

Interactions, Specifications, DLR Probabilities and the Ruelle Operator in the One-Dimensional Lattice

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Abstract

In this paper, we describe several different meanings for the concept of Gibbs measure on the lattice \mathbb{N} in the context of finite alphabets (or state space). We compare and analyze these “in principle” distinct notions: DLR-Gibbs measures, Thermodynamic Limit and eigenprobabilities for the dual of the Ruelle operator (also called conformal measures).

Among other things we extended the classical notion of a Gibbsian specification on \mathbb{N} in such way that the similarity of many results in Statistical Mechanics and Dynamical System becomes apparent. One of our main result claims that the construction of the conformal Measures in Dynamical Systems for Walters potentials, using the Ruelle operator, can be formulated in terms of Specification. We also describe the Ising model, with $1/r^{2+\varepsilon}$ interaction energy, in the Thermodynamic Formalism setting and prove that its associated potential is in Walters space - we present an explicit expression. We also provide an alternative way for obtaining the uniqueness of the DLR-Gibbs measures.

1 Introduction

The basic idea of the Ruelle Operator remounts to the transfer matrix method introduced by Kramers and Wannier [KW41] and (independently) by Montroll [Mon41], on an effort to compute the partition function of the Ising model. In a very famous work published by Lars Onsager in 1944 [Ons44], the transfer matrix method was generalized to the two-dimensional lattice and was employed to successfully compute the partition function for the first neighbors Ising model. As a byproduct, he obtained the critical point at which the model passes through a phase transition. These two historical and remarkable chapters of the theory of transfer operators are related to the study of their actions on finite-dimensional vector spaces.

In a seminal paper in 1968, David Ruelle [Rue68] introduced the transfer operator for an one-dimensional statistical mechanics model with infinite range

interactions. This paved the way to the study of transfer operators in infinite-dimensional vector spaces. In this paper, Ruelle proved the existence and uniqueness of the Gibbs measure for a lattice gas system with a potential depending on infinitely many coordinates.

Nowadays, the transfer operators are called Ruelle operators (mainly in Thermodynamic Formalism) and play an important role in Dynamical Systems and Mathematical Statistical Mechanics. They are actually useful tools in several other branches of mathematics.

Roughly speaking, the famous Ruelle-Perron-Frobenius Theorem states that the Ruelle operator for a potential with a certain regularity, acting on a suitable Banach space, has a unique simple positive eigenvalue (equal to the spectral radius) and associated to it a positive eigenfunction. For Hölder continuous potentials the proof of this theorem can be found in [Bal00, PP90, Rue68]. In 1978, Walters obtained the Ruelle-Perron-Frobenius Theorem for a more general setting [Wal78], allowing expansive and mixing dynamical systems together with potentials with summable variation.

The Ruelle operator was successfully used to study the problem of existence and uniqueness of equilibrium states, introduced in [Rue67, Wal75], for a very general class of potentials f , see also [Led74]. Under some regularity conditions on f one can show the uniqueness of the equilibrium states, see [Bal00, Bow08, PP90, Rue68, Rue04, Wal01, Wal05, Wal07] and references therein. Some important properties of the equilibrium probability can be derived from the Ruelle operator and this operator turns out to be a very important tool on topological dynamics and differentiable dynamical systems, with applications to the study of invariant measures for an Anosov diffeomorphism [Bow08, Sin72] and the meromorphy of Zelberg's zeta function [Rue02].

The so called DLR Gibbs measures were introduced in 1968 and 1969 independently by Dobrushin [Dob68] and Lanford and Ruelle [LR69]. The abstract formulation in terms of specifications was developed five years later in [Dob70, Föl75, Pre76]. An important stage in the development of the theory was established by the works of Preston [Pre76] and Gruber, Hintermann and Merlini [GHM77] around 1977, Ruelle (1978) [Rue78] and Israel (1979) [Isr79]. Preston's work was more focused on the abstract measure theory, while Gruber et al. concentrated on specific methods for Ising type models, Israel dealt with the variational principle and Ruelle worked towards Gibbsian formalism in Ergodic Theory.

Dobrushin began the study of non-uniqueness of the DLR Gibbs measure and proposed its interpretation as a phase transition. He proved the famous Dobrushin Uniqueness Theorem in 1968, ensuring the uniqueness of the Gibbs measures for a very general class of interactions at very high temperatures ($\beta \ll 1$). This result, together with the rigorous proof of non-uniqueness of the Gibbs measures for the two-dimensional Ising model at low temperatures, is a great triumph

of the DLR approach in the study of phase transition in Statistical Mechanics. Some accounts of the general results on the Gibbs Measure theory (from the Statistical Mechanics' viewpoint) can be found in [Bov06, Ell06, vEFS93, FV17, Geo11, Rue78].

In Section 2 of [Sar09] the author introduces a concept of DLR-Gibbs measure in the context of topological Markov shifts. Afterwards the concept and existence of Thermodynamic Limit were discussed in such context. Here the definitions of DLR-Gibbs measures and Thermodynamic Limit are similar to the ones considered in [Sar09]. We shall remark that in reference [Sar09] (see Definition 1.4) the concept of Gibbs measure is considered in the sense of Bowen. Here we will not work with this concept of Gibbs measure.

The present work aims to explain how to use DLR-Gibbs measures to obtain the conformal measures considered in Thermodynamic Formalism. In order to do that we introduce a notion of specification associated to continuous potential. In particular, we show how to construct an absolutely uniformly summable specification for any Hölder potential and use this construction to motivate the specifications considered here. The main results of this paper are Theorems A and B, in Section 6, which prove the equivalence between the conformal measures considered in Thermodynamic Formalism and DLR-Gibbs measures, for potentials in the Walters space.

The Preprint [CL16] approaches similar problems (as described here) but in different setting. For example, potentials can be continuous functions and the alphabet can be any compact metric space (which includes uncountable alphabets). But, on the other hand, the strong equivalence proved here in Theorem B is no longer true in this setting.

2 Ruelle Operator and Conformal Measures

In this paper \mathbb{N} denotes the set of positive integers, \mathcal{A} is a finite alphabet and $\Omega \equiv \mathcal{A}^{\mathbb{N}}$ denotes the symbolic space endowed with its standard metric d given by $d(x, y) = 2^{-N}$, where $N = \inf\{i \in \mathbb{N} : x_i \neq y_i\}$. The Borel σ -algebra of Ω is denoted by \mathcal{F} . The dynamics here is given by $\sigma : \Omega \rightarrow \Omega$, the left-shift mapping. The space of all real continuous bounded functions on Ω endowed with its standard supremum norm $\|\cdot\|_{\infty}$ is denoted simply by $C(\Omega)$. We use the notation $\mathcal{P}(\Omega) \equiv \{\nu : \mathcal{F} \rightarrow [0, 1] : \nu \text{ is a probability measure}\}$ for the set of all Borel probability measures over Ω .

Definition 1 (Ruelle Operator). Let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function. The Ruelle operator associated to f , notation $\mathcal{L}_f : C(\Omega) \rightarrow C(\Omega)$ is defined on the function ψ as follows

$$\mathcal{L}_f(\psi)(x) = \sum_{y \in \Omega; \sigma(y)=x} \exp(f(y)) \psi(y).$$

Normally we call f a potential and \mathcal{L}_f the transfer operator associated to the potential f . The dual of the Ruelle operator \mathcal{L}_f^* acts on the set of Borel finite signed measures over Ω as follows $\mathcal{L}_f^*(\nu)(\psi) = \nu(\mathcal{L}_f(\psi))$ for all $\psi \in C(\Omega)$.

Fix $0 < \alpha < 1$. We say that a function $f : \Omega \rightarrow \mathbb{R}$ is α -Hölder continuous if

$$\text{Hol}_\alpha(f) \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} < +\infty.$$

The space of all real α -Hölder continuous functions on Ω is denote by $C^\alpha(\Omega)$. When we say that f is Hölder continuous function we mean $f \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$. For any $n \geq 1$ we define the n -th variation of a function $f : \Omega \rightarrow \mathbb{R}$ by $\text{var}_n(f) = \sup \{|f(x) - f(y)| : x, y \in \Omega \text{ and } x_i = y_i \text{ for all } 1 \leq i \leq n\}$. We say that a function $f : \Omega \rightarrow \mathbb{R}$ is in the **Walters** space, notation $W(\Omega)$, if the following condition is satisfied

$$\lim_{p \rightarrow \infty} \sup_{n \geq 1} \text{var}_{n+p}(S_n(f)) = 0, \quad \text{where } S_n(f) = f + \dots + f \circ \sigma^{n-1}. \quad (1)$$

We remark that for any $0 < \alpha < 1$ we have $C^\alpha(\Omega) \subset W(\Omega) \subset C(\Omega)$.

Theorem 2 (Ruelle-Perron-Frobenius (RPF) for Walters Potentials). *Let f be a potential in $W(\Omega)$. Then there exists a strictly positive function $\psi_f \in W(\Omega)$ and a strictly positive eigenvalue λ_f such that $\mathcal{L}_f(\psi_f) = \lambda_f \psi_f$. The eigenvalue λ_f is simple and it is equal to the spectral radius of the operator. Moreover, there exists a unique probability measure ν_f over Ω such that $\mathcal{L}_f^*(\nu_f) = \lambda_f \nu_f$.*

Proof. For a proof see [Bou01, Wal01, Wal05, Wal07]. □

Definition 3. Let $f \in C(\Omega)$ a continuous potential and $\rho(\mathcal{L}_f)$ the spectral radius of \mathcal{L}_f acting on $C(\Omega)$. The set of all Borel probability measures ν over Ω , satisfying $\mathcal{L}_f^* \nu = \rho(\mathcal{L}_f) \nu$ is denoted by $\mathcal{G}^*(f)$.

Note that if $f \in W(\Omega)$, then follows from Theorem 2 that $\rho(\mathcal{L}_f) = \lambda_f$ and

$$\mathcal{G}^*(f) = \{\nu \in \mathcal{P}(\Omega) : \mathcal{L}_f^* \nu = \lambda_f \nu\}$$

is a singleton.

3 Interactions and Continuous Potentials

In the classical literature on Statistical Mechanics the concept of interaction is prominent. In what follows we described it but only in the generality needed in this paper. For a comprehensive exposition on this topic see [Geo11].

From now on the notation $A \Subset \mathbb{N}$ means that A is an empty or finite subset of \mathbb{N} . If for each $A \Subset \mathbb{N}$ we associated a function $\Phi_A : \Omega \rightarrow \mathbb{R}$ then we have a family

of functions defined on Ω and indexed on the finite parts of \mathbb{N} . We denote such family simply by $\Phi = \{\Phi_A\}_{A \in \mathbb{N}}$ and Φ will be called an interaction. We shall remark that is usual Φ to have several finite subsets A 's for which the associated function Φ_A is identically zero.

The space of interactions has natural structure of a vector space where the sum of two interactions Φ and Ψ , is given by the interaction $(\Phi + \Psi) \equiv \{\Phi_A + \Psi_A\}_{A \in \mathbb{N}}$ and $\lambda\Phi = \{\lambda\Phi_A\}_{A \in \mathbb{N}}$, for any $\lambda \in \mathbb{R}$. This vector space is too big for our purposes so we focus in a proper subspace of it.

Before proceed we shall remark that we can also consider interactions defined on a general countable set V . If $V = \mathbb{Z}$, for example, then the family Φ is now indexed over the collection of all $A \subseteq \mathbb{Z}$. In this case we say that the interaction is defined on the lattice \mathbb{Z} . We focus here on interactions Φ defined on the lattice \mathbb{N} , in order to relate the DLR-Gibbs measures and the Thermodynamic Formalism.

Definition 4 (Uniformly Absolutely Summable Interaction). An interaction $\Phi = \{\Phi_A\}_{A \in \mathbb{N}}$ is called *uniformly absolutely summable* (UAS) interaction if it satisfies:

1. for each $A \in \mathbb{N}$ the function $\Phi_A : \Omega \rightarrow \mathbb{R}$ depends only on the coordinates with indexes in A ;
2. $\|\Phi\| \equiv \sup_{n \in \mathbb{N}} \sum_{A \in \mathbb{N}; A \ni n} \sup_{x \in \Omega} |\Phi_A(x)| < \infty$.

Example 5 (Dyson Model on \mathbb{N}). Consider the alphabet $\mathcal{A} = \{-1, 1\}$ and a fixed $\alpha > 1$. Then the interaction Φ given by

$$\Phi_A(x) = \begin{cases} \frac{x_n x_m}{|n - m|^\alpha}, & \text{if } A = \{n, m\} \text{ and } m \neq n; \\ 0, & \text{otherwise,} \end{cases}$$

is an UAS interaction. In fact, for any $A \in \mathbb{N}$ we have that $\Phi_A \equiv 0$ if $\#A \neq 2$. On the other hand, if $A = \{m, n\}$ with $m \neq n$ we have that Φ_A depends only on the coordinates x_n and x_m . The regularity condition is verified as follows

$$\|\Phi\| \equiv \sup_{n \in \mathbb{N}} \sum_{A \in \mathbb{N}; A \ni n} \sup_{x \in \Omega} |\Phi_A(x)| = \sup_{n \in \mathbb{N}} \sum_{m \in \mathbb{N} \setminus \{n\}} \sup_{x \in \Omega} \frac{|x_n x_m|}{|m - n|^\alpha} \leq 2\zeta(\alpha),$$

where ζ is the Riemann zeta function.

In order to state our next result we introduce some notations. Let x, y and $z \in \Omega$ and $n, m \in \mathbb{N}$. We use the notation $x_1^n y_{n+1}^{n+m} z_{n+m+1}^\infty$ to denote a point in Ω , whose its coordinates are $(x_1, \dots, x_n, y_{n+1}, \dots, y_{n+m}, z_{n+m+1}, \dots)$. For each $k \geq 1$ and $n \geq 0$ consider the arithmetic progression $A(k, n) \equiv \{k, \dots, 2k + n\}$. For each $f \in C(\Omega)$ and $y \in \Omega$ we define $f_{A(k, n)} : \Omega \rightarrow \mathbb{R}$ as follows

$$f_{A(k, n)}(x) = f(x_k^{2k+n} y_{2k+n+1}^\infty) - f(x_k^{2k+n-1} y_{2k+n}^\infty)$$

if $n \geq 1$ and $f_{A(k, 0)}(x) = f(x_k^{2k} y_{2k+1}^\infty) - f(y)$.

Lemma 1. Let $f \in C(\Omega)$, $y \in \Omega$ and $\Phi^f \equiv \{\Phi_A^f\}_{A \in \mathbb{N}}$ be the interaction given by $\Phi_A^f(x) = f_{A(k,n)}(x)$ if $A = A(k, n)$ and 0 otherwise. Then for all $x \in \Omega$ we have

$$f(x) = f(y) + \sum_{A \in \mathbb{N}; A \ni 1} \Phi_A^f(x).$$

Proof. For any $n \geq 0$ we have $\sum_{j=0}^n \Phi_{A(1,j)}^f(x) = (f(x_1^2 y_3^\infty) - f(y)) + (f(x_1^3 y_4^\infty) - f(x_1^2 y_3^\infty)) + \dots + (f(x_1^{n+2} y_{n+3}^\infty) - f(x_1^{n+1} y_{n+2}^\infty)) = f(x_1^{n+2} y_{n+3}^\infty) - f(y)$. From the continuity of f and the previous equation the lemma follows. \square

Proposition 1. Let $0 < \alpha < 1$, $f \in C^\alpha(\Omega)$ and Φ^f as in previous lemma. Then Φ^f is an UAS interaction.

Proof. Note that for all $n \geq 1$ we have

$$\sum_{A \in \mathbb{N}; A \ni n} \|\Phi_A^f(x)\|_\infty \leq \sum_{k=1}^n \sum_{m=k+1}^\infty \|\Phi_{A(k,m)}^f(x)\|_\infty$$

Since $f \in C^\alpha(\Omega)$, for all $k \in \{1, \dots, n\}$ and $m \geq k+1$ we have $\|\Phi_{A(k,m)}^f(x)\|_\infty \leq \text{Hol}_\alpha(f) 2^{-\alpha(k+m-1)}$. Therefore

$$\|\Phi^f\| \equiv \sup_{n \in \mathbb{N}} \sum_{A \in \mathbb{N}; A \ni n} \|\Phi_A^f\|_\infty \leq \frac{2^\alpha \text{Hol}_\alpha(f)}{(1 - 2^{-\alpha})^2}. \quad \square$$

Let Φ be a UAS interaction. For each $n \in \mathbb{N}$ let $\Lambda_n \equiv \{1, \dots, n\}$. The function

$$H_n(x) = \sum_{\substack{A \in \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi_A(x) \quad (2)$$

is called the Hamiltonian associated to the interaction Φ in the volume Λ_n . In Mathematical Statistical Mechanics the Gibbs measures (called here DLR-Gibbs measures) associated to an interaction is normally constructed by means of $(H_n)_{n \geq 1}$. Before explain this construction we obtain a formula for H_n when $\Phi \equiv \Phi^f$ is a UAS interaction.

Proposition 2. Let $f \in C(\Omega)$ and assume that Φ^f defined as in Lemma 1 is a UAS interaction. Then, there is a constant C so that for all $n \in \mathbb{N}$ the Hamiltonian H_n defined by (2) satisfies

$$H_n(x) = f(x) + \dots + f(\sigma^{n-1}x) + nC.$$

Proof. From the definition of Φ^f and the UAS property we have

$$H_n(x) = \sum_{\substack{A \in \mathbb{N} \\ A \cap \Lambda_n \neq \emptyset}} \Phi_A^f(x) = \sum_{k=1}^n \sum_{m=0}^{\infty} \Phi_{A(k,m)}(x).$$

By using similar argument as in Lemma 1 we can prove that the inner sum in rhs above is given by $f(\sigma^{k-1}x) - f(y)$. By taking $C \equiv f(y)$ and then summing the last expression with k varying from 1 to n the proposition follows. \square

Aiming to have an equivalent description of conformal measures associated to a Walters potential f and the DLR-Gibbs measure associated to Φ^f we develop below the theory of DLR-Gibbs measures (within our setting) dispensing the UAS hypothesis.

4 Specifications and DLR-Gibbs Measures

From now on, the Hamiltonian H_n is assumed to be of form

$$H_n(x) = f(x) + f(\sigma x) + \dots + f(\sigma^{n-1}x) + nC, \quad (3)$$

where $f \in C(\Omega)$. In this section we extend some classical results about Gibbsian specifications to the case where the Hamiltonian has the above form. The motivation to extend the DLR theory on this direction becomes natural in view of the results of the previous section and this extension is crucial to show the equivalence stated in Theorem B.

Lemma 2. *For any $n, r \in \mathbb{N}$, x, y and $z \in \Omega$ we have*

$$H_{n+r}(y_1^{n+r} z_{n+r+1}^\infty) - H_n(y_1^{n+r} z_{n+r+1}^\infty) = H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty) - H_n(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty).$$

Proof. From definition of H_n follows that

$$H_{n+r}(y_1^{n+r} z_{n+r+1}^\infty) - H_n(y_1^{n+r} z_{n+r+1}^\infty) = \sum_{j=n}^{n+r-1} f(\sigma^j(y_1^{n+r} z_{n+r+1}^\infty)).$$

Since rhs above is equals to $H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty) - H_n(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty)$ the lemma is proved. \square

Definition 6. Given a continuous potential f we define a family of probability kernels $(K_n)_{n \geq 1}$, where for each $n \in \mathbb{N}$ the kernel $K_n : \mathcal{F} \times \Omega \rightarrow \mathbb{R}$ is given by

$$K_n(F, y) = \frac{1}{Z_n^y} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} 1_F(x) \exp(H_n(x)), \text{ where } Z_n^y = \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} \exp(H_n(x)).$$

Note that the constant C in (3) is irrelevant for the definition of K_n , therefore, without loss of generality, we can assume that $C = 0$. Let $\pi_n : \Omega \rightarrow \mathcal{A}$ be the canonical projection in the n -th coordinate and \mathcal{T}_n the sigma-algebra generated by the projections $\{\pi_j : j \geq n+1\}$. Then for any $f \in C(\Omega)$ and for all $n \in \mathbb{N}$ it is easy to see that the kernel K_n satisfies:

- a) $y \mapsto K_n(F, y)$ is \mathcal{T}_n -measurable;
- b) $F \mapsto K_n(F, y)$ is a Borel probability measure;
- c) $y \mapsto \int_{\Omega} g(x) dK_n(x, y)$ is continuous for any $g \in C(\Omega)$.

Theorem 7 (Compatibility Conditions). *If $(K_n)_{n \geq 1}$ is a family of probability kernels as in Definition 6, then for each fixed $z \in \Omega$ and for any integers $r, n \geq 1$ we have*

$$\int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] dK_{n+r}(y, z) = \int_{\Omega} g(y) dK_{n+r}(y, z), \quad \forall g \in C(\Omega).$$

Proof. Follows from the definition of K_n that for any $g \in C(\Omega)$

$$\begin{aligned} \int_{\Omega} g(x) dK_n(x, (y_1^{n+r} z_{n+r+1}^{\infty})) &= \frac{1}{Z_n^{(y_1^{n+r} z_{n+r+1}^{\infty})}} \sum_{\substack{x \in \Omega \\ \sigma^n(x) = (y_{n+1}^{n+r} z_{n+r+1}^{\infty})}} g(x) \exp(H_n(x)) \\ &\equiv h(y_1^{n+r} z_{n+r+1}^{\infty}). \end{aligned}$$

We are using above the notation $h(y_1^{n+r} z_{n+r+1}^{\infty})$ for the sake of compatibility, but note that this quantity do not depends on y_1, \dots, y_n .

Therefore to prove the theorem is enough to show that

$$\frac{1}{Z_{n+r}^z} \sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} h(y) \exp(H_{n+r}(y)) = \frac{1}{Z_{n+r}^z} \sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} g(y) \exp(H_{n+r}(y)).$$

Since $Z_{n+r}^z > 0$, the above equation is equivalent to

$$\sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} h(y) \exp(H_{n+r}(y)) = \sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} g(y) \exp(H_{n+r}(y)). \quad (4)$$

In order to prove the theorem we show in the sequel that (4) holds. Indeed, from the definition of h , we have that the l.h.s above is given by

$$\sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} \frac{1}{Z_n^{(y_1^{n+r} z_{n+r+1}^{\infty})}} \sum_{\substack{x \in \Omega \\ \sigma^n(x) = (y_{n+1}^{n+r} z_{n+r+1}^{\infty})}} g(x) \exp(H_n(x) + H_{n+r}(y)).$$

From Lemma 2 follows that the above expression is equal to

$$\sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} \frac{\exp(H_n(y_1^{n+r} z_{n+r+1}^\infty))}{Z_n^{(y_1^{n+r} z_{n+r+1}^\infty)}} \sum_{\substack{x \in \Omega \\ \sigma^n(x) = (y_{n+1}^{n+r} z_{n+r+1}^\infty)}} g(x) \exp(H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty)).$$

Note that the above expression is equals to

$$\sum_{y_1, \dots, y_{n+r} \in \mathcal{A}} \frac{\exp(H_n(y_1^{n+r} z_{n+r+1}^\infty))}{Z_n^{(y_1^{n+r} z_{n+r+1}^\infty)}} \sum_{x_1, \dots, x_n \in \mathcal{A}} g(x) \exp(H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty)).$$

By interchanging summation order we can rewrite the above expression as

$$\sum_{y_{n+1}, \dots, y_{n+r} \in \mathcal{A}} \sum_{y_1, \dots, y_n \in \mathcal{A}} \frac{\exp(H_n(y_1^{n+r} z_{n+r+1}^\infty))}{Z_n^{(y_1^{n+r} z_{n+r+1}^\infty)}} \sum_{x_1, \dots, x_n \in \mathcal{A}} g(x) \exp(H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty)).$$

Since the third sum above do not depend on y_1, \dots, y_n and

$$\sum_{y_1, \dots, y_n \in \mathcal{A}} \frac{\exp(H_n(y_1^{n+r} z_{n+r+1}^\infty))}{Z_n^{(y_1^{n+r} z_{n+r+1}^\infty)}} = 1$$

the previous expression is equal to

$$\sum_{\substack{y_{n+1}, \dots, y_{n+r} \in \mathcal{A} \\ x_1, \dots, x_n \in \mathcal{A}}} g(x) \exp(H_{n+r}(x_1^n y_{n+1}^{n+r} z_{n+r+1}^\infty)) = \sum_{\substack{y \in \Omega \\ \sigma^{n+r}(y) = \sigma^{n+r}(z)}} g(y) \exp(H_{n+r}(y)).$$

The last expression shows that (4) holds and the theorem is proved. \square

Notice that the collection $(K_n)_{n \in \mathbb{N}}$ is similar to but not exactly a quasilocal specification as in the literature of Mathematical Statistical Mechanics, see for example [vEFS93, Geo11, Pre76]. It is possible to extend this collection to a classical quasilocal specification, but the point here is to obtain similar results to the classical theory of DLR-Gibbs measures in this more general setting. For the extension argument see [CL16].

Proposition 3. *Let $(K_n)_{n \geq 1}$ be as in Definition 6 and $z \in \Omega$ a fixed point. If the sequence $K_{n_j}(\cdot, z) \rightarrow \mu^z$ (weak-* topology), when $j \rightarrow \infty$, then for any continuous function $g : \Omega \rightarrow \mathbb{R}$, we have*

$$\int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] d\mu^z(y) = \int_{\Omega} g d\mu^z.$$

Proof. For any fixed $n \in \mathbb{N}$, the mapping

$$\Omega \ni y \longmapsto \int_{\Omega} g(x) dK_n(x, y)$$

is continuous. From the compatibility condition and the definition of weak-* topology follows that

$$\begin{aligned} \int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] d\mu^z(y) &= \lim_{j \rightarrow \infty} \int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] dK_{n_j}(y, z) \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} g(y) dK_{n_j}(y, z) \\ &= \int_{\Omega} g d\mu^z. \end{aligned} \quad \square$$

Definition 8 (DLR-Gibbs Measures). Let $(K_n)_{n \in \mathbb{N}}$ be as in Definition 6. The set of DLR Gibbs measures associated to a continuous potential f is defined as

$$\mathcal{G}^{DLR}(f) \equiv \left\{ \mu \in \mathcal{P}(\Omega) : \begin{array}{l} \mu(F|\mathcal{T}_n)(y) = K_n(F, y) \text{ for } \mu - \text{a.a. } y, \\ \forall F \in \mathcal{F} \text{ and } \forall n \in \mathbb{N} \end{array} \right\}.$$

The DLR equations play an important role in Statistical Mechanics.

Theorem 9 (DLR-equations). *Let $(K_n)_{n \in \mathbb{N}}$ be as in Definition 6. A Borel probability measure $\mu \in \mathcal{P}(\Omega)$ belongs to $\mathcal{G}^{DLR}(f)$ iff for all $n \in \mathbb{N}$ and any continuous function $g : \Omega \rightarrow \mathbb{R}$, we have*

$$\int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] d\mu(y) = \int_{\Omega} g d\mu.$$

Proof. We follow closely the reference [Geo11]. Suppose that $\mu \in \mathcal{G}^{DLR}(f)$ then it follows from the definition of $\mathcal{G}^{DLR}(f)$ and the basic properties of the conditional expectation that for all $n \in \mathbb{N}$ we have

$$\int_{\Omega} g d\mu = \int_{\Omega} \mu(g|\mathcal{T}_n)(y) d\mu(y) = \int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] d\mu(y).$$

Conversely, we assume that the DLR-equations are valid for all $n \in \mathbb{N}$ and for any continuous function g . Let $g = 1_E h$, where $E \in \mathcal{T}_n$ is a cylinder set and h is an arbitrary continuous function. Then g is continuous and

$$\int_{\Omega} 1_E(y) \left[\int_{\Omega} h(x) dK_n(x, y) \right] d\mu(y) = \int_{\Omega} \left[\int_{\Omega} 1_E(x) h(x) dK_n(x, y) \right] d\mu(y) = \int_E h d\mu,$$

where in the first equality we used that the function 1_E do not depends on its n first coordinates and definition of $K_n(\cdot, y)$. From the Dominate Convergence Theorem follows that the class of E 's satisfying the above identity is a monotone class, and from Monotone Class Theorem follows that the above identity holds for any measurable set $E \in \mathcal{T}_n$. Since the mapping

$$y \mapsto \int_{\Omega} h(x) dK_n(x, y)$$

is \mathcal{T}_n -measurable and $E \in \mathcal{T}_n$ is an arbitrary measurable set, we have, from the definition of conditional expectation and last equality, that

$$\int_{\Omega} h(x) dK_n(x, y) = \mu(h|\mathcal{T}_n)(y) \quad \mu \text{ a.e.}$$

Using again the Dominate Convergence Theorem for conditional expectation and Monotone Class Theorem we can show that the above equality holds for $h = 1_F$ where F is a measurable set in \mathcal{F} , so the result follows. \square

From item c) that appears before Theorem 7 and DLR-equations follows that $\mathcal{G}^{DLR}(f)$ is a closed subset of $\mathcal{P}(\Omega)$, with respect to the weak-* topology. Since $\mathcal{P}(\Omega)$ endowed with this topology is a compact Hausdorff space follows that $\mathcal{G}^{DLR}(f)$ is compact.

Let $f \in C(\Omega)$ and $(K_n)_{n \in \mathbb{N}}$ as in Definition 6. For each $y \in \Omega$ we define \mathcal{C}_y as being the set of all the cluster points, in the weak-* topology, of the set $\{K_n(\cdot, y) : n \geq 1\}$. We call $\mu \in \mathcal{C}_y$ a Thermodynamic Limit obtained from the boundary condition y .

Definition 10. The closure, in the weak-* topology, of the convex hull of the set $\cup_{y \in \Omega} \mathcal{C}_y$ will be denoted by $\mathcal{G}^{TL}(f)$.

Proposition 4. For any $f \in C(\Omega)$ we have that the set $\mathcal{G}^{TL}(f)$ is always non-empty. Moreover, $\mathcal{G}^{TL}(f) \subset \mathcal{G}^{DLR}(f)$.

Proof. For any compact metric space Ω we have that $\mathcal{P}(\Omega)$ is compact, with respect to the weak-*. Since this topology is metrizable follows that $\mathcal{P}(\Omega)$ is sequentially compact. Therefore the subset $\{K_n(\cdot, y) : n \geq 1\} \subset \mathcal{P}(\Omega)$ has at least one cluster point μ^y , thus proving that $\mathcal{G}^{TL}(f) \neq \emptyset$. The inclusion is straightforward application of Proposition 3 and Theorem 9. \square

Examples where one can get different Thermodynamic Limits $\mu \in \mathcal{C}_y$ depending of the boundary condition y appear in [CL15].

5 Specifications and Ruelle Operator

In this section we establish relevant relations, in this work, between the kernels $(K_n)_{n \in \mathbb{N}}$ given by Definition 6 and Ruelle operator \mathcal{L}_f .

We first recall that the n -th iterated of Ruelle operator applied to any $\psi \in C(\Omega)$ and calculated at y is given by the following formula

$$\mathcal{L}_f^n(\psi)(y) = \sum_{\substack{x \in \Omega; \\ \sigma^n(x)=y}} \exp(S_n(f)(x))\psi(x).$$

Proposition 5. *Let $f \in C(\Omega)$. For any cylinder set $F \in \mathcal{F}$ and $n \in \mathbb{N}$ we have*

$$K_n(F, y) = \frac{1}{Z_n^y} \sum_{\substack{x \in \Omega; \\ \sigma^n(x)=\sigma^n(y)}} 1_F(x) \exp(H_n(x)) = \frac{\mathcal{L}_f^n(1_F)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))}.$$

Proof. The first equality is simply definition of K_n . From definition we have $H_n(x) = S_n(f)(x)$ so the second equality above follows from the formula for the n -th iterated of Ruelle operator since

$$\mathcal{L}_f^n(1_F)(\sigma^n(y)) = \sum_{\substack{x \in \Omega; \\ \sigma^n(x)=\sigma^n(y)}} \exp(S_n(f)(x))1_F(x) = \sum_{\substack{x \in \Omega; \\ \sigma^n(x)=\sigma^n(y)}} 1_F(x) \exp(H_n(x))$$

and

$$\mathcal{L}_f^n(1)(\sigma^n(y)) = \sum_{\substack{x \in \Omega; \\ \sigma^n(x)=\sigma^n(y)}} \exp(S_n(f)(x)) = Z_n^y. \quad \square$$

Lemma 3. *Let f be a continuous potential. For all $n, m \in \mathbb{N}$, $z \in \Omega$ and $\psi \in C(\Omega)$ we have*

$$\mathcal{L}_f^{n+m}(\psi)(\sigma^{n+m}(z)) = \mathcal{L}_f^{n+m} \left(\frac{\mathcal{L}_f^n(\psi)(\sigma^n(\cdot))}{\mathcal{L}_f^n(1)(\sigma^n(\cdot))} \right) (\sigma^{n+m}(z)).$$

Proof. The proof follows from Proposition 5 and Theorem 7 (compatibility conditions for $(K_n)_{n \in \mathbb{N}}$). \square

6 Main Results

Lemma 4. *Let $f \in W(\Omega)$ and $(K_n)_{n \in \mathbb{N}}$ as in Definition 6. Given $g \in C(\Omega)$ and $\varepsilon > 0$ there is $n_0 \equiv n_0(f, g) \in \mathbb{N}$ such that if $n \geq n_0$ then*

$$\sup_{y, z \in \Omega} \left| \int_{\Omega} g(x) dK_n(x, y) - \int_{\Omega} g(x) dK_n(x, z) \right| = \mathcal{O}(\varepsilon).$$

Proof. Given $\varepsilon > 0$, follows from the Walters condition (1) that there is $n_1 \in \mathbb{N}$ so that if $n \geq n_1$, then $|S_n(f)(x_1^n y_{n+1}^\infty) - S_n(f)(x_1^n z_{n+1}^\infty)| \leq \log(1 + \varepsilon)$, for all x, y and $z \in \Omega$. Therefore

$$-\log(1 + \varepsilon) \leq S_n(f)(x_1^n y_{n+1}^\infty) - S_n(f)(x_1^n z_{n+1}^\infty) \leq \log(1 + \varepsilon)$$

which implies that

$$(1 + \varepsilon)^{-1} \leq \frac{\exp(S_n(f)(x_1^n y_{n+1}^\infty))}{\exp(S_n(f)(x_1^n z_{n+1}^\infty))} \leq 1 + \varepsilon.$$

From the above inequality follows that $(1 + \varepsilon)^{-1} Z_n^z \leq Z_n^y \leq (1 + \varepsilon) Z_n^z$. Since g is a continuous function and its domain Ω is a compact set follows that g is uniformly continuous, and so there is $n_2 \in \mathbb{N}$ such that if $n \geq n_2$ then $|g(x_1^n z_{n+1}^\infty) - g(x_1^n y_{n+1}^\infty)| < \varepsilon$, for all x, y and $z \in \Omega$. For all $n \geq n_0 \equiv \max\{n_1, n_2\}$ we have

$$\begin{aligned} \int_{\Omega} g(x) dK_n(x, z) &= \frac{1}{Z_n^z} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(z)}} g(x) \exp(H_n(x)) \\ &\leq \frac{(1 + \varepsilon)^2}{Z_n^y} \sum_{\substack{x \in \Omega; \\ \sigma^n(x) = \sigma^n(y)}} (g(x) + \varepsilon) \exp(H_n(x)) \\ &= (1 + \varepsilon)^2 \int_{\Omega} g(x) dK_n(x, y) + (1 + \varepsilon)^2 \varepsilon \\ &= \int_{\Omega} g(x) dK_n(x, y) + \mathcal{O}(\varepsilon). \end{aligned}$$

By a similar reasoning we obtain the reverse inequality. \square

Corollary 1. *Let $f \in W(\Omega)$ and $(K_n)_{n \in \mathbb{N}}$ as in Definition 6. If $(y_n)_{n \in \mathbb{N}}$ is a sequence in Ω such that $y_n \rightarrow y^*$ and $K_n(\cdot, y_n) \rightarrow \tilde{\nu}$, then $K_n(\cdot, y^*) \rightarrow \tilde{\nu}$.*

Proof. For any fixed $g \in C(\Omega)$ we have

$$\begin{aligned} \left| \int_{\Omega} g(x) dK_n(x, y^*) - \int_{\Omega} g(x) d\tilde{\nu}(x) \right| &\leq \left| \int_{\Omega} g(x) dK_n(x, y^*) - \int_{\Omega} g(x) dK_n(x, y_n) \right| \\ &\quad + \left| \int_{\Omega} g(x) dK_n(x, y_n) - \int_{\Omega} g(x) d\tilde{\nu}(x) \right|. \end{aligned}$$

Given $\varepsilon > 0$ follows from Lemma 4 that the first term in rhs above is smaller than ε if n is large enough. The second term can also be made smaller than ε since $K_n(\cdot, y_n) \rightarrow \tilde{\nu}$. \square

Theorem A. *Let f be a continuous potential and $(K_n)_{n \in \mathbb{N}}$ as in Definition 6. Then $\mathcal{G}^{DLR}(f) = \mathcal{G}^{TL}(f)$.*

Proof. The inclusion $\mathcal{G}^{TL}(f) \subset \mathcal{G}^{DLR}(f)$ is the content of Proposition 4. Suppose by contradiction that there exists $\mu \in \mathcal{G}^{DLR}(f)$ which is not in $\mathcal{G}^{TL}(f)$. By using the compactness of $\mathcal{G}^{DLR}(f)$ and the classical hyperplane separation theorem we can ensure the existence of a continuous function $g : \Omega \rightarrow \mathbb{R}$ and $\epsilon > 0$ such that

$$\int_{\Omega} g d\mu < \int_{\Omega} g d\nu - \epsilon, \quad \forall \nu \in \mathcal{G}^{TL}(f).$$

From Theorem 9, for any $n \in \mathbb{N}$, we have

$$\int_{\Omega} \left[\int_{\Omega} g(x) dK_n(x, y) \right] d\mu(y) = \int_{\Omega} g d\mu.$$

Therefore, for each $n \in \mathbb{N}$, we have from the previous inequality that there is $y_n \in \Omega$ such that

$$\int_{\Omega} g(x) dK_n(x, y_n) < \int_{\Omega} g d\nu - \epsilon.$$

Up to subsequences, we can suppose that $K_n(\cdot, y_n) \rightharpoonup \tilde{\nu}$ and $y_n \rightarrow y^*$. From Corollary 1 follows that $K_n(\cdot, y^*) \rightharpoonup \tilde{\nu}$ and consequently $\tilde{\nu} \in \mathcal{G}^{TL}(f)$ which is contradiction, thus showing that $\mathcal{G}^{DLR}(f) = \mathcal{G}^{TL}(f)$. \square

Theorem B. *If $f \in W(\Omega)$ then $\mathcal{G}^{TL}(f) = \mathcal{G}^{DLR}(f) = \mathcal{G}^*(f)$.*

Proof. If $f \in W(\Omega)$ then we know that $\mathcal{G}^*(f)$ is a singleton ([Wal07]), $\#\mathcal{G}^{TL}(f) \geq 1$ and $\mathcal{G}^{TL}(f) = \mathcal{G}^{DLR}(f)$ (Theorem A) so it is enough to prove that $\mathcal{G}^{TL}(f) \subset \mathcal{G}^*(f)$. From Proposition 5 we have for any $g \in C(\Omega)$ and $y \in \Omega$ fixed

$$\frac{\mathcal{L}_f^n(g)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))} = \int_{\Omega} g(x) dK_n(x, y).$$

Assume K_n converges, up to a subsequence, to some probability measure ν . Then

$$\int_{\Omega} g d\nu = \lim_{n \rightarrow \infty} \int_{\Omega} g(x) dK_n(x, y) = \lim_{n \rightarrow \infty} \frac{\mathcal{L}_f^n(g)(\sigma^n(y))}{\mathcal{L}_f^n(1)(\sigma^n(y))} = \int_{\Omega} g d\nu_f,$$

where the above limit is computed in [Wal07] and $\nu_f \in \mathcal{G}^*(f)$. Since the function $g \in C(\Omega)$ in above equation is arbitrary, follows that $\nu = \nu_f$, thus finishing the proof. \square

7 Ising Model and Walters Condition

In this section we briefly discuss the long-range Ising model in Thermodynamic Formalism setting and apply the above results to ensure the uniqueness of the DLR-Gibbs measures of this model.

The long-range Ising model on the lattice \mathbb{N} with $1/r^{2+\varepsilon}$ interaction energy, is usually defined by the means of the interaction Φ of Example 5 with $\alpha = 2 + \varepsilon$, i.e.,

$$\Phi_A(x) = \begin{cases} \frac{x_n x_m}{|n - m|^{2+\varepsilon}}, & \text{if } A = \{n, m\} \text{ and } m \neq n; \\ 0, & \text{otherwise.} \end{cases}$$

A straightforward computation shows that the potential $f : \Omega \rightarrow \mathbb{R}$ given by

$$f(x) = \sum_{n \geq 2} \frac{x_1 x_n}{(n-1)^{2+\varepsilon}}$$

is according to Lemma 1 the potential corresponding to Φ . It is simple to show that f is not α -Hölder continuous for any $0 < \alpha < 1$. On the other hand, we have that f is in the Walters class for any $\varepsilon > 0$. Indeed, it is easy to see that for $n, p \in \mathbb{N}$ we have

$$\text{var}_{n+p}(S_n(f)) = (n+p)^{-2-\varepsilon+1} + (n+p-1)^{-2-\varepsilon+1} + \dots + p^{-2-\varepsilon+1}.$$

Therefore, for p fixed, we have

$$\sup_{n \in \mathbb{N}} \text{var}_{n+p}(S_n(f)) \leq \text{const.} \sum_{j=p}^{\infty} j^{-2-\varepsilon+1} \leq \text{const.} p^{-\varepsilon},$$

which proves that the potential f is in Walters space.

Now, we can apply Theorem B to ensure that this Ising model has a unique DLR-Gibbs measure and therefore it has no phase transition in the sense of multiples DLR-Gibbs measures.

8 Concluding Remarks

In this paper we have compared the definitions of Gibbs measures in terms of the Ruelle operator and specifications. We show how to obtain for potentials in the Walters and Hölder class the Gibbs measures usually considered in the Thermodynamic Formalism via the DLR formalism and prove that the measures obtained from both approaches are the same.

Both approaches have their advantages. For example, using the Ruelle operator we were able to prove some uniform convergence theorems for $K_n(\cdot, y)$.

The literature about absolutely uniformly summable interactions is vast and this approach allow us to consider non translation invariant potentials and other lattices than \mathbb{N} . We also show that the long-range Ising model on \mathbb{N} can be studied using the Ruelle operator, at least when the interaction energy is of the form $1/r^\alpha$ with $\alpha > 2$. In these cases, we have proved that the unique Gibbs measure of this model satisfies $\mathcal{G}^{DLR}(\Phi) = \mathcal{G}^{TL}(\Phi) = \mathcal{G}^*(f)$, but on the other hand, it is not clear how to treat the cases $1 < \alpha \leq 2$ by using the Ruelle operator and what kind of information is obtainable through this approach. It is worth pointing out that treating this model with the DLR approach is fairly standard, so the connection made here suggests that more understanding of the DLR Specification theory can shed light on more general spaces where one can efficiently use the Ruelle Operator. Another important feature of the DLR-measure Theory is that it is also readily applicable to standard Borel spaces, which includes compact and non-compact spaces [Geo11]. Some of the results obtained here can be extended to compact and metric alphabets, but measurability issues have to be taken into account and some theorems requires different approach, although the main ideas are contained here, see [CL16].

Acknowledgments

We would like to express our thanks to Aernout van Enter, Anthony Quas, José Siqueira and Rodrigo Bissacot for their suggestions and comments on an early version of this paper. The authors thanks CNPq and FEMAT for the financial support.

References

- [Bal00] V. Baladi. *Positive transfer operators and decay of correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
- [Bou01] T. Bousch. La condition de Walters. *Ann. Sci. École Norm. Sup. (4)*, 34(2):287–311, 2001.
- [Bov06] A. Bovier. *Statistical mechanics of disordered systems*, volume 18 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2006. A mathematical perspective.
- [Bow08] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.

- [CL15] L. Cioletti and A. O. Lopes. Phase transitions in one-dimensional translation invariant systems: a Ruelle operator approach. *J. Stat. Phys.*, 159(6):1424–1455, 2015.
- [CL16] L. Cioletti and A. O. Lopes. Ruelle operator for continuous potentials and DLR-Gibbs measures. *Preprint arXiv:1608.03881*, 2016.
- [Dob68] R. L. Dobrushin. Description of a random field by means of conditional probabilities and conditions for its regularity. *Teor. Veroyatnost. i Primenen.*, 13:201–229, 1968.
- [Dob70] R. L. Dobrushin. Prescribing a system of random variables by conditional distributions. *Theory of Probability & Its Applications*, 15(3):458–486, 1970.
- [Ell06] R. S. Ellis. *Entropy, large deviations, and statistical mechanics*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.
- [Föll75] H. Föllmer. Phase transition and Martin boundary. pages 305–317. *Lecture Notes in Math.*, Vol. 465, 1975.
- [FV17] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction*. Cambridge University Press, To appear 2017.
- [Geo11] H-O. Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2011.
- [GHM77] C. Gruber, A. Hintermann, and D. Merlini. *Group analysis of classical lattice systems*. Springer-Verlag, Berlin-New York, 1977. With a foreword by Ph. Choquard, *Lecture Notes in Physics*, Vol. 60.
- [Isr79] Robert B. Israel. *Convexity in the theory of lattice gases*. Princeton University Press, Princeton, N.J., 1979. Princeton Series in Physics, With an introduction by Arthur S. Wightman.
- [KW41] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. I. *Phys. Rev. (2)*, 60:252–262, 1941.
- [Led74] F. Ledrappier. Principe variationnel et systèmes dynamiques symboliques. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 30:185–202, 1974.

- [LR69] O. E. Lanford, III and D. Ruelle. Observables at infinity and states with short range correlations in statistical mechanics. *Comm. Math. Phys.*, 13:194–215, 1969.
- [Mon41] E. W. Montroll. Statistical mechanics of nearest neighbor systems. *The Journal of Chemical Physics*, 9(9):706–721, 1941.
- [Ons44] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, 65:117–149, 1944.
- [PP90] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188):268, 1990.
- [Pre76] C. Preston. *Random fields*. Lecture Notes in Mathematics, Vol. 534. Springer-Verlag, Berlin-New York, 1976.
- [Rue67] D. Ruelle. A variational formulation of equilibrium statistical mechanics and the Gibbs phase rule. *Comm. Math. Phys.*, 5:324–329, 1967.
- [Rue68] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Comm. Math. Phys.*, 9:267–278, 1968.
- [Rue78] D. Ruelle. *Thermodynamic formalism*, volume 5 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1978. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota.
- [Rue02] D. Ruelle. Dynamical zeta functions and transfer operators. *Notices Amer. Math. Soc.*, 49(8):887–895, 2002.
- [Rue04] D. Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.
- [Sar09] O. Sarig. Lecture notes on thermodynamic formalism for topological markov shifts. *Penn State*, 2009.
- [Sin72] Ja. G. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [vEFS93] A. C. D. van Enter, R. Fernández, and A. D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory. *J. Statist. Phys.*, 72(5-6):879–1167, 1993.

- [Wal75] P. Walters. A variational principle for the pressure of continuous transformations. *Amer. J. Math.*, 97(4):937–971, 1975.
- [Wal78] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. *Trans. Amer. Math. Soc.*, 236:121–153, 1978.
- [Wal01] P. Walters. Convergence of the Ruelle operator for a function satisfying Bowen’s condition. *Trans. Amer. Math. Soc.*, 353(1):327–347, 2001.
- [Wal05] P. Walters. Regularity conditions and Bernoulli properties of equilibrium states and g -measures. *J. London Math. Soc. (2)*, 71(2):379–396, 2005.
- [Wal07] P. Walters. A natural space of functions for the Ruelle operator theorem. *Ergodic Theory Dynam. Systems*, 27(4):1323–1348, 2007.