

ON THE DIMENSION OF THE SPACE OF HARMONIC FUNCTIONS ON TRANSITIVE SHIFT SPACES

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ABSTRACT. In this paper, we show a new relation between phase transition in Statistical Mechanics and the dimension of the space of harmonic functions (SHF) for a transfer operator. This is accomplished by extending the classical Ruelle-Perron-Frobenius theory to the realm of low regular potentials defined on either finite or infinite (uncountable) alphabets. We also give an example of a potential having a phase transition where the Perron-Frobenius eigenvector space has dimension two. We discuss entropy and equilibrium states, in this general setting, and show that if the SHF is non-trivial, then the associated equilibrium states have full support. We also obtain a weak invariance principle in cases where the spectral gap property is absent. As a consequence, a functional central limit theorem for non-local observables of the Dyson model is obtained.

1. INTRODUCTION

This paper is about the Perron-Frobenius eigenvector space of a class of transfer operators arising in Ergodic Theory and Equilibrium Statistical Mechanics. For this class of operators, defined by a general continuous potential, our main result relates the dimension of this subspace with the number of extreme conformal measures. Consequently, we show that the geometric multiplicity of this eigenvector space can only be greater than one if a first-order phase transition takes place, as in Dobrushin [10] and Lanford and Ruelle [23].

Here we are mostly interested in the case where the potential is continuous but does not have the usual regularity properties. In such cases, little is known about the Perron-Frobenius eigenvector space. The quest for a theory that can handle such a general class of potentials is the primary motivation of the present article. Besides, considering low regular potentials, we also allow the phase space to be any product space of the form $X = E^{\mathbb{N}}$, where E is a compact metric space. Therefore the Perron-Frobenius operator considered here leads to more general results than in the classical case. These generalizations bring powerful ideas of Thermodynamic Formalism to the study of decay of correlations, invariance principles and phase transitions of long-range continuous spins systems on the one-dimensional lattice. Our results apply, for example, to the $O(n)$ -models (including Ising, XY and Heisenberg models), with long-range interactions which are formally defined in the following way.

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Let $\mathbb{K}_n = (\mathbb{V}_n, \mathbb{E}_n)$ be the complete graph, where the set of vertices is $\mathbb{V}_n \equiv \{1, \dots, n\}$. A configuration of the spin $O(n)$ -model is an element of the product space $(\mathbb{S}^{n-1})^{\mathbb{V}_n}$, where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the $(n-1)$ -dimensional unit sphere. The Hamiltonian of the model is defined by $\mathcal{H}_{\mathbb{V}_n}(x) = \sum_{i,j \in \mathbb{E}_n} J(|i-j|) \langle x_i, x_j \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n , and $J(|i-j|)$ are real constants, which depend only on the distance between i and j . At the inverse temperature β , we define the finite-volume Gibbs measures $\mu_{\mathbb{V}_n, \beta}$ to be the probability measure on $(\mathbb{S}^{n-1})^{\mathbb{V}_n}$ given by

$$d\mu_{\mathbb{V}_n, \beta} = \frac{1}{Z_{\mathbb{V}_n, \beta}} \exp(-\beta \mathcal{H}_{\mathbb{V}_n}) \prod_{i \in \mathbb{V}_n} dp,$$

where $Z_{\mathbb{V}_n, \beta}$ is the usual partition function and dp is the uniform distribution on \mathbb{S}^{n-1} . By taking the weak limit of these measures, when $n \rightarrow \infty$, an infinite-volume measure μ_β can be defined, and one may ask whether phase transition occurs at some critical temperature, what is the decay ratio of the correlations and so on. In Subsection 3.3 we show how to describe the $O(n)$ -models in our generalized Thermodynamic Formalism setting, and consequently how to apply to these statistical mechanics models the results obtained here.

Before proceeding, we fix some notations and then state our main results. Let (E, d_E) be a compact metric space, $X = E^{\mathbb{N}}$ the product space equipped with a metric d inducing the product topology, and $\mathcal{B}(X)$ the Borel sigma-algebra on X . As usual, $(C(X), \|\cdot\|_\infty)$ denotes the Banach space of all real-valued continuous functions on X , endowed with the supremum norm. For an arbitrary metric space Y let us denote by $\mathcal{M}_s(Y)$, $\mathcal{M}(Y)$ and $\mathcal{M}_1(Y)$ the spaces of all finite Borel signed, positive and probability measures on Y , respectively. The support of $\nu \in \mathcal{M}(Y)$ is the closed set $\text{supp}(\nu) \equiv Y \setminus \bigcup \{A : A \text{ is open and } \nu(A) = 0\}$. The class of transfer operators considered here are associated with the dynamics given by the left shift-map $\sigma : X \rightarrow X$, defined by $\sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$, a continuous potential $f : X \rightarrow \mathbb{R}$, and an *a priori* Borel probability measure $p \in \mathcal{M}_1(E)$. The transfer operator associated with a continuous potential f will be denoted by $\mathcal{L} : C(X) \rightarrow C(X)$. It sends $\varphi \mapsto \mathcal{L}\varphi$, which is defined for any $x \in X$, as follows:

$$\mathcal{L}\varphi(x) \equiv \int_E \exp(f(ax)) \varphi(ax) dp(a), \quad \text{where } ax \equiv (a, x_1, x_2, \dots). \quad (1)$$

Since X is compact \mathcal{L}^* acts on $\mathcal{M}_s(X)$. Furthermore, denoting by $\rho(\mathcal{L})$ the spectral radius of \mathcal{L} it follows from classical arguments, that

$$\mathcal{G}^* \equiv \{\nu \in \mathcal{M}_1(X) : \mathcal{L}^* \nu = \rho(\mathcal{L}) \nu\}$$

is always a non-empty convex and compact set in the weak- $*$ -topology (see [5]). This set is sometimes called the set of leading eigenmeasures, but here following Denker and Urbanski [9] a probability measure $\nu \in \mathcal{G}^*$ will be called a $\rho(\mathcal{L})$ -conformal measure. In the particular case where $\rho(\mathcal{L}) = 1$, we call ν simply a conformal measure. The set of extreme points of \mathcal{G}^* is denoted by $\text{ex}(\mathcal{G}^*)$.

The existence of leading eigenfunctions is a much harder problem. Despite the efforts of many authors we still do not know, in general, when the set of leading eigenfunctions $\mathcal{H} \equiv \{h \in C(X) : \mathcal{L}h = \rho(\mathcal{L})h\}$ is non-trivial. If f has some regularity properties, such as α -Hölder continuity, then the Ruelle-Perron-Frobenius theorem ensures that \mathcal{H} is non-trivial. On the other hand, for some continuous but

low regular potentials, \mathcal{H} might be trivial, see [4, 17]. The existence of such counterexamples leads us to reformulate this problem for some suitable extensions of the transfer operator (1) defined on $L^1(\nu) \equiv L^1(X, \mathcal{B}(X), \nu)$, where $\nu \in \mathcal{G}^*$.

As we will see if p is fully supported on E , then for any continuous potential f , we have that any $\rho(\mathcal{L})$ -conformal measure ν is fully supported on X . Consequently, there is a continuous linear embedding from $C(X)$ to $L^1(\nu)$. Moreover, for each $\nu \in \mathcal{G}^*$ the transfer operator \mathcal{L} admits a bounded linear extension $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$ such that $\rho(\mathcal{L}) = \rho(\mathbb{L})$. By replacing f by $f - \log \rho(\mathcal{L})$, we can assume without loss of generality that $\rho(\mathbb{L}) = 1$. In this way the operator \mathbb{L} defines a Markov process. The eigenspace $\mathbb{H} \equiv \{h \in L^1(\nu) : \mathbb{L}h = h\}$ is called the space of harmonic functions (SHF) for \mathbb{L} . Although we are using a shorter notation, the reader should have in mind that $\mathbb{H} \equiv \mathbb{H}(f, p, \nu)$. Our first main result is the following

Theorem 1.1. *Let f be an arbitrary continuous potential, $p \in \mathcal{M}_1(E)$ such that $\text{supp}(p) = E$, and $m \in \mathcal{G}^*$ a conformal measure. Then $0 \leq \dim(\mathbb{H}) \leq \#\text{ex}(\mathcal{G}^*)$.*

The proof of Theorem 1.1 reveals that SHF's dimension is maximized when the conformal measure m is taken as the barycenter of \mathcal{G}^* , that is, for all $\nu \in \mathcal{G}^*$ we have $\dim(\mathbb{H}(f, p, \nu)) \leq \dim(\mathbb{H}(f, p, m))$. We also remark that the result of Theorem 1.1 is optimal in the sense that there are continuous potentials f 's for which the SHF's are trivial, and cases where the upper bound is saturated. Multidimensional SHF's for a transfer operator have appeared before: on functional equations related to ergodic theory and Markov chains [8]; on hyperbolic maps with metastable states [11, 14]; on multiresolution wavelet theory [20, 25]; and on the Ruelle-Perron-Frobenius Theorem in the context of non-forward topologically transitive subshifts of finite type [1, 21]. However, the mechanism behind the multiplicity of the SHF's here is different from all the ones described above because our dynamics is forward topologically transitive.

The other two main results are the functional central limit theorem for non-local observables of the Dyson model (Example 4.4), and the following theorem which provides an example of a potential for which the associated SHF is two-dimensional.

Theorem 1.2. *Let $E = \{-1, 1\}$, p the counting measure on E , $\beta > 0$, $f : X \rightarrow \mathbb{R}$ given by $f(x) \equiv x_1 \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=2}^{N+1} x_k$, μ_γ the Bernoulli measure defined by $\mu_\gamma(\{x_k = +1\}) = q$, and $\mu_\gamma(\{x_k = -1\}) = 1 - q$, where $\gamma = 2q - 1$; and $\gamma(\beta), -\gamma(\beta)$ solutions of the equation $\gamma = \tanh(\beta\gamma)$, then the following hold:*

- (1) for $0 < \beta \leq 1$:
 - (a) the operator $\mathbb{L}_{\beta f} : L^1(\mu_0) \rightarrow L^1(\mu_0)$ has norm $\|\mathbb{L}_{\beta f}\| = 2$ and the symmetric Bernoulli measure μ_0 is a generalized conformal measure, associated to βf , in the sense of Definition 3.1;
 - (b) the eigenspace of $\mathbb{L}_{\beta f}$, associated to eigenvalue 2, has dimension one and is spanned by $\mathbb{1}$.
- (2) for $\beta > 1$:
 - (a) the operator $\mathbb{L}_{\beta f} : L^1(\nu) \rightarrow L^1(\nu)$, where $\nu = t\mu_{\gamma(\beta)} + (t-1)\mu_{-\gamma(\beta)}$ and $t \in (0, 1)$, has operator norm $\|\mathbb{L}_{\beta f}\| = 2 \cosh(\beta\gamma(\beta)) > 2$ and ν is a generalized conformal measure associated to βf ;
 - (b) for any non-trivial convex combination $\nu = t\mu_{\gamma(\beta)} + (t-1)\mu_{-\gamma(\beta)}$, the eigenspace of $\mathbb{L}_{\beta f}$ is two-dimensional and is spanned by $\{\mathbb{1}_{B_+}, \mathbb{1}_{B_-}\}$, where $B_\pm = \{x \in X : m(x) = \pm\gamma(\beta)\}$ and m is given by (11).

The paper is organized as follows. In Section 2, we begin by discussing some of the general properties of the eigenmeasures associated with the transfer operator (1). After, we show how to use the theory of Markov process to establish upper and lower bounds for the dimension of the SHF. In Section 3, we present an example of a potential such that the associated SHF has dimension exactly two. In Section 4, we obtain a weak invariance principle in case where the spectral gap property is absent. In the appendix, we show that $\text{spec}(\mathbb{L}) = \overline{B(0, \rho(\mathcal{L}))}$.

2. CONFORMAL MEASURES, MARKOV PROCESSES, INVARIANT SETS AND HARMONIC FUNCTIONS

This section is divided into subsections organized as the section's title.

2.1. Conformal Measures. In this subsection, we prove several facts about the $\rho(\mathcal{L})$ -conformal measures (or leading eigenmeasures). Some of them are generalizations to our setting of classical results that go back at least to Sullivan and Patterson.

Theorem 2.1. *Let $p \in \mathcal{M}_1(E)$ be an a priori measure such that $\text{supp}(p) = E$. Then for any $f \in C(X)$ and $\nu \in \mathcal{G}^*$ we have that $\text{supp}(\nu) = X$.*

Proof. It is enough to show that $\nu(B(x, r)) > 0$, where $B(x, r)$ is an open ball in X centered in x with radius r . Since d induces the product topology there are $n \in \mathbb{N}$ and $R \in \mathbb{R}$ such that $B(x, r) \supseteq B_E(x_1, R) \times \dots \times B_E(x_n, R) \times E^{\mathbb{N}} \equiv B(R)$.

For each fixed $a \in E$ consider the auxiliary continuous function $\psi_a : E \rightarrow [0, 1]$ given by $\psi_a(b) = \max\{1 - (2/R)d_E(b, B_E(a, R/2)), 0\}$. Notice that, $\mathbf{1}_{B_E(a, \frac{R}{2})} \leq \psi_a \leq \mathbf{1}_{B_E(a, R)}$. Therefore $\mathbf{1}_{B(R/2)}(y) \leq \prod_{k=1}^n \psi_{x_k}(y_k) \leq \mathbf{1}_{B(R)}(y)$, for all $y \in X$.

By using elementary properties of the transfer operator we get

$$\begin{aligned} \nu(B(x, r)) &\geq \frac{1}{\rho^n(\mathcal{L})} \int_X \int_{E^n} \exp \left[\sum_{k=1}^n f(a_k \dots a_n y) \right] \prod_{k=1}^n \psi_{x_k}(a_k) dp^n(a) d\nu(y) \\ &\geq \left(\frac{\min_{x \in X} \exp(f(x))}{\rho(\mathcal{L})} \right)^n \prod_{k=1}^n p(B_E(x_k, R/2)) > 0, \end{aligned}$$

where the existence of a positive minimum follows from the compactness of X and the continuity of f , and $p(B_E(x_k, R/2)) > 0$ by hypothesis. \blacksquare

If p and ν are as in Theorem 2.1, then $\text{supp}(\nu) = X$ and so there is a linear continuous embedding $q : (C(X), \|\cdot\|_\infty) \hookrightarrow (L^1(\nu), \|\cdot\|_1)$.

Since q is an embedding it follows that the map $\mathbb{L} : q(C(X)) \rightarrow L^1(\nu)$ given by $\mathbb{L}[\varphi] \equiv [\mathcal{L}\varphi]$, where $[\varphi]$ means the ν -equivalence class of φ , is a well-defined linear map. By using that $\nu \in \mathcal{G}^*$ we get $\|\mathcal{L}\varphi - \mathcal{L}\psi\|_1 \leq \rho(\mathcal{L})\|\varphi - \psi\|_1$, for any pair $\varphi, \psi \in C(X)$. Since the embedded copy of $C(X)$ in $L^1(\nu)$ is a dense subset it follows that \mathbb{L} can be uniquely extended to a bounded positive linear operator on $L^1(\nu)$. By abusing notation, we will say that \mathbb{L} is an extension of \mathcal{L} . Using again that $\nu \in \mathcal{G}^*$ we get $\mathbb{L}^*\mathbf{1} = \rho(\mathcal{L})\mathbf{1}$ ν -a.e., where $\mathbb{L}^* : L^\infty(\nu) \rightarrow L^\infty(\nu)$ is the adjoint of \mathbb{L} . Furthermore, $\|\mathbb{L}\|_{\text{op}} \equiv \sup\{\langle \mathbf{1}, |\mathbb{L}\varphi| \rangle_\nu : \|\varphi\|_1 = 1\} \leq \sup\{\langle \mathbb{L}^*\mathbf{1}, |\varphi| \rangle_\nu : \|\varphi\|_1 = 1\} = \rho(\mathcal{L})$. By taking $\varphi = \mathbf{1}$, we can see from the last inequality that the first supremum is attained and therefore $\|\mathbb{L}\|_{\text{op}} = \rho(\mathcal{L})$. From the Gelfand's formula for the spectral radius and the previous reasoning we can also conclude that $\rho(\mathbb{L}) = \rho(\mathcal{L})$.

In what follows, we show that $\sigma : X \rightarrow X$ is a non-singular endomorphism on $(X, \mathcal{B}(X), \nu)$, where ν is an arbitrary $\rho(\mathcal{L})$ -conformal measure. Recall that the map $\sigma : X \rightarrow X$ is a non-singular endomorphism (or equivalently ν is a (backward) quasi-invariant measure with respect to σ) on $(X, \mathcal{B}(X), \nu)$ if for every $C \in \mathcal{B}(X)$ we have that $\nu(\sigma^{-1}(C)) = 0$ if and only if $\nu(C) = 0$.

Proposition 2.2. *Let f be a continuous potential, $\nu \in \mathcal{G}^*$ a conformal measure and p the a priori measure used to define \mathcal{L} . Then the conformal measure ν is quasi-invariant.*

Proof. It is enough to prove that there are positive constant K_1 and K_2 such that the following inequality holds:

$$K_1(p \times \nu)(B) \leq \nu(B) \leq K_2(p \times \nu)(B), \quad \forall B \in \mathcal{B}(X). \quad (2)$$

We first show its validity for the family of rectangles $\mathcal{R} = \{U \times V : U \subseteq E \text{ and } V \subseteq X \text{ are open sets}\}$.

Let $B \in \mathcal{R}$ of the form $B = U \times V$. Since U is open in E , there is an increasing sequence of continuous functions $\psi_n : E \rightarrow [0, 1]$ such that $\psi_n \uparrow \mathbf{1}_U$ pointwisely and, therefore, in $L^1(p)$. Similarly, there is an increasing sequence of continuous functions $\phi_n : X \rightarrow [0, 1]$ (Urysohn functions) such that $\phi_n \uparrow \mathbf{1}_V$, pointwisely and in $L^1(\nu)$. Therefore for any $x \in X$ we have that $\Psi_n(x) \equiv \psi_n(x_1)\phi_n(\sigma(x)) \uparrow \mathbf{1}_B(x)$. Clearly $\Psi_n \in C(X)$ and we have

$$\nu(B) = \int_X \mathbf{1}_B d\nu \geq \int_X \Psi_n d\nu \geq \frac{e^{-\|f\|_\infty}}{\rho(\mathcal{L})} \int_E \psi_n(a) dp(a) \int_X \phi_n(x) d\nu(x).$$

By taking the limit when $n \rightarrow \infty$, one can conclude that $\nu(B) \geq e^{-\|f\|_\infty} \rho(\mathcal{L})^{-1} p(U)\nu(V) = e^{-\|f\|_\infty} \rho(\mathcal{L})^{-1} (p \times \nu)(B)$. This shows that the first inequality in (2) holds for any open rectangle with $K_1 = e^{-\|f\|_\infty} \rho(\mathcal{L})^{-1}$.

In order to prove the second inequality in (2), we consider a product of a closed rectangles of the form $W = C \times D$. By using a similar reasoning, we can consider a decreasing sequence of Urysohn functions, and obtain the inequality $\nu(W) \leq e^{\|f\|_\infty} \rho(\mathcal{L})^{-1} (p \times \nu)(W)$. By taking $K_2 = \rho(\mathcal{L})^{-1} e^{\|f\|_\infty}$ and recalling that any Borel measure on a metric space is regular, it follows that (2) holds for every Borel set. \blacksquare

As a consequence of the above proposition, we have that the Koopman operator $L^\infty(\nu) \ni \varphi \mapsto \varphi \circ \sigma \in L^\infty(\nu)$ is well-defined. Moreover, for any $\psi \in L^\infty(\nu)$ the following identity holds $\mathbb{L}^* \psi = \psi \circ \sigma$ ν -a.e.. This will be used in the appendix.

2.2. Markov Processes and Invariant Sets. In this section, we use the theory of Markov processes to study the properties of the operator \mathbb{L} . In doing so, we identify a close relation among the extreme conformal measures, the invariant sets with respect to the process, and non-negative harmonic functions that form a basis for the SHF.

From now on, we assume f is an arbitrary continuous potential, $\text{supp}(p) = E$, and $\rho(\mathcal{L}) = 1$. Therefore the extension $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$, for any $\nu \in \mathcal{G}^*$, defines a Markov process in the analytical sense [18], which means that it is a positive contraction on $L^1(\nu)$, more precisely.

Definition 2.3 (Markov Processes). A Markov process is defined as an ordered quadruple (X, \mathcal{F}, μ, T) , where the triple (X, \mathcal{F}, μ) is a sigma-finite measure space

with a positive measure μ and T is a bounded linear operator acting on $L^1(\mu)$ satisfying:

- (1) T is a contraction: $\sup\{\|T\varphi\|_1 : \|\varphi\|_1 \leq 1\} \equiv \|T\|_{\text{op}} \leq 1$;
- (2) T is a positive operator, that is, if $\varphi \geq 0$, then $T\varphi \geq 0$.

Here, the sigma-algebra \mathcal{F} will be the Borel sigma-algebra $\mathcal{B}(X)$, T is the extension $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$ of the transfer operator \mathcal{L} , and $\mu = \nu$. Condition (2), the positivity property of \mathbb{L} , is inherited from \mathcal{L} , and the condition (1) follows from $\|\mathbb{L}\|_{\text{op}} = \rho(\mathcal{L})$, and the assumption $\rho(\mathcal{L}) = 1$.

Observe that $\mathbb{L}^*\mathbb{1} = \mathbb{1}$ and therefore $\mathbb{1}$ is always an eigenfunction of \mathbb{L}^* . On the other hand, the existence of a positive or non-negative eigenfunction for \mathbb{L} itself is a much more delicate issue.

In the general theory of Markov processes, a measurable set $B \in \mathcal{B}(X)$ satisfying $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$ is sometimes called an invariant set for the process. Next, we provide a short list of important properties of such invariant sets that will be used ahead.

Proposition 2.4. *If $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$, for some $B \in \mathcal{B}(X)$ and ν is a conformal measure, then the Borel measure ν_B given by $A \mapsto \nu(A \cap B)$ is an eigenmeasure for \mathcal{L}^* . Moreover, if $\nu(B) \neq 0$ then the conditional measure $A \mapsto \nu(A \cap B)/\nu(B) \equiv \nu(A|B)$ is an element of \mathcal{G}^* .*

Proof. By using that $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$ and that ν is conformal we have for any continuous function φ that $\int_X \mathcal{L}\varphi d\nu_B = \langle \mathbb{1}_B, \varphi \rangle_\nu = \int_X \varphi d\nu_B$. If $\nu(B) \neq 0$ we have immediately that $\nu(\cdot \cap B)/\nu(B) = \nu(\cdot|B)$ is a conformal measure. \blacksquare

Proposition 2.5 (Invariant sets are dense in X). *If $B \in \mathcal{B}(X)$ is an invariant set, in the sense that $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$, then B is dense in X .*

Proof. Since we are assuming B is an invariant set with respect to \mathbb{L} it follows from Proposition 2.4 that $\nu(\cdot|B)$ is also a conformal measure. Fix a point $x \in X$. By applying Theorem 2.1 to $\nu(\cdot|B)$ we can conclude that, for every $r > 0$, the conditional probability $\nu(B(x, r)|B) > 0$. Therefore, at least one point of B is in $B(x, r)$, otherwise $\nu(B(x, r)|B) = 0$. Since this holds for every $x \in X$ and $r > 0$, it follows that any invariant set B is dense in X . \blacksquare

Lemma 2.6. *If $\nu \in \mathcal{G}^*$ is an extreme point and B is an invariant set for the Markov process induced by \mathbb{L} , then $\nu(B) = 0$ or 1.*

Proof. The proof of this Lemma is an straightforward consequence of Proposition 2.4. \blacksquare

2.3. Harmonic Functions. In this subsection, we present some properties of harmonic functions needed in our applications.

Proposition 2.7. *Let $\nu \in \mathcal{G}^*$ and h a harmonic function for \mathbb{L} . Then the set $\{h > 0\}$ is an invariant set. Moreover, if $\nu \in \text{ex}(\mathcal{G}^*)$ and $\nu(\{h > 0\}) > 0$ then h is positive ν -almost everywhere.*

Proof. We first prove that if h is a harmonic function then $\mathbb{L}h^\pm = h^\pm$. For, let $B = \{h > 0\}$. By using the linearity of \mathbb{L} and $\mathbb{L}h = h$, we get

$$\mathbb{L}(h^+) = (h^+ + \mathbb{L}h^-)\mathbb{1}_B + (\mathbb{L}h^- - h^-)\mathbb{1}_{B^c}. \quad (3)$$

By multiplying both sides of (3) by $\mathbb{1}_{B^c}$, we get from the positivity of \mathbb{L} that $0 \leq \mathbb{1}_{B^c}\mathbb{L}(h^+) = (\mathbb{L}h^- - h^-)\mathbb{1}_{B^c}$. Therefore $0 \leq \int_X (\mathbb{L}h^- - h^-)\mathbb{1}_{B^c} d\nu = \int_X \mathbb{1}_{B^c}\mathbb{L}h^- -$

$h^- d\nu \leq \|\mathbb{L}h^-\|_1 - \|h^-\|_1 \leq 0$, where in the last inequality we used the contraction property of \mathbb{L} . This shows that $(\mathbb{L}h^- - h^-)\mathbb{1}_B = 0$ ν -a.e.. Replacing this in (3) we get the equality $\mathbb{L}h^+ = (h^+ + \mathbb{L}h^-)\mathbb{1}_B$. Therefore $\|\mathbb{L}h^+\|_1 = \|h^+\|_1 + \|\mathbb{L}h^-\mathbb{1}_B\|_1$. Applying again the contraction property we have that $\|\mathbb{L}h^-\mathbb{1}_B\|_1 = 0$. This implies $\mathbb{L}h^-\mathbb{1}_B = 0$ ν -a.e.. Finally, from the identity $\mathbb{L}h^+ = (h^+ + \mathbb{L}h^-)\mathbb{1}_B$, it follows that $\mathbb{L}h^+ = \mathbb{1}_B h^+ = h^+$. Consequently, $\mathbb{L}h^- = h^-$.

Now we prove that B is an invariant set. Indeed,

$$\langle \mathbb{L}^*\mathbb{1}_B, h^+ \rangle_\nu = \langle \mathbb{1}_B, \mathbb{L}h^+ \rangle_\nu = \langle \mathbb{1}_B, h^+ \rangle_\nu. \quad (4)$$

Note that $\mathbb{L}^*\mathbb{1}_B \leq \mathbb{1}$, because the adjoint of a positive contraction is also a positive contraction. Since $0 \leq \mathbb{L}^*\mathbb{1}_B \leq \mathbb{1}$, it follows from (4) that $\mathbb{1}_B \mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$. From these observations, we get that $\mathbb{L}^*\mathbb{1}_B \geq \mathbb{1}_B$, since \mathbb{L}^* is positive. Therefore $\|\mathbb{L}^*\mathbb{1}_B\|_1 \geq \|\mathbb{1}_B\|_1$. As we already mentioned, \mathbb{L}^* is a contraction with respect to the $L^\infty(\nu)$ -norm. Moreover, the operator \mathbb{L}^* acts as a contraction, with respect to the $L^1(\nu)$ -norm, on the linear manifold spanned by the characteristic functions. Indeed, $\|\mathbb{L}^*\mathbb{1}_B\|_1 = \langle \mathbb{L}^*\mathbb{1}_B, \mathbb{1} \rangle_\nu = \langle \mathbb{1}_B, \mathbb{L}\mathbb{1} \rangle_\nu \leq \langle \mathbb{1}_B, \mathbb{1} \rangle_\nu = \|\mathbb{1}_B\|_1$.

Since $0 \leq \mathbb{1}_B \leq \mathbb{L}^*\mathbb{1}_B$ and $\|\mathbb{L}^*\mathbb{1}_B\|_1 = \|\mathbb{1}_B\|_1$, it follows that the equation $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$ holds.

If $\nu \in \text{ex}(\mathcal{G}^*)$, then follows from the invariance $\{h > 0\}$ and Lemma 2.6 that $\nu(\{h > 0\}) = 0$ or 1. Therefore if $\nu(\{h > 0\}) > 0$, then h has to be positive ν -almost everywhere. \blacksquare

Theorem 2.8. *If ν is an extreme point in \mathcal{G}^* then $\dim(\mathbb{H}) = 0$ or 1.*

Proof. Suppose that there are linearly independent harmonic functions u and v for \mathbb{L} . By Proposition 2.7, we can assume that $u > 0$ and $v > 0$ a.e.. For an arbitrary function u denote $\int_X u d\nu \equiv \nu(u)$ and consider $w \equiv u - (\nu(u)/\nu(v))v$. It follows that w is also a harmonic function and that $\nu(w) = 0$. By applying again Proposition 2.7 we conclude that w must be identically zero. \blacksquare

By using Liverani-Saussol-Vaianti maps, we can construct an example where $\dim(\mathbb{H}) = 0$. First, we consider $Y = [0, 1]$ and the map $T_\alpha : Y \rightarrow Y$ given by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & \text{if } x \in [0, \frac{1}{2}); \\ 2x - 1, & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \quad (5)$$

where $\alpha \geq 2$. Let $f_\alpha : Y \rightarrow \mathbb{R}$ be the geometric potential given by $f_\alpha(x) = -\log |T'_\alpha(x)|$ and $\mathcal{L} : C(Y) \rightarrow C(Y)$ given by

$$\mathcal{L}\phi(x) = \sum_{y \in T_\alpha^{-1}(x)} \frac{1}{|T'_\alpha(y)|} \phi(y). \quad (6)$$

It is clear that $\text{Leb} \in \mathcal{G}^*$ and it is well-known that there is no Lebesgue-integrable leading eigenfunction for \mathbb{L} . Finally, by considering a suitable Markov partition we can restate this result within our setting with $X = \{0, 1\}^{\mathbb{N}}$ and p the normalized counting measure.

Corollary 2.9. *Let f be any continuous potential and $\mathcal{L} : C(X) \rightarrow C(X)$ be a transfer operator constructed from this potential and a fully supported a priori probability measure p on E . Then $\dim(\mathcal{H}) = 0$ or 1.*

Proof. We can assume that $\rho(\mathcal{L}) = 1$. Since X is a compact metric space we have that \mathcal{G}^* is convex, compact and necessarily not empty. Take any $\nu \in \text{ex}(\mathcal{G}^*)$. Suppose that u and v are linearly independent continuous harmonic functions for \mathcal{L} . From Theorem 2.1 we get that ν is fully supported. From this fact we conclude that $[u]_\nu \neq [v]_\nu$. Furthermore, they are linearly independent in $L^1(\nu)$ and also harmonic functions of \mathbb{L} , contradicting Theorem 2.8. \blacksquare

We finish this section by showing the one-to-one correspondence between extreme conformal measures and the atoms of the sigma-algebra of the invariant sets with respect to \mathbb{L} and its main consequence, which is the proof of Theorem 1.1.

Lemma 2.10. *Let $\nu, \mu \in \text{ex}(\mathcal{G}^*)$ be distinct measures and m a non-trivial convex combination of them, $m = t\nu + (1-t)\mu$. Then there is a $\mathcal{B}(X)$ -measurable set, B , for which $\nu(B) = 1$, $\mu(B^c) = 1$ and the following equations hold*

$$\mathbb{L}^* \mathbf{1}_B = \mathbf{1}_B \quad \text{and} \quad \mathbb{L}^* \mathbf{1}_{B^c} = \mathbf{1}_{B^c}, \quad (7)$$

where \mathbb{L} is the extension of \mathcal{L} to $L^1(m)$. Moreover B is unique up to a m -null set.

Proof. In the appendix of reference [5] the authors adapted Theorem 7.7 item (c) of [13] to our setting. This result says that any extreme point in \mathcal{G}^* is uniquely determined by the values it takes on the elements of the tail sigma-algebra \mathcal{T} . Since ν and μ are distinct and determined by the values taken on \mathcal{T} , there is at least one element $B \in \mathcal{T}$ such that $\mu(B) \neq \nu(B)$. But from Corollary 10.5 in [5] we also know that any extreme point in \mathcal{G}^* is trivial on \mathcal{T} , meaning that, for every $B \in \mathcal{T}$, $\nu(B) = 0$ or $\nu(B) = 1$. Then, supposing that $\mu(B) = 0$, we have that $\nu(B) = 1$. We now have two disjoint sets, B and B^c with $\nu(B) = 1$ and $\mu(B^c) = 1$.

Following the computations of Proposition 2.4, we will show that $\mathbb{L}^* \mathbf{1}_B = \mathbf{1}_B$. Actually, it is enough to prove that $\langle \mathbb{L}^* \mathbf{1}_B, \varphi \rangle_m = \langle \mathbf{1}_B, \varphi \rangle_m$ for every $\varphi \in C(X, \mathbb{R})$. Indeed, for an arbitrary continuous function φ we have

$$\begin{aligned} \langle \mathbb{L}^* \mathbf{1}_B, \varphi \rangle_m &= \int_X \mathbf{1}_B \mathcal{L} \varphi \, dm = \int_X \mathcal{L} \varphi \, dm(\cdot \cap B) = \int_X \mathcal{L} \varphi \, d(t\nu) = \int_X \varphi \, d(t\mathcal{L}^* \nu) \\ &= \int_X \varphi \, d(t\nu) = \int_X \mathbf{1}_B \varphi \, dm = \langle \mathbf{1}_B, \varphi \rangle_m. \end{aligned}$$

The third equality above holds because $m(\cdot \cap B) = t\nu$. Again by using the density of $C(X)$ in $L^1(m)$, we conclude that $\mathbb{L}^* \mathbf{1}_B = \mathbf{1}_B$. Recalling that $\mathbb{L}^* \mathbf{1} = \mathbf{1}$ we get from previous identity that $\mathbb{L}^* \mathbf{1}_{B^c} = \mathbf{1}_{B^c}$. Uniqueness of B can be derived by standard measure theoretical arguments. \blacksquare

Remark 2.11. In general, when $m = \sum_{i=1}^n t_i \nu_i$, is a convex combination of distinct extreme conformal measures, there are disjoint invariant sets B_j 's such that $\nu_j(B_i) = \delta_{ij}$.

Proof of Theorem 1.1. The arguments in this proof involve, simultaneously, different extensions of transfer operator $\mathcal{L} : C(X) \rightarrow C(X)$. To avoid confusions these extensions will be indexed by the conformal measure as in notation \mathbb{L}_ν . For each conformal measure $m, \nu, \mu \in \mathcal{G}^*$ we have that the extensions $\mathbb{L}_m, \mathbb{L}_\nu$ and \mathbb{L}_μ , define themselves Markov processes.

If $\#\text{ex}(\mathcal{G}^*) = +\infty$ we are done. Thus in what follows we assume that the cardinality of the set of extreme points of \mathcal{G}^* is finite.

In case $\text{ex}(\mathcal{G}^*)$ is a singleton we already know that $\dim(\mathbb{H}) \leq 1$. For simplicity we will present the argument in case $\#\text{ex}(\mathcal{G}^*) = 2$.

Let \mathbb{L}_m be the extension of \mathcal{L} to $L^1(m)$, where $m = t\nu + (1-t)\mu$. As we observed before there is a unique (modulo- m) set $B \in \mathcal{B}(X)$ such that $\nu(B) = 1$, $\mu(B^c) = 1$, $\mathbb{L}_m^* \mathbb{1}_B = \mathbb{1}_B$ and $\mathbb{L}_m^* \mathbb{1}_{B^c} = \mathbb{1}_{B^c}$.

Note that one of the following three possibilities occur:

- i*) the eigenvalue problem $\mathbb{L}_m[u]_m = [u]_m$ has only the trivial solution, i.e., $[u]_m = 0$;
- ii*) any harmonic function $[u]_m$ for \mathbb{L}_m is such that $[\mathbb{1}_B u]_m \neq 0$, but $[\mathbb{1}_{B^c} u]_m = 0$ and vice-versa;
- iii*) there is a harmonic function $[u]_m$ such that both $[\mathbb{1}_B u]_m \neq 0$, and $[\mathbb{1}_{B^c} u]_m \neq 0$.

Of course, in the first case the dimension of \mathbb{H} is zero and the theorem is proved. We will show next that in the second case, the maximal eigenspace is one-dimensional. In this case we will say that the harmonic functions are supported on either B or B^c , depending on where u does not vanish. Finally, in the third case we will show that \mathbb{H} is spanned by two linearly independent functions $\{[\mathbb{1}_B u]_m, [\mathbb{1}_{B^c} u]_m\}$, and therefore will be a two-dimensional subspace of $L^1(m)$.

Let us assume that *iii*) holds. We are choosing to handle this case firstly because the arguments involved in it work similarly in case *ii*).

We are going to show that if $[v]_m$ is any other harmonic function then $[v]_m = \alpha[\mathbb{1}_B u]_m + \beta[\mathbb{1}_{B^c} u]_m$, for some $\alpha, \beta \in \mathbb{R}$.

Firstly, we will show that both $[\mathbb{1}_B u]_m$ and $[\mathbb{1}_{B^c} u]_m$ are two linearly independent harmonic functions of \mathbb{L}_m . The linear independence of these two functions is obvious. Lets us show that $[\mathbb{1}_B u]_m$ is a harmonic function for \mathbb{L}_m . Note that

$$\mathbb{L}_m[\mathbb{1}_B u]_m = [u]_m - \mathbb{L}_m[\mathbb{1}_{B^c} u]_m = [\mathbb{1}_B u]_m + [\mathbb{1}_{B^c} u]_m - \mathbb{L}_m[\mathbb{1}_{B^c} u]_m. \quad (8)$$

Recalling that $\mathbb{L}_m^* \mathbb{1}_B = \mathbb{1}_B$ and using the above equality, we obtain

$$\begin{aligned} \|\mathbb{1}_B u\|_{L^1(m)} &= \langle \mathbb{1}_B, [\mathbb{1}_B u] + [\mathbb{1}_{B^c} u] - \mathbb{L}_m[\mathbb{1}_{B^c} u] \rangle_m \\ &= \|u \mathbb{1}_B\|_{L^1(m)} - \langle \mathbb{1}_B, \mathbb{L}_m[\mathbb{1}_{B^c} u] \rangle_m, \end{aligned}$$

which implies $\mathbb{L}_m[\mathbb{1}_{B^c} u]_m = 0$ in B . Similarly, we get $\mathbb{L}_m[\mathbb{1}_B u]_m = 0$ in B^c . By plugging this back into the equation (8) we get that $\mathbb{L}_m[\mathbb{1}_B u]_m = [\mathbb{1}_B u]_m$ and consequently $\mathbb{L}_m[\mathbb{1}_{B^c} u]_m = [\mathbb{1}_{B^c} u]_m$.

From definition of m we get $\mu(B) = 0$. Since $\mathbb{L}_m[\mathbb{1}_B u]_m = [\mathbb{1}_B u]_m$ we have that $\mathbb{L}_\nu[\mathbb{1}_B u]_\nu = [\mathbb{1}_B u]_\nu$. The conformal measure $\nu \ll m$ and therefore we get from item *iii*) that $[\mathbb{1}_B u]_\nu \neq 0$. Since $\nu \in \text{ex}(\mathcal{G}^*)$ we have $[\mathbb{1}_B u]_\nu$ is positive ν -almost everywhere.

Now, let $[v]_m$ be an arbitrary harmonic function for \mathbb{L}_m . By repeating the above steps we conclude that $[\mathbb{1}_B v]_\nu$ is also a ν -almost everywhere positive harmonic function for \mathbb{L}_ν , and so there is some $\alpha \in \mathbb{R}$ such that $[\mathbb{1}_B v]_\nu = \alpha[\mathbb{1}_B u]_\nu$. From the definition of B and m we conclude that the last equality actually implies $[\mathbb{1}_B v]_m = \alpha[\mathbb{1}_B u]_m$. Proceeding similarly for $[\mathbb{1}_{B^c} v]_\nu$, we get that $[v]_m = \alpha[\mathbb{1}_B u]_m + \beta[\mathbb{1}_{B^c} u]_m$, which finishes the proof of the theorem. \blacksquare

3. APPLICATIONS AND EXAMPLES

3.1. The Support of Equilibrium States. When working with uncountable alphabets, there are two interesting alternatives generalizing the measure-theoretic entropy. Both give rise to concepts of equilibrium states carrying physical interpretations. The first one appeared in Statistical Mechanics and it is thoroughly developed in [13, 19, 28]. The other one appeared in the Dynamical Systems literature in reference [24] in the context of shifts in compact metric alphabets. Here we adopt the Statistical Mechanics viewpoint. Given μ and ν , two arbitrary finite measures on X , and \mathcal{A} , a sub-sigma-algebra of \mathcal{F} , we define

$$\mathcal{H}_{\mathcal{A}}(\mu|\nu) = \begin{cases} \int_X \frac{d\mu|_{\mathcal{A}}}{d\nu|_{\mathcal{A}}} \log \left(\frac{d\mu|_{\mathcal{A}}}{d\nu|_{\mathcal{A}}} \right) d\nu, & \text{if } \mu \ll \nu \text{ on } \mathcal{A}; \\ \infty, & \text{otherwise.} \end{cases}$$

This is in general a non-negative extended real number, and $\mathcal{H}_{\mathcal{A}}(\mu|\nu)$ is called *relative entropy* of μ with respect to ν on \mathcal{A} . Let $\mathbf{p} = \prod_{i \in \mathbb{N}} p$ be the product measure constructed from our a priori measure p . The entropy we want to consider will be denoted by \mathbf{h} , and for each shift-invariant probability measure $\mu \in \mathcal{M}_{\sigma}(X)$, it is defined as the limit $\mathbf{h}(\mu) \equiv -\lim_{n \rightarrow \infty} (1/n) \mathcal{H}_{\mathcal{F}_n}(\mu|\mathbf{p})$, where \mathcal{F}_n is the sigma-algebra generated by the projections $\{\pi_j : X \rightarrow E : 1 \leq j \leq n\}$. The existence of such limit follows from a subadditivity argument. Although $\mathbf{h}(\mu)$ is always a non-positive number, it is related to the measure-theoretic entropy $h_{\mu}(\sigma)$ by the formula $\mathbf{h}(\mu) + \log |E| = h_{\mu}(\sigma)$ when the alphabet E is finite and the a priori measure is taken as the normalized counting measure. Therefore both entropies determine the same set of equilibrium states in this particular context.

Back to the general case, Proposition 15.14 in [13] ensures that in our context (compact metric alphabets) the mapping $\mathcal{M}_{\sigma}(X) \ni \mu \mapsto \mathbf{h}(\mu)$ is affine and upper semi-continuous, relative to the weak-* topology, and therefore, for any continuous potential f , there is at least one solution for the generalized version of the variational principle

$$\sup_{\mu \in \mathcal{M}_{\sigma}(X)} \left\{ \mathbf{h}(\mu) + \int_X f d\mu \right\}.$$

Next, we show how to use the abstract results obtained in the previous section to get information on the support of the equilibrium states. If E is a compact alphabet and f is a sufficiently regular (Hölder, Walters or Bowen) potential, then the set of equilibrium states is a singleton. Moreover, the unique equilibrium state is obtained in the traditional way by taking a suitable scalar multiple of the harmonic function h for \mathcal{L} and the unique conformal measure in $\mathcal{G}^* = \{\nu\}$. It is natural to expect that this construction also works for general continuous potentials that are less regular than those mentioned above.

Suppose that f is a low regular potential, but \mathcal{G}^* is a singleton. This last assumption is not so restrictive since this property holds generically in $C(X)$. If a harmonic function $h = \mathbb{L}h$ does exist then we get the existence of a fully supported equilibrium state μ . Indeed, the ν -a.e. positivity of h and Theorem 2.1 which ensures that $\text{supp}(\nu) = X$ immediately implies $\text{supp}(\mu) = \text{supp}(h\nu) = X$.

3.2. Two Dimensional Space of Harmonic Functions. In this section, we construct an explicit example of a transfer operator, denoted by $\mathbb{L}_{\beta f}$, where $\beta > 0$ and f is a potential given by (10), for which the maximal eigenspace is two-dimensional. The potential f and the results in this section are inspired in the Curie-Weiss (mean-field) model for ferromagnetism. It is one of the simplest models in Equilibrium Statistical Mechanics exhibiting the phase transition phenomenon, see [12].

The aim is to illustrate the results presented in the previous sections in a concrete and simple example, especially including the construction of a basis for the maximal eigenspace of this transfer operator. There is a special feature of this example, the potential discontinuity. This leads us to introduce a proper replacement for $\rho(\mathcal{L})$ -conformal measures. The measures that are going to play the same role as those in \mathcal{G}^* , will be called here generalized conformal measures. This concept is introduced in Definition 3.1, and before that, a motivation is presented.

In what follows $E = \{-1, 1\}$, $p = \sum_{e \in E} \delta_e$ is the counting measure on E and $\nu \in \mathcal{M}_1(X)$ is a quasi-invariant measure. Fix a bounded and $\mathcal{B}(X)$ -measurable potential $f : X \rightarrow \mathbb{R}$. Then the mapping sending a ν -integrable function φ to $L\varphi$ given by

$$L\varphi(x) = \sum_{a \in E} \exp(f(ax)) \varphi(ax), \quad \text{where } ax \equiv (a, x_1, x_2, \dots) \quad (9)$$

defines a linear operator acting on the space of ν -integrable functions. Since we are assuming that ν is quasi-invariant we have that the operator L induces a positive and continuous linear operator $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$.

We have seen that, for every continuous potentials $f \in C(X)$, there is at least one measure for which $\mathcal{L}^*\nu = \rho(\mathcal{L})\nu$. As mentioned early, this implies $\mathbb{L}^*\mathbb{1} = \rho(\mathcal{L})\mathbb{1}$. Conversely, if ν is a fully supported quasi-invariant probability measure on X and \mathbb{L} is an extension of \mathcal{L} satisfying $\mathbb{L}^*\mathbb{1} = \rho(\mathcal{L})\mathbb{1}$ then $\mathcal{L}^*\nu = \rho(\mathcal{L})\nu$. Therefore for continuous potentials,

$$\mathcal{L}^*\nu = \rho(\mathcal{L})\nu \iff \mathbb{L}^*\mathbb{1} = \rho(\mathcal{L})\mathbb{1} \quad \nu - a.e.$$

This equivalence is the motivation for the following definition.

Definition 3.1 (Generalized Conformal Measures). Let $X = E^{\mathbb{N}}$ be a product space, where E is a finite set and $f : X \rightarrow \mathbb{R}$ a bounded and a $\mathcal{B}(X)$ -measurable potential. A quasi-invariant measure $\nu \in \mathcal{M}_1(X)$ such that the operator \mathbb{L} induced by L satisfies $\mathbb{L}^*\mathbb{1} = \|\mathbb{L}\|_{\text{op}}\mathbb{1}$ ν -a.e. will be called a generalized conformal measure associated to the potential f .

The above definition could be formulated for a general compact metric alphabet E , but this particular form is enough for our needs in this section and avoids some unnecessary technicalities. We should also remark that although the space X is compact, in this setting there is no guarantee, in general, that the set of generalized conformal measure is not empty.

In this section the potential f will be given by

$$f(x) \equiv x_1 \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=2}^{N+1} x_k. \quad (10)$$

Clearly, $(\beta f)_{\beta > 0}$ is a family of bounded and $\mathcal{B}(X)$ -measurable potentials. But, differently from the other sections of this paper, every potential in this family is discontinuous, with respect to the product topology.

The following discussion aims to convince the reader that some product measures are natural candidates to be generalized conformal measures, and to explain how to compute the respective eigenvalues, in this case.

Firstly, we consider the family of product measures μ_γ on $\{-1, 1\}^{\mathbb{N}}$, parameterized by $\gamma \in (-1, 1)$, and defined by $\mu_\gamma(\{x_k = +1\}) = q$ and $\mu_\gamma(\{x_k = -1\}) = 1 - q$, where the parameter $\gamma = 2q - 1$ is the expected value of the coordinate functions, that is, $\mathbb{E}_{\mu_\gamma}[x_k] = \gamma$, for every $k \in \mathbb{N}$.

Clearly, for any choice of $\gamma \in (-1, 1)$, we have that μ_γ is quasi-invariant and therefore $L_{\beta f}$ induces an operator on $L^1(\mu_\gamma)$. Next, we compute the operator norm of $L_{\beta f}$. By definition

$$\|L_{\beta f}\| = \int_X \sum_{a \in \{-1, 1\}} \exp(\beta f(ax)) d\mu_\gamma(x) = \int_X \sum_{a \in \{-1, 1\}} \exp(\beta a m(ax)) d\mu_\gamma(x)$$

where

$$m(x) \equiv \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=2}^{N+1} x_k = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N x_k = \gamma, \quad \mu_\gamma - \text{a.e.} \quad (11)$$

by the Law of Large Numbers. Of course, for any $a \in \{-1, 1\}$ we have $m(ax) = m(x)$ and therefore

$$\|L_{\beta f}\| = \int_X e^{-\beta\gamma} + e^{\beta\gamma} d\mu_\gamma(x) = 2 \cosh(\beta\gamma).$$

So far the parameter γ is free, but in order to μ_γ to be a generalized conformal measure for βf the equality $\langle \mathbb{1}, L_{\beta f} u \rangle_{\mu_\gamma} = 2 \cosh(\beta\gamma) \langle \mathbb{1}, u \rangle_{\mu_\gamma}$ must hold for any $u \in L^1(\mu_\gamma)$. For this, it is enough that, $\langle \mathbb{1}, L_{\beta f} \mathbb{1}_B \rangle_{\mu_\gamma} = \cosh(\beta\gamma) \langle \mathbb{1}, \mathbb{1}_B \rangle_{\mu_\gamma}$ for any indicator function $\mathbb{1}_B$, where $B \in \mathcal{B}(X)$.

Developing the left-hand side of the last equation we get

$$\begin{aligned} \int_X L_{\beta f} \mathbb{1}_B d\mu_\gamma &= \int_X \sum_{a \in \{-1, +1\}} \exp(a\beta m(ax)) \mathbb{1}_B(ax) d\mu_\gamma(x) \\ &= \frac{e^{-\beta\gamma} \mu_\gamma(B \cap [-1])}{\mu_\gamma([-1])} + \frac{e^{\beta\gamma} \mu_\gamma(B \cap [+1])}{\mu_\gamma([+1])}. \end{aligned}$$

By taking $B = [+1]$ and next $B = [-1]$, in the previous identity, and using $\langle \mathbb{1}, L_{\beta f} \mathbb{1}_B \rangle_{\mu_\gamma} = \cosh(\beta\gamma) \langle \mathbb{1}, \mathbb{1}_B \rangle_{\mu_\gamma}$, we see that the following relations must be satisfied

$$e^{\pm\beta\gamma} = \int_X L_{\beta f} \mathbb{1}_{[\pm 1]} d\mu_\gamma = 2 \cosh(\beta\gamma) \int_X \mathbb{1}_{[\pm 1]} d\mu_\gamma = 2 \cosh(\beta\gamma) \mu_\gamma([\pm 1]).$$

Since $q = \mu_\gamma([+1]) = e^{\beta\gamma}/2 \cosh(\beta\gamma)$ and $1 - q = \mu_\gamma([-1]) = e^{-\beta\gamma}/2 \cosh(\beta\gamma)$, we finally get that γ has to be a solution of the following equation

$$\gamma = 2q - 1 = \frac{e^{\beta\gamma} - e^{-\beta\gamma}}{2 \cosh(\beta\gamma)} = \tanh(\beta\gamma).$$

The equation $\gamma = \tanh(\beta\gamma)$ has either one or three solutions, depending on the value of β . If $0 < \beta \leq 1$ then $\gamma = 0$ is the unique solution. Otherwise, if $\beta > 1$ then there is some $\gamma(\beta) \in (0, 1)$ such that $-\gamma(\beta), 0$ and $\gamma(\beta)$ are all the solutions

of the equation. Now, we can justify the previous steps. Firstly, take γ satisfying $\gamma = \tanh(\beta\gamma)$. Secondly, note that in the previous computation of $\langle \mathbb{1}, \mathbb{L}_{\beta\gamma} \mathbb{1}_B \rangle_{\mu_\gamma}$ we can use the values we got for $\mu_\gamma([-1])$ and $\mu_\gamma([+1])$. Therefore, we have that

$$\int_X \mathbb{L}_{\beta\gamma} \mathbb{1}_B d\mu_\gamma = \cosh(\beta\gamma) [\mu_\gamma(B \cap [-1]) + \mu_\gamma(B \cap [+1])] = \cosh(\beta\gamma) \int_X \mathbb{1}_B d\mu_\gamma.$$

Since the last identity holds for any measurable set B it follows that $\langle \mathbb{1}, \mathbb{L}_{\beta\gamma} \mathbb{1}_B \rangle_{\mu_\gamma} = \langle \mathbb{1}, \mathbb{1}_B \rangle_{\mu_\gamma}$ and so μ_γ is indeed a generalized conformal measure associate with the potential βf . This reasoning actually proves the following proposition.

Proposition 3.2 (Generalized Conformal Measures – Curie-Weiss Model). *Let f be the potential defined given by (10) and for each $\gamma \in (-1, 1)$ let μ_γ be a Bernoulli measure as defined above. Then*

- (1) μ_γ is a generalized conformal measure, if and only if, γ is a solution of the equation $\gamma = \tanh(\beta\gamma)$;
- (2) for any solution γ of the above equation, $2 \cosh(\beta\gamma)$ is an eigenvalue of $\mathbb{L}_{\beta f}^*$.

By Proposition 3.2 if $0 < \beta < 1$, then μ_0 , the symmetric Bernoulli measure with parameter $q = 1/2$, is a generalized conformal measure associated with the eigenvalue 2. But on the other hand, if $\beta > 1$, this measure still is an eigenmeasure associated with the eigenvalue 2, but now there is two other Bernoulli measures $\mu_{\pm\gamma(\beta)}$ associate with a strictly bigger eigenvalue $2 \cosh(\beta\gamma(\beta))$.

Now let us move the discussion to the eigenfunctions. We first observe that for any fixed $\beta > 0$, the operator $\mathbb{L}_{\beta f} : L^1(\mu_0) \rightarrow L^1(\mu_0)$, has the constant function as an eigenfunction associated with the eigenvalue $\lambda = 2$, that is, $\mathbb{L}_{\beta f} \mathbb{1} = 2\mathbb{1}$.

However, for $\beta > 1$, which is above the critical point of the original Curie-Weiss model, we can see more interesting phenomena, such as multidimensional eigenspaces. Since β is fixed, in what follows we will write $\mu_\pm \equiv \mu_{\pm\gamma(\beta)}$ to lighten the notation.

Consider the operator $\mathbb{L}_{\beta f} : L^1(\nu) \rightarrow L^1(\nu)$, where $\nu \equiv \nu(t) \equiv t\mu_+ + (t-1)\mu_-$ is a nontrivial convex combination of μ_\pm . The measurable sets $B_+ = \{x \in X : m(x) = \gamma(\beta)\}$ and $B_- = \{x \in X : m(x) = -\gamma(\beta)\}$ are chosen in such way they form a measurable partition of the space $X = B_+ \cup B_- \cup N$ up to a ν -null set N . Note that $\mu_+(B_+) = 1$ and $\mu_-(B_+) = 0$. Proceeding as before it is simple to argue that $\mathbb{1}_{B_\pm}$ are two linearly independent eigenfunctions for the adjoint of the operator $\mathbb{L}_{\beta f}$, that is, $\mathbb{L}_{\beta f}^* \mathbb{1}_{B_+} = 2 \cosh(\beta\gamma(\beta)) \mathbb{1}_{B_+}$ and $\mathbb{L}_{\beta f}^* \mathbb{1}_{B_-} = 2 \cosh(\beta\gamma(\beta)) \mathbb{1}_{B_-}$.

Regarding the operator $\mathbb{L}_{\beta f}$ itself, it turns out that the characteristic functions $\mathbb{1}_{B_\pm}$ are also eigenfunctions, more precisely, $\mathbb{L}_{\beta f} \mathbb{1}_{B_+} = 2 \cosh(\beta\gamma(\beta)) \mathbb{1}_{B_+}$ and $\mathbb{L}_{\beta f} \mathbb{1}_{B_-} = 2 \cosh(\beta\gamma(\beta)) \mathbb{1}_{B_-}$. To see this, remember that for any point $x \in B_+$, $m(x) = \gamma(\beta)$, and so

$$\mathbb{L}_{\beta f} \mathbb{1}_{B_+}(x) = \sum_{a \in \{-1, 1\}} \mathbb{1}_{B_+}(x) \exp(\beta a m(ax)) = (2 \cosh \beta\gamma) \mathbb{1}_{B_+}(x).$$

The same is true for B_- , since $m(B_-) = -\gamma(\beta)$. Moreover, with a proper rewording, the proof of Theorem 1.1 can be adapted to this discontinuous case showing that these are the only linear independent eigenfunctions of $\mathbb{L}_{\beta f}$ (with the measure ν). This proves Theorem 1.2.

In order to construct a bi-dimensional SHF for a transfer operator associated with a continuous potential it is enough take g -functions for which the set of its g -measures is not a singleton.

3.3. Phase Transitions in Lattice Spin Systems. Let E be a standard Borel space, $X = E^{\mathbb{N}}$ and $\mathcal{B}(X)$ the product sigma-algebra on X . The space X is equipped with the product topology. For each $i \in \mathbb{N}$, let $\pi_i : X \rightarrow E$ be the standard projection as defined in Subsection 3.1. For each $\Lambda \Subset \mathbb{N}$ (finite subset) we consider the following sub-sigma-algebras $\mathcal{B}_\Lambda \equiv \sigma(\pi_j : j \in \Lambda)$ and $\mathcal{T}_\Lambda \equiv \sigma(\cup_{\Gamma} \mathcal{B}_\Gamma : \Gamma \Subset \Lambda^c)$.

A probability kernel γ_Λ is called a proper probability kernel from \mathcal{T}_Λ to \mathcal{B}_Λ if

- i) $\gamma_\Lambda(\cdot|x)$ is probability measure on $(X, \mathcal{B}(X))$ for any $x \in X$;
- ii) $\gamma_\Lambda(A|\cdot)$ is \mathcal{T}_Λ -measurable for any $A \in \mathcal{B}(X)$;
- iii) $\gamma_\Lambda(A \cap B|x) = 1_B(x)\gamma_\Lambda(A|x)$ for any $A \in \mathcal{B}(X)$, $B \in \mathcal{T}_\Lambda$ and $x \in X$.

The family $\gamma \equiv (\gamma_\Lambda)_{\Lambda \Subset \mathbb{N}}$ is said to be consistent if

$$\int_X \gamma_\Lambda(A|x) d\gamma_\Gamma(\cdot|x) = \gamma_\Gamma(A, x), \text{ whenever } \emptyset \subsetneq \Lambda \subset \Gamma.$$

A *specification* with parameter set \mathbb{N} and state space E is a family $\gamma \equiv (\gamma_\Lambda)_{\Lambda \Subset \mathbb{N}}$ such that γ_Λ is a proper probability kernel from \mathcal{T}_Λ to \mathcal{B}_Λ and $(\gamma_\Lambda)_{\Lambda \Subset \mathbb{N}}$ satisfies the consistency condition.

Let γ be a specification with parameter set \mathbb{N} and state space E . The set of all Borel probability measures defined by

$$\mathcal{G}^{\text{DLR}}(\gamma) = \{\mu \in \mathcal{M}_1(X) : \mu(A|\mathcal{T}_\Lambda)(x) = \gamma_\Lambda(A, x) \text{ } \mu - a.s.\}$$

will be called the set of DLR-Gibbs Measures determined by γ . In this context of Statistical Mechanics, if $\#\mathcal{G}^{\text{DLR}}(\gamma) > 1$, then we say that we have phase transition.

Theorem 3.3 (Georgii, [13]). *Let $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{N}}$ be a specification with parameter set \mathbb{N} and state space E . Then the following statements are equivalent:*

- (1) $\mu \in \mathcal{G}^{\text{DLR}}(\gamma)$;
- (2) $\mu(A) = \int_X \gamma_\Lambda(A, x) d\mu(x) \equiv \mu\gamma_\Lambda(A)$, for all $A \in \mathcal{F}$ and $\Lambda \Subset \mathbb{N}$;
- (3) *there is a cofinal collection $\{\Gamma_\alpha : \Gamma_\alpha \Subset \mathbb{N}, \forall \alpha \in I\}$ (i.e., directed by inclusion and, for all $\Lambda \Subset \mathbb{N}$, there is an index $\alpha \in I$ such that $\Lambda \Subset \Gamma_\alpha$) satisfying $\mu(A) = \mu\gamma_\Lambda(A)$, for all $A \in \mathcal{F}$.*

Example 3.4. $O(n)$ -models. Let $E = \mathbb{S}^{n-1}$ and take as the a priori measure p the normalized surface area measure on E . The Hamiltonian of the $O(n)$ -model on the volume \mathbb{V}_n with boundary condition $x \in X$ is given by

$$\mathcal{H}_{\mathbb{V}_n}^x(y) = - \sum_{ij \in \mathbb{E}_n} J(|i-j|)\langle y_i, y_j \rangle - \sum_{i \in \mathbb{V}_n; j \in \mathbb{N} \setminus \mathbb{V}_n} J(|i-j|)\langle y_i, x_j \rangle,$$

where we assume that $\sum_n |J(n)| < +\infty$. Consider the potential $f \in C(X)$ given by $f(x) = \sum_n J(n)\langle x_1, x_{n+1} \rangle$ and the transfer operator given by

$$\mathcal{L}\varphi(x) \equiv \int_{\mathbb{S}^{n-1}} \exp(f(ax))\varphi(ax) dp(a).$$

Note that for each $n \in \mathbb{N}$ the Birkhoff sum $S_n(f)(x) = -\mathcal{H}_{\mathbb{V}_n}^x(x)$. This equality is crucial to relate the dynamical system and statistical mechanics descriptions of this model. Consider the probability kernel

$$\gamma_{\mathbb{V}_n}(A|x) \equiv \frac{\mathcal{L}^n(\mathbf{1}_A)(\sigma^n x)}{\mathcal{L}^n(\mathbf{1})(\sigma^n x)}. \quad (12)$$

In [5] the authors show that the family of kernels $(\gamma_{\mathbb{V}_n})_{n \in \mathbb{N}}$ can be naturally extended to a specification $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{N}}$. By Theorem 3.3 we have that $\mathcal{G}^{\text{DLR}}(\gamma)$ does not depend on the choice of this extension, since the collection $(\Lambda_n)_{n \in \mathbb{N}}$ is a

cofinal collection. The set $\mathcal{G}^{\text{DLR}}(\gamma)$ contains the closure of the convex hull of all the thermodynamic limits defined similarly as in the introduction. Moreover, for any continuous potential f , the equality $\mathcal{G}^{\text{DLR}}(\gamma) = \mathcal{G}^*$ holds.

Therefore, if for some $\nu \in \mathcal{G}^*$ the SHF for $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$ has dimension greater than one, then ν can not be an extreme measure in \mathcal{G}^* . By Krein-Milman Theorem it follows that $\#\mathcal{G}^* > 1$ and so $\#\mathcal{G}^{\text{DLR}}(\gamma) > 1$, that is, we have a first-order phase transition in the sense of Dobrushin. The sufficient condition for phase transition, presented in this section, motivates the following interesting question. Is there any continuous potential f for which there are two distinct non-trivial critical points $0 < \beta_c^1 < \beta_c^2 < +\infty$ such that

$$\dim(\mathbb{H}_{\beta f}) = \begin{cases} 0, & \text{if } 0 < \beta < \beta_c^1; \\ 1, & \text{if } \beta_c^1 < \beta < \beta_c^2; \\ N, & \text{if } \beta_c^2 < \beta, \end{cases}$$

where $\mathbb{H}_{\beta f}$ is the SHF for $\mathbb{L} : L^1(\nu_{\beta f}) \rightarrow L^1(\nu_{\beta f})$, $\nu_{\beta f}$ is the barycenter of $\mathcal{G}(\beta f)$, and $N > 1$. This question is actually motivated by the following observation. The phase transition in the sense of Dobrushin is related to the dichotomy of the cardinality of $\mathcal{G}^{\text{DLR}}(\gamma^\beta)$ be either one or infinite, accordingly the inverse temperature β be greater or less than a critical point β_c . This translates in a similar statement as in the above question, but for only one critical point β_c^2 (due to the compactness of E and Dobrushin's uniqueness theorem at high temperatures) and the dimensions possibly jumping from 1 to N .

And so a positive answer to this question could show that SHF's are capable of detecting new types of phase transitions.

4. FUNCTIONAL CENTRAL LIMIT THEOREM

This section is dedicate to prove a FCLT (also known as invariance principle) for a Markov process which arises naturally in our context. We use the version obtained here to prove a new result in Statistical Mechanics.

Often proving a FCLT for a Markov process relies on verifying some analytical condition on the associated transfer operator, see for instance [2, 15, 16, 26]. The approach we will use consists in solving *Poisson's equation*. This technique is due to Gordin and Lifsic, see [15].

In our case this theory is applied as follows. We will assume that f is a normalized potential, meaning that $\mathcal{L}\mathbf{1} = \mathbf{1}$. For each measurable set A , the expression

$$p(x, A) = \mathbb{L}(\mathbf{1}_A)(x),$$

where the operator \mathbb{L} is the extension of \mathcal{L} , defines a transition probability kernel of some Markov chain $(Z_n)_{n \in \mathbb{N}}$ taking values in X . A straightforward computation shows that the induced transfer operator satisfies $P\varphi(x) = \mathbb{L}(\varphi)(x)$. Therefore, any probability measure fixed by the adjoint operator \mathcal{L}^* is a stationary measure for P . As usual, we denote by E_μ the expectation with respect to the joint law of the Markov Chain $(Z_n)_{n \in \mathbb{N}}$ with stationary measure μ .

The distributional relation between the Markov chain $(Z_n)_{n \in \mathbb{N}}$ and the shift map is given by the following lemma, whose proof can be found in reference [16, p.85].

Lemma 4.1. *Let $(Z_n)_{n \in \mathbb{N}}$ be the Markov chain defined above, $n \geq 1$, $g : X^n \rightarrow \mathbb{R}$ a positive measurable function, and μ a stationary measure. Then we have*

$$\int_X g(x, \sigma(x), \dots, \sigma^{n-1}(x)) d\mu(x) = E_\mu[g(Z_n, Z_{n-1}, \dots, Z_1)].$$

Here we focus on functions g of the form $g = \mathbb{1}_A \circ h$, where $A = (-\infty, t]$ is some suitable interval on the real line, $h : X^n \rightarrow \mathbb{R}$ is given by $h(z_1, \dots, z_n) = \phi(z_1) + \dots + \phi(z_n)$, with $\phi : X \rightarrow \mathbb{R}$ being a positive function in some Banach space, for example, the space of Hölder continuous functions.

Theorem 4.2. *Let P be the transfer operator induced by the extension \mathbb{L} associated with a continuous and normalized potential and $\mu \in \mathcal{G}^*$. Let $\phi : X \rightarrow \mathbb{R}$ be a non-constant observable in $L^2(\mu)$ satisfying $\mu(\phi) = 0$. If there exists a solution $v \in L^2(\mu)$ for Poisson's equation $(\text{Id} - \mathbb{L})v = \phi$, then the stochastic process $Y_n(t)$, given by*

$$Y_n(t) = \frac{1}{\varrho \sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor} \phi \circ \sigma^j, \quad 0 \leq t < \infty, \quad (13)$$

where $\varrho = \mu(v^2) - \mu(Pv^2)$, converges in distribution to the Wiener measure in $D[0, \infty)$.

The proof of the above theorem is done by reducing the problem to the martingale case. See [2] for this reduction and [3] for a FCLT for martingale differences.

Example 4.3 (Spectral Gap). In this example, we show how to apply the above theorem to a transfer operator whose action on the space of Hölder continuous functions have the spectral gap property.

Let $X = E^{\mathbb{N}}$, where E is a compact metric space, and $f : X \rightarrow \mathbb{R}$ be an α -Hölder potential, that is, f is a potential satisfying

$$\text{Hol}_\alpha(f) \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

Up to summing a coboundary term, we can assume that f is a normalized potential. In [6] the authors showed that the transfer operator acts on the space of α -Hölder continuous functions with a spectral gap.

Let ϕ be an arbitrary α -Hölder continuous observable for which $\mu(\phi) = 0$. In this case, we can always get a solution of Poisson's equation $(\text{Id} - P)v = \phi$. Indeed, following the notation of reference [6], and taking $\psi = \mathbb{1}$ and $\varphi = \phi$ in Theorem 3.1 we get $\|\mathbb{L}^n \phi\|_\infty \leq C s^n$, for constants $0 < s < 1$ and $C > 0$, and as a consequence $\|\mathbb{L}^n \phi\|_2 \leq C s^n$, which implies that $v = -\sum_{n=0}^{\infty} \mathbb{L}^n \phi$ is a well-defined element of $L^2(\mu)$ and also a solution of Poisson's equation. Therefore the Theorem 4.2 applies.

The next example shows the validity of the FCLT for a large class of observables in a situation where we do not have the spectral gap property.

Example 4.4 (Absence of Spectral Gap). In this example we consider a Dyson type model for ferromagnetism on the one-sided lattice. Before presenting this model, we need to introduce some notation.

We start by remembering that a *modulus of continuity* is a continuous, increasing and concave function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that $\omega(0) = 0$. We say that $f : X \rightarrow \mathbb{R}$ is ω -Hölder continuous if there is a constant $C > 0$ such that $|f(x) -$

$f(y)| \leq C\omega(d(x, y))$ for any $x, y \in X$. The smallest constant C satisfying the above inequality will be denoted by $\text{Hol}_\omega(f)$. Denote by $C^\omega(X)$ the space of all such functions. By considering the norm $\|f\|_\omega \equiv \|f\|_\infty + \text{Hol}_\omega(f)$, it is easy to check that $(C^\omega(X), \|\cdot\|_\omega)$ is a Banach algebra.

Note that, for $\alpha, \beta \geq 0$, the function $\omega_{\alpha+\beta \log}(r) \equiv r^\alpha \log(r_0/r)^{-\beta}$ defines a modulus of continuity. We denote the space of $\omega_{\alpha+\beta \log}$ -Hölder continuous functions by $C^{\alpha+\beta \log}(X)$. In particular, if $\alpha = 0$, we simply write $C^{\beta \log}(X)$.

Let $X = \{-1, 1\}^{\mathbb{N}}$, endowed with the metric $d(x, y) = 2^{-N(x, y)}$, where the number $N(x, y) \equiv \inf\{i \in \mathbb{N} : x_j = y_j, 1 \leq j \leq i-1 \text{ and } x_i \neq y_i\}$. The *Dyson* potential on the one-sided lattice is given by the following expression

$$f(x) = \sum_{n=2}^{\infty} \frac{x_1 x_n}{n^{2+\varepsilon}}.$$

One can easily show that f is not a Hölder continuous function with respect to $d(x, y)$. Additionally, the transfer operator \mathcal{L}_f , associated with a Dyson potential f , does not leave the space of Hölder continuous functions invariant. But, on the other hand, it leaves invariant a bigger subspace of $C(X)$, called the Walters space. Although there exists \bar{f} in the Walters space cohomologous to f , neither \mathcal{L}_f nor $\mathcal{L}_{\bar{f}}$ acts with the spectral gap property on this subspace, see [7]. However, we will see that the absence of spectral gap will be not an obstruction to solve Poisson's equation if our system is sufficiently mixing.

The aim of this example is to prove that the stochastic process defined in (13) converges in distribution to the Wiener measure in $D[0, \infty)$ for any observable $\phi \in C^{\varepsilon \log}$, and $\varepsilon > 2$.

Firstly, we show that $f \in C^{\varepsilon \log}(X)$. Indeed,

$$\begin{aligned} |f(x) - f(y)| &\leq 2(N(x, y) + 1)^{-2-\varepsilon} \left(1 + \int_1^\infty \left(\frac{N(x, y) + 1}{N(x, y) + t + 1}\right)^{2+\varepsilon} dt\right) \\ &\leq 2N(x, y)^{-2-\varepsilon} \left(N(x, y) + 2^3 \frac{N(x, y)}{1 + \varepsilon}\right) \leq 20 \log(2^{N(x, y)})^{-\varepsilon} \\ &= \omega \circ d(x, y), \end{aligned}$$

where $\omega(r) = \log(r_0/r)^{-\varepsilon}$.

The previous estimate holds for any $\varepsilon > 0$, but to solve Poisson's equation later, we will need to restrict ourselves to $\varepsilon > 2$. Observe that, from the definition of d , we have $d(a_1 \dots a_j x, a_1 \dots a_j y) = 2^{-j} d(x, y)$. By using this identity and the previous inequality, we conclude, for $0 \leq j \leq n-1$, that

$$|f(\sigma^j(a_1 \dots a_n x)) - f(\sigma^j(a_1 \dots a_n y))| \leq \omega(2^{n-j} d(x, y)).$$

Therefore, $|\sum_{j=0}^{n-1} f(\sigma^j(a_1 \dots a_n x)) - f(\sigma^j(a_1 \dots a_n y))| \leq \sum_{j=0}^{n-1} \omega(2^{-j} d(x, y))$. We recall that the summands on the rhs can be written as

$$\omega(2^{-j} d(x, y)) = \left[\log\left(\frac{r_0}{2^{-j} d(x, y)}\right)\right]^{-\varepsilon} = \left[j \log 2 + \log\left(\frac{r_0}{d(x, y)}\right)\right]^{-\varepsilon}.$$

Similarly, there exists a positive constant C , such that

$$\sum_{j=0}^{n-1} \omega(2^{-j} d(x, y)) \leq C \left[\log\left(\frac{r_0}{d(x, y)}\right)\right]^{1-\varepsilon} = C\tilde{\omega}(r).$$

Therefore for every $n \in \mathbb{N}$, we have

$$\left| \sum_{j=0}^{n-1} f(\sigma^j(a_1 \dots a_n x)) - f(\sigma^j(a_1 \dots a_n y)) \right| \leq C\tilde{\omega}(d(x, y)). \quad (14)$$

As consequences of (14), we have $f \in C^{(\varepsilon-1)\log}(X)$, and the Dyson potential is a *flat potential*, with respect to the *natural coupling*, and $\tilde{\omega}$, see reference [22] Definitions 2.1.3 and 5.2.

The previous discussion allows us to apply Theorems 3.2 and 4.1 of [22] obtaining a strictly positive eigenfunction $h \in C^{(\varepsilon-1)\log}(X)$ associated with $\rho(\mathcal{L}_f)$. By using this eigenfunction, we can construct a normalized potential $\bar{f} \in C^{(\varepsilon-1)\log}(X)$ cohomologous to f given by $\bar{f} = f + \log h - \log h \circ \sigma - \log \rho(\mathcal{L}_f)$. Moreover, one can check that \bar{f} is also a flat potential. Recall that $\mathcal{L}_{\bar{f}}\mathbb{1} = \mathbb{1}$ and $\mathcal{L}_{\bar{f}}^*\mu_f = \mu_f$.

Therefore, we are in conditions to apply Theorem 5.8 of [22, p.31] to prove the existence of a constant $D > 0$ such that, for any $\phi \in C^{(\varepsilon-1)\log}(X)$ satisfying $\int_X \phi d\mu_f = 0$, we have that $\|\mathcal{L}_{\bar{f}}^n \phi\|_\infty \leq D/n^{\varepsilon-1}$. Since we are assuming $\varepsilon > 2$, we get $\sum_{n=2}^\infty \|\mathcal{L}_{\bar{f}}^n \phi\|_\infty \leq \sum_{n=2}^\infty Cn^{-(\varepsilon-1)} < \infty$, which implies that $v = -\sum_{n=0}^\infty \mathcal{L}_{\bar{f}}^n \phi$ is a well defined element of $L^2(\mu_f)$ and also a solution of Poisson's equation which allows us to apply Theorem 4.2, thus showing the validity of a FCLT for this example.

APPENDIX A. THE SPECTRUM OF \mathbb{L}

The spectrum of \mathbb{L} is already known when E is a finite alphabet see, for example, [27]. However the generalization presented here for uncountable alphabets is new. Recall that when talking about the spectrum of the extension of the transfer operator \mathbb{L} , we are actually referring to the spectrum of its standard complexification, but for the sake of simplicity we will keep the same notation for both operators.

Proposition A.1. *Let f be a general continuous potential and suppose that the a priori measure satisfies the full support condition $\text{supp}(p) = E$. Let $\nu \in \mathcal{G}^*$ an arbitrary conformal measure and $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$ the extension of the transfer operator associated to the potential f . Then $\text{spec}(\mathbb{L}) = \{\lambda \in \mathbb{C} : |\lambda| \leq \rho(\mathcal{L})\}$.*

Proof. Without loss of generality we can assume that $\rho(\mathcal{L}) = 1$. Therefore the spectral radius of $\mathbb{L} : L^1(\nu) \rightarrow L^1(\nu)$ is also equal to one. Since $\text{spec}(\mathbb{L}) = \text{spec}(\mathbb{L}^*)$ and the spectrum of a bounded operator is a closed subset of the complex plane, it is enough to show that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \text{spec}(\mathbb{L}^*)$. And this is a consequence of the operator $\mathbb{L}^* - \lambda \text{Id}$ to be not onto whenever $|\lambda| < 1$, hence it is not invertible.

The main idea is to show that $\text{Im}(\mathbb{L}^* - \lambda \text{Id})$ can not contain an essentially bounded measurable function which is positive on a set A , with $\nu(A) > 0$ and $\sigma(A) = X$; and identically zero on a set B , which is disjoint from A and also have positive measure $\nu(B) > 0$.

Let us first construct the sets A and B . They can be chosen as two disjoint open cylinder sets of the following form. Take two distinct points $a, b \in E$ and $0 < r < (1/2)d_E(a, b)$. Define $A \equiv B_E(a, r) \times E^{\mathbb{N}}$ and $B \equiv B_E(b, r) \times E^{\mathbb{N}}$. Since A and B are open sets of X it follows from Theorem 2.1 that $\nu(A), \nu(B) > 0$. By construction $\sigma(A) = X$ and $A \cap B = \emptyset$.

Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < 1$ and suppose by contradiction that there is some complex function $\psi = |\psi| \exp(i \arg(\psi))$ such that $\mathbb{L}^* \psi - \lambda \psi = \mathbb{1}_B$. Since $\nu(B) > 0$ it

follows that ψ can not be identically zero. As we mentioned before $L^*\psi = \psi \circ \sigma$, ν -almost everywhere. Multiplying the above equation by $\mathbb{1}_A$ we obtain the following identity

$$\mathbb{1}_A |\psi \circ \sigma| \exp(i \arg(\psi \circ \sigma)) - \lambda \mathbb{1}_A |\psi| \exp(i \arg(\psi)) = 0, \quad \nu - a.e.$$

Therefore there is a measurable subset $X' \subset X$ such that $\nu(X') = 1$ and the above equality holds for every $x \in X'$. By taking the modulus on the last expression, and after the essential supremum we get

$$\operatorname{ess\,sup}_{x \in X' \cap A} |\psi \circ \sigma(x)| \leq |\lambda| \operatorname{ess\,sup}_{x \in X' \cap A} |\psi(x)| \leq |\lambda| \operatorname{ess\,sup}_{x \in X} |\psi(x)| \equiv |\lambda| \|\psi\|_\infty.$$

By the definition of A , we have $\sigma(X' \cap A) = \sigma(X')$. From Proposition 2.2 it follows that $\sigma(X')$ contains a set of ν -measure one. Therefore it follows from the definition of essential supremum that the left hand side above is equal to $\|\psi\|_\infty$. But, this implies that $1 \leq |\lambda|$ which is an absurdity. \blacksquare

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