Homework Assignment 1

Leandro Cioletti
Advanced Topics in Analysis

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Exercise 1. Let (V,τ) be a topological vector space. Prove that if $\mathscr{A} \subset V$ is a convex set, then $\overline{\mathscr{A}}$ (closure of $\mathscr{A}$) is a convex set.

Exercise 2. Let (V,τ) be a topological vector space. Prove that if $\mathscr{A} \subset V$ is a convex set, then $\mathrm{Int}(\mathscr{A})$ (interior of $\mathscr{A}$) is a convex set.

Exercise 3. Let (V,τ) be a topological vector space and $\mathscr{A} \subset V$ a convex set such $\mathrm{Int}(\mathscr{A}) \neq \emptyset$. Under these assumptions show that $\overline{\mathrm{Int}(\mathscr{A})} = \overline{\mathscr{A}}$.

Exercise 4. Prove Jensen's inequality. Let $F : V \to \overline{\mathbb{R}}$ be a convex function. Then for every finite set $u_1, \ldots, u_n \in V$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$, we have

$$F(\lambda_1 u_1 + \ldots + \lambda_n u_n) \leq \lambda_1 F(u_1) + \ldots + \lambda_n F(u_n).$$

Exercise 5. Let $\varphi \in \Gamma_0(\mathbb{R})$ an **even** function and $\varphi^* \in \Gamma_0(\mathbb{R})$ the Legendre-Fenchel transform of φ. Let $(V, \|\cdot\|)$ be a Banach space and consider the functions $F : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ defined by $F(u) = \varphi(\|u\|)$ and $G(u) = \varphi^*(\|u\|_*)$. Prove that $F^* = G$

Exercise 6. Let $\alpha, \alpha^* \in (1,+\infty)$ be such that $1/\alpha + 1/\alpha^* = 1$. Prove that the functions

$$\varphi(t) \equiv \frac{1}{\alpha}|t|^\alpha \qquad \text{and} \qquad \varphi^*(t) = \frac{1}{\alpha^*}|t|^{\alpha^*}$$

belong to $\Gamma_0(V)$ and are conjugate convex functions. Use the previous exercise to conclude that if $(V, \|\cdot\|)$ is a Banach space and $F, G : V \to \mathbb{R}$ functions given by

$$F(u) = \frac{1}{\alpha}\|u\|^\alpha \qquad \text{and} \qquad G(u) = \frac{1}{\alpha^*}\|u^*\|_*^{\alpha^*}.$$

Then F and G are conjugate convex functions.

Exercise 7. Prove the **Fenchel-Young Inequality**. Let (V,τ) be a locally convex Hausdorff space and $F : V \to (-\infty, +\infty]$. Show that for any $u^* \in V^*$ and $u \in \operatorname{dom} F$ we have

$$\langle u, u^* \rangle \leq F(x) + F^*(u^*).$$

Moreover the equality holds if and only if $u^* \in \partial F(u)$.

Exercise 8. Let $p, q \in (1, +\infty)$ and suppose that $1/p + 1/q = 1$. By using the function $F(u) = (1/p)\|u\|^p$ obtain a direct proof of the following inequality

$$\langle u, u^* \rangle \leq \frac{\|u\|^p}{p} + \frac{\|u^*\|^q}{q}.$$

Exercise 9. Prove the following theorem. Let $(V, \|\cdot\|)$ be a Banach space. Suppose that $F : V \to \overline{\mathbb{R}}$ is a lower semicontinuous function. Prove that F is continuous over the interior of its effective domain.

Exercise 10. Prove the **Principle of Uniform Boundedness**. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and for each $i \in I$ an idex set, let $T_i : X \to Y$ be a bounded linear operator. Suppose that $\sup_{i\in I} \|T_i(x)\|_Y < +\infty$ for each $x \in X$. Then $\sup_{i\in I} \|T_i\| < +\infty$, where $\|T_i\|$ denotes de operator norm.

Exercise 11. Show that the Legendre-Fenchel transform of $F : \mathbb{R} \to \mathbb{R}$ given by $F(x) = e^x$ is the function

$$F^*(u^*) = \begin{cases} u^* \log u^* - u^*, & \text{if } u^* > 0; \\ 0, & \text{if } u^* = 0; \\ +\infty, & \text{if } u^* < 0. \end{cases}$$

Exercise 12. Let (V, τ) be a locally convex Hausdorff space and for each $i \in I$, an arbitrary index set, let $F_i : V \to \overline{\mathbb{R}}$ be an arbitrary function. Prove that

$$\big(\inf_{i\in I} F_i\big)^* = \sup_{i\in I} F_i^* \qquad \text{and} \qquad \big(\sup_{i\in I} F_i\big)^* \leq \inf_{i\in I} F_i^*.$$

Exercise 13. Prove the **Extreme Value Theorem**. Let (K, τ) be a Hausdorff compact topological space and $F : K \to \overline{\mathbb{R}}$ be a lower semicontinuous function. Then

$$S \equiv \big\{u \in K : F(u) = \inf_{v\in K} F(v)\big\}$$

is a non-empty closed subset of K.

Exercise 14. Let (V, τ) be a locally convex Hausdorff space and $\mathscr{A}$ be a closed convex subset of V. Suppose that $F : \mathscr{A} \to \overline{\mathbb{R}}$ is a convex, lower semicontinuous, and proper function. Prove that the so-called solution set

$$S = \big\{u \in \mathscr{A} : F(u) = \inf_{v\in \mathscr{A}} F(v)\big\}$$

is a closed and convex set which is possibly empty.

Exercise 15. Let (X, d) be a non-empty compact metric space and $(C(X, \mathbb{R}), \|\cdot\|_\infty)$ be the Banach space of all real-valued continuous functions on X endowed with the norm $\|f\|_\infty \equiv \sup_{x\in X} |f(x)|$. We denote by $\mathscr{B}(X)$ the sigma-algebra on X, generated by the open sets. Let $\mathscr{M}_1(X) \equiv \{\mu : \mathscr{B}(X) \to \mathbb{R} : \mu \text{ is a probability measure}\}$. Use the Riesz-Markov Theorem and Banach-Alaoglu Theorem to prove that $\mathscr{M}_1(X)$ is convex and compact subset, with respect to the weak-$*$-topolgy.

Exercise 16. Let $(V, \|\cdot\|)$ be a Banach space and $F : V \to \overline{\mathbb{R}}$ a convex function. Prove that if F is continuous at u, then $\partial F(u)$ is a non-empty, convex and weak-$*$ compact of V^*. Moreover the relation $v \longmapsto \partial F(v)$ is locally bounded at u, that is, there exist $M > 0$ and a neighborhood $\mathcal{O}$ of $u \in V$ such that $\|u^*\| \leq M$, whenever $v \in \mathcal{O}$ and $u^* \in \partial F(v)$.

Exercise* 17. Let $(V, \|\cdot\|)$ be a separable Banach space and $F : V \to \mathbb{R}$ a continuous convex function. Then the set of points u, where $F'(u)$ (Gâteaux derivative of F at u) exists is a dense G_δ in V.

Exercise 18. For $x = (x_1, x_2, \ldots) \in \ell^\infty(\mathbb{N})$ define a seminorm $F : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ as follows

$$F(x) = \limsup_{n \to \infty} |x_n|.$$

Then F is continuous, but nowhere Gâteaux differentiable. Compare this example with the previous exercise.

Exercise 19. Consider the Hilbert space $\ell^2(\mathbb{N})$ endowed with its natural norm.
(i) Prove that $\mathscr{C} \equiv \{x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N}) : |x_n| \leq 2^{-n}\}$ is compact and convex subset. Consider the function

$$F(x) = \begin{cases} -\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + x_n\right)^{\frac{1}{2}}, & \text{if } x \in \mathscr{C}; \\ +\infty, & \text{otherwise.} \end{cases}$$

(ii) Prove that F is continuous and convex.
(iii) Prove that $\partial F(x) = \emptyset$ for any $x \in \mathscr{C}$ such that $x_n > -2^{-n}$ for infinitely many n.
(iv) Conclude that F is lower semicontinuous, but not continuous at any point of $\mathscr{C}$.

Exercise 20. Consider the function $\phi : [0, +\infty) \to \mathbb{R}$ given by

$$\phi(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{2}; \\ 2t - 1, & \text{if } \frac{1}{2} \leq t. \end{cases}$$

From ϕ we define a function $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \phi\Big(\sum_{k \geq n} x_k^2\Big).$$

Prove that F is continuous and convex on $\ell^2(\mathbb{N})$ but it is not bounded on the unit ball.