, for all $\alpha \in \Gamma$. If $f : X \to Y$ is a function such that for all $\alpha \in \Gamma$ we have $f|_{A_\alpha}$ is continuous, then f is continuous.*

Proof. Suppose that A_α is an open set for each $\alpha \in \Gamma$. If V is an open set in (Y, τ_Y), then follows from basic properties of inverse image that

$$f^{-1}(V) = \bigcup_{\alpha \in \Gamma} (f|_{A_\alpha})^{-1}(V).$$

From hypothesis, for each $\alpha \in \Gamma$, we have that $(f|_{A_\alpha})^{-1}(V) \in \tau_{A_\alpha}$, the relative topology. Since A_α is open in X it follows that $\tau_{A_\alpha} \subset \tau$ and so $(f|_{A_\alpha})^{-1}(V) \in \tau$, thus finishing the proof. ■

When we write $f : X \to Y$, we specified the domain of f as being X, but the image of f is not determined, except that it must be contained in Y. The next theorem says essentially, that is not necessary to modify this procedure when dealing with continuous functions.

THEOREM 1.37. *Suppose that* $(Y, \tau_Y) \subset (Z, \tau_Z)$ *and* $f : X \to Y$ *is a function. Then* f *is a continuous map from* X *to* Y *if and only if* $\tilde{f} : X \to Z$ *given by* $\tilde{f}(x) = f(x)$*, for all* $x \in X$*, is a continuous function.*

LEMMA 1.38 (The pasting Lemma). *Let* (X, τ_X) *and* (Y, τ_Y) *be topological spaces and* $A, B \subset X$ *closed sets in* (X, τ) *such that* $A \cup B = X$*. Let* $f : A \to Y$ *and* $g : B \to Y$ *be continuous functions. If* $f(x) = g(x)$ *for all* $x \in A \cap B$*, then* f *and* g *can be used to construct a new continuous function* $h : X \to Y$ *given by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B \setminus A. \end{cases}$$

PROOF. Let C be a closed subset of (Y, τ_Y). From the elementary properties of inverse image we get that

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous, $f^{-1}(C)$ is a closed set in A and, therefore closed in X, see Theorem 1.28 item *iii)*. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Their union $h^{-1}(C)$ is thus closed in X. ■

Summarizing we have discussed the following techniques to construct continuous functions:

- Inclusion (Proposition 1.35).
- Compositions (Theorem 1.32).
- Restricting the domain (Theorem 1.34).
- Restricting or expanding the range (Theorem 1.37).
- Local formulation of Continuity (Lemma 1.36).
- The pasting lemma (Lemma 1.38).

1.6. Homeomorphisms

A function from a topological space (X, τ_X) to a topological space (Y, τ_Y) can make "information" disappear in two ways. The first in set-theoretical sense, which means Y will have fewer (or at least, no more) points than X. The second is topological, meaning that the topological space (Y, τ_Y) will have fewer (or at least, no more) open sets than (X, τ_X), in the sense that each open $V \in Y$ is the image of an open set (for example, $U = f^{-1}(V)$) in X, but there may well be open sets $U \in \tau_X$ such that $f(U)$ is not open in (Y, τ_Y). The maps which preserve X set-theoretically and topologically are called homeomorphisms.

DEFINITION 1.39 (Homeomorphism). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* between X and Y, if f is injective, surjective (or onto) and continuous, and $f^{-1} : Y \to X$ is also continuous.

DEFINITION 1.40 (Embedding). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is called a *embedding* of X into Y, if f is injective and continuous, and $f^{-1} : f(X) \to X$ is also continuous.

Note that X is embedded into Y by f if and only if the function f is a homeomorphism between X and some subspace of Y.

Evidently, a continuous map $f : X \to Y$ is a homeomorphism if and only if there is a continuous map $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are the identity maps on X and Y, respectively.

The reader can easily verify the next theorem as a direct consequence of Theorem 1.30

THEOREM 1.41. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$ a bijection. The following are equivalent:*

i) f is a homeomorphism;
ii) if $U \subset X$, then $f(U)$ is open in Y if and only if U is open in X;
iii) if $A \subset X$, then $f(A)$ is closed in Y if and only if A is closed in X;
iv) if $A \subset X$, then $f(\overline{A}) = \overline{f(A)}$.

Homeomorphic topological spaces are, for the purposes of a topologist, the same. That is, there is nothing about homeomorphic spaces X and Y having to do only with their respective topologies which we can use to distinguish them. If we denote "X is homeomorphic to Y" by $X \sim Y$, then "$\sim$" defines an equivalence relation. Indeed,

i) $X \sim X$;
ii) if $X \sim Y$, then $Y \sim X$;
iii) if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.

In general, to prove that two spaces are homeomorphic, one constructs a homeomorphism. To establish that two spaces are not homeomorphic, one must find a topological property possessed by one and not the other. The definition of "topological property" makes it clear why this work. A *topological property* is a property of topological spaces which, if possessed by X, is possessed by all spaces homeomorphic to X.

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