Applications of Variable Discounting Dynamic Programming to Iterated Function Systems and Related Problems

L. Cioletti
Departamento de Matemática - UnB
70910-900, Brasília, Brazil
cioletti@mat.unb.br

Elismar R. Oliveira
Departamento de Matemática - UFRGS
91509-900, Porto Alegre, Brazil
elismar.oliveira@ufrgs.br

Abstract

We study existence and uniqueness of the fixed points solutions of a large class of non-linear variable discounted transfer operators associated to a sequential decision-making process. We establish regularity properties of these solutions, with respect to the immediate return and the variable discount. In addition, we apply our methods to reformulating and solving, in the setting of dynamic programming, some central variational problems on the theory of iterated function systems, Markov decision processes, discrete Aubry-Mather theory, Sinai-Ruelle-Bowen measures, fat solenoidal attractors, and ergodic optimization.

1 Introduction

The abstract theory of dynamic programming (DP for short) is a powerful tool for analysis of decision-making problems. This paper aims to strength some of fundamental theorems in this theory in order to prove new results on existence and uniqueness of variational problems, arising in Ergodic Theory and Iterated Function Systems, within a unified framework.

2010 *Mathematics Subject Classification*: 37Axx; 37Dxx; 49Lxx
Keywords: Bellman-Hamilton-Jacobi equations, Dynamic Programming, Thermodynamic Formalism, Ergodic Theory, Transfer Operator, Eigenfunctions.

As motivation and to illustrate the applicability of our theorems, we explain below how to reformulate some of very important variational problems on:

- decision problems for iterated function systems (IFS);
- Markov decision process;
- discrete Aubry-Mather theory;
- Sinai-Ruelle-Bowen (SRB) measures and fat solenoidal attractors;
- ergodic optimization;

in the language of DP so that their solutions can be obtained by straightforward applications of our main results.

Since the pioneering work of Bellman [Bel57] the application of this theory has been growing fast, and nowadays it is a well developed subject and a standard tool for some researchers in pure and applied Mathematics. It has also been used in engineering problems, optimal control theory and machine learning, just to name a few, see [BC88, Ber13, BM84, CM91, CV87, dVVM$^+$08, GAGK12, JMN14b, Liu01, TBS10] and references therein.

Successful applications of this theory in Dynamical Systems were obtained by the so-called *discounted methods*. In [Bou01] this method is applied to several problems on thermodynamic formalism as well as in the study of maximizing measures (ergodic optimization) for expanding endomorphisms on metric spaces. In [BCL$^+$11, LMMS15] this method was adapted to study Statistical Mechanics models in one-dimensional one-sided lattices. In these works, the authors proved existence of particular discounted limits of solutions of the Bellman equation and obtain the maximal eigenvalue and a positive eigenfunctions of the Ruelle operator, and subactions.

Infinite dimensional linear DP problems are considered in [Gom05], [Gom08], [BG10] and [GO12] in both discrete and continuous setting. In these works a connection between DP and the theory of viscosity solutions [Fat97a, Fat97b, Fat98a, Fat98b] are explored to obtain new results on Aubry-Mather problem [Mn92, Mn96] and related topics as the Hamilton-Jacobi equation.

Here we extended the recently developed theory of variable discount in DP [JMN14b] to broaden its range of applications. Two central problems in our paper are the following ones. Given a sequential decision-making process $S = \{X, A, \Psi, f, u, \delta\}$ (Definition 2.1), we study the existence and uniqueness of the fixed point solutions of the variable discounted Bellman's equation

$$v(x) = \sup_{a\in\Psi(x)} u(x,a) + \delta(v(f(x,a)))$$

as well as the fixed point solutions of the variable discounted transfer operator

$$w(x) = \ln \int_{a\in\Psi(x)} \exp\big(\, u(x,a) + \delta(w(f(x,a)))\,\big)\, d\nu_x(a).$$

In addition, regularity properties of the solutions v and w, with respect to u and the variable discount δ are determined.

After discussing some results on variable discount, we present new results on aggregator function associated to the Ruelle operator. We believe that our results about discounted limits, in Section 3.3, provide truly new insights into the behavior of decision-making problems. These insights are clear when the variable discount function vanishes, because it allows the *future rewards* function play a major role. It is remarkable fact that the Bellman equation survives on this general setting and produces a new equation capable of explaining the behavior of decision-making problems.

In our opinion this paper will be of potential interest to the readers working on variational problems in Dynamical Systems such as ergodic optimization, thermodynamical formalism, Aubry-Mather theory, Lagrangian mechanics, Hamilton-Jacobi equations via viscosity solutions, etc. Nonetheless, some methods presented here can be useful in Analysis, random dynamics and many other related fields.

In what follows, we explain within DP framework the statement of some central problems on the topics mentioned in the beginning of this section. Before proceed, we shall introduce some basic notations. Here $X = (X, d)$ always denotes a complete metric space and $C(X, \mathbb{R})$, $C_b(X, \mathbb{R})$ stands for the space of all real continuous and real bounded continuous functions on X, respectively. Of course, if X is compact then $C(X, \mathbb{R}) = C_b(X, \mathbb{R})$. Both $C(X, \mathbb{R})$ and $C_b(X, \mathbb{R})$ are endowed with their standard supremum norm and regarded as Banach spaces. The space of all Borel probability measures over X is denoted by $\mathscr{P}(X)$. If X is a compact space and $T : X \to X$ is a continuous mapping, then we denote by $\mathscr{P}_T(X)$ the space of all T-invariant Borel probability measures defined over X. These spaces are endowed with their standard weak-$*$ topology.

Decision Problems for IFS

A deterministic decision problem controlled by an IFS $\{\phi_a : X \to X,\, a \in A\}$ can be described as follows. The state of the system at time n is a point $x_n \in X$ and determined by the following rules. We give an initial state x_0. At each discrete time $n \geq 0$, a point $a \in A$ (the set of possible actions) is chosen (by some agent) and the state changes from x_n to a new state $x_{n+1} := \phi_a x_n$. There is a reward, given by a real valued function $c(x_n, a)$, associated to taking action a, when system is in the state x_n and also a discount factor $0 < \lambda < 1$, which represents the relevance of the first choices. In this setting, an infinite horizon decision problem takes the form

$$V(x_0) \;=\; \sup\left\{ \sum_{n=0}^{\infty} \lambda^n c(x_n, a_n) :\; (a_0, a_1, \ldots) \in A^{\mathbb{N}} \text{ and } \; x_{n+1} = \phi_{a_n} x_n \right\}$$

The dynamic programming theory explains how to break this decision problem into smaller subproblems, leading to Bellman's principle of optimality

$$V(x) = \max_{a \in A} \left\{ c(x,a) + \lambda V(\phi_a x) \right\},$$

known as Bellman's equation.

Markov Decision Process (MDP)

An example of stochastic dynamic programming problem is a Markov Decision Process (MDP) controlled by an IFS $\{\phi_a : X \to X,\, a \in A\}$. A sample of this decision process is a feasible history $(x_0, a_0, x_1, a_1, \ldots)$, where $x_{n+1} = \phi_{a_n} x_n$. In this setting, we fix an ordered quadruple (X, A, p, r), where X is a set of states, A is a set of available actions, p is a probability measure such that $p(x_{n+1} = \phi_{a_n} x_n | x_n = x, a_n = a)$ is the probability that x evolves from $x_n = x$ to the state $x_{n+1} = \phi_a x$, by taking the action $a_n = a$, and r is a function $r : X \times A \to \mathbb{R}$, where $r(x,a)$ is a reward for taking action a at the state x. Note that in this context, the aggregation function will be a random variable.

A central problem in MDP is to find a *policy* for the decision maker, which is a function $\pi : X \to A^{\mathbb{N}}$, specifying the actions $(a_0, a_1, \ldots)$ that should be taken, when the system is in the state x. The goal is to find a policy π maximizing the expected discounted sum, over an infinite horizon

$$\mathbb{E}[\sum_{k=0}^{\infty} \lambda^k r(x_k, a_k)],$$

where the expectation is taken with respect to the law of the Markov chain defined by the above transition rates, and λ is the discount factor satisfying $0 < \lambda < 1$ and $x_{k+1} = \phi_{a_k} x_k$, $x_0 = x$. When there exists a solution $V(x) = \max_\pi \mathbb{E}[\sum_{k=0}^{\infty} \lambda^k c(x_k, y_k)]$ for this problem it satisfies the stochastic discounted Bellman equation

$$V(x) = \max_{a \in A} \left\{ c(x,y) + \lambda \mathbb{E}[V(\phi_a x)] \right\}.$$

For a comprehensive survey on MDP, see [Put94].

Discrete Aubry-Mather Problem

In Lagrangian Mechanics, the Aubry-Mather problem [Mn92, Mn96] consists in finding probability measures defined on the tangent fiber bundle TM of a manifold M that minimizes the action of a convex and superlinear Lagrangian $L : TM \to \mathbb{R}$, of class C^2, that is,

$$\inf_\mu \int_{TM} L(x,v)\, d\mu(x,v).$$

In [Gom05] the author considers the case $M = \mathbb{T}^n$, the n-dimensional torus and the dynamics $f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ given by $f(x,v) = x+v$. Define the discrete differential operator with respect to f, acting on a function $g \in C(\mathbb{T}^n, \mathbb{R})$ as follows $d_x g(v) := g(f(x,v)) - g(x)$. The minimization is taken over the set of holonomic probability measures

$$\mathcal{H} := \left\{ \mu \in \mathscr{P}(TM) \,\middle|\, \int_{TM} d_x g(v)\, d\mu(x,v) = 0, \quad \forall g \in C(\mathbb{T}^n, \mathbb{R}) \right\}.$$

By Fenchel-Rockafellar duality theorem, see [Roc66] and [Gom05], we have

$$-\inf_{\mu \in \mathcal{H}} \int_{TM} L(x,v)\, d\mu(x,v) = \inf_{g \in C(\mathbb{T}^n,\mathbb{R})} \sup_{(x,v) \in \mathbb{T}^n \times \mathbb{R}^n} -d_x g(v) - L(x,v).$$

This problem is related to one of finding the solutions of the discrete Hamilton-Jacobi-Bellman equation

$$\overline{H} = \sup_{v \in \mathbb{R}^n} -d_x g(v) - L(x,v),$$

commonly solved by using viscosity solutions methods, which is a dynamic programming problem associated to Bellman's operator

$$T_\alpha(u) = \inf_{v \in \mathbb{R}^n} e^{-\alpha} u(f(x,v)) + L(x,v),$$

for $\alpha > 0$. This operator defines a uniform contraction on a suitable Banach space and its unique fixed point is the unique viscosity solution of the Bellman's equation

$$u_\alpha(x) = \inf_{v \in \mathbb{R}^n} e^{-\alpha} u_\alpha(f(x,v)) + L(x,v).$$

In [Gom05] it is shown that $u_\alpha(x) - \min u_\alpha \to u(x)$ and $(1 - e^{-\alpha}) \min u_\alpha \to \overline{H}$, when $\alpha \to 0$, and furthermore it is shown that the limit function u satisfies the equation $u(x) = \inf_{v \in \mathbb{R}^n} u(x+v) + L(x,v) + \overline{H}$, that is,

$$\overline{H} = \sup_{v \in \mathbb{R}^n} u(x) - u(f(x,v)) - L(x,v) = \sup_{v \in \mathbb{R}^n} -d_x u(v) - L(x,v) = H(x, d_x u),$$

where the Hamiltonian H is the Legendre transform of $-L$. Actually, in [Gom05], the discount is $T_\alpha(u) = e^{-\alpha} \inf_{v \in \mathbb{R}^n} u(f(x,v)) + L(x,v)$, but the reasoning is exactly the same in both cases.

SRB-measures and Fat Solenoidal Attractors

Sums controlled by IFS are also used to characterize the boundary of attractors, of certain skew maps, and to show when the SRB-measures are absolutely continuous. We recall that a skew map is a map $F : X \times Y \to X \times Y$ of the form $F(x,y) = (F_1(x), F_2(x,y))$, where F_1 is a self-map of X. In [Tsu01], the author study the attractor of the map $F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ given by $F(x,y) = (T(x), \lambda y + f(x))$, where $T(x) = 2x \mod 1$, $y \in \mathbb{R}$ and $f : \mathbb{S}^1 \to \mathbb{R}$ is a C^2 potential.

For a fixed $a = (a_0, a_1, \ldots) \in \{0,1\}^{\mathbb{N}}$ define $\phi_{k,a}x := \phi_{a_{k-1}} \circ \phi_{a_{k-2}} \circ \ldots \circ \phi_{a_0}x$, where ϕ_i, $i = 0, 1$, are the inverse branches of $T(x) = 2x \mod 1$. A straightforward computation shows that for any $n \in \mathbb{N}$ we have

$$F^n(\phi_{n,a}x, y) = (x, \lambda^n y + \lambda^0 f(\phi_{a_0}x) + \cdots + \lambda^n f(\phi_{a_n} \cdots \phi_{a_0}x)).$$

The expression in rhs above lead us naturally to consider the discounted controlled sums given by $S(x, a) := \sum \lambda^k f(\phi_{k,a}x)$. In [Tsu01] (see also [BKRLU06] for topological properties of the attractor) the author gives a description of the SRB measure, by analyzing $S(x, a)$ and conjectured that the optimal return function $\sup_a S(x, a)$ can be used to describe the boundary of the attractor. This conjecture was partially solved in [LO14], assuming that the potential f satisfies a certain twist condition. A natural question arises when we change the skew map F by a non uniform hyperbolic one, with variable discount such as $G(x, y) = (T(x), \ln(1 + y) + f(x))$ (note that $\{1, 2\}$ is always contained in the spectrum of $DG(x, 0)$). This situation requires a variable discounted dynamic programming approach.

Ergodic Optimization

A central problem in ergodic optimization consists in finding an optimal invariant measure attaining the supremum

$$m = \sup_{\mu \in \mathscr{P}_T(X)} \int_X f \, d\mu,$$

where (X, d) is a metric space, $T : X \to X$ is a continuous transformation and $f : X \to \mathbb{R}$ is a given potential.

For example, in case where $X = \mathbb{R}/\mathbb{Z}$ and the transformation $T : X \to X$ is the double mapping, the ergodic optimization problem can be viewed as a decision problem for IFS as follows. We take the IFS $\{\phi_0, \phi_1\}$, where $\phi_0 x = (1/2)x$ and $\phi_1 x = (1/2)x + 1/2$, the set of possible actions is $A = \{0, 1\}$ and the immediate return $c(x, a) := f(\phi_a x)$.

Under fairly general conditions on the potential f, we can prove several theorems about the support of maximizing measures. For example, the solutions of Bellman's equation

$$b(x) = \max_{a \in A} \{f(\phi_a x) + \lambda \, b(\phi_a x)\},$$

can be characterized if the potential f satisfies a twist condition. By taking the limit when $\lambda \to 1$, we obtain a subaction V satisfying

$$V(x) = \max_{a \in A} \{f(\phi_a x) - m + V(\phi_a x)\}.$$

The support of a maximizing measure ν, notation $\operatorname{supp} \nu$, is contained in the set where we have the equality in the above expression, see [Bou01, Gar17] and the recent survey [Jen18].

2 Sequential Decision-Making Processes

In this section we introduce very general setting to handle some variational problems in DP. The applications discussed here will be obtained by considering additional regularity conditions and specializing the spaces, functions and so on. Our starting point will be the following definition.

Definition 2.1 (Sequential Decision-Making Process). *A sequential decision-making process is an ordered sextuple $S = \{X, A, \Psi, f, u, \delta\}$, where*

- *X is a complete metric space, called state space;*
- *A is a general metric space, called set of all available actions;*
- *$\Psi : X \to 2^A$ is a set-valued function. For all $x \in X$ the set $\Psi(x) \subseteq A$ is always assumed to be a non-empty compact set and called the set of all feasible actions for an agent x. We shall assume that Ψ is continuous, with respect to the Hausdorff topology on the not-empty compact subsets of A.*
- *$f : X \times A \to X$ is a continuous map, called transition law for the system;*
- *$u : X \times A \to \mathbb{R}$ is continuous function and $u(x, a)$ is called the immediate reward or return associated with taking the action a in the state x;*
- *$\delta : D \subset \mathbb{R} \to \mathbb{R}$, is an increasing continuous function called discount function. It represents the relevance of taking an action at the next step.*

Although linear discount, by a factor $\beta \in (0, 1)$, can be employed to solve several problems in DP, it may not be a suitable tool to handle some other complicated problems. A natural alternative would be consider variable discount factor or even a variable discount function. In order to give a precise definition of this concept, let us introduce the notion of a generalized modulus of contraction.

2.1 Variable Discount Functions

Definition 2.2. *A generalized modulus of contraction for a function $\delta : D \subset \mathbb{R} \to \mathbb{R}$ is an increasing function $\gamma : [0, \infty) \to [0, \infty)$ such that for all $t \geq 0$ the n-th iterate $\gamma^n(t) \to 0$, when $n \to \infty$, and*

$$|\delta(t_2) - \delta(t_1)| \leq \gamma(|t_2 - t_1|)$$

for any $t_1, t_2 \in D$.

Any function γ as above satisfies $\gamma(0) = 0$. Indeed, if $\gamma(0) = \gamma_0 > 0$ the monotonicity of γ implies that $\liminf \gamma^n(t) \geq \gamma_0 > 0$. By using a similar reasoning, we can prove that $\gamma(t) < t$, for all $t > 0$.

Definition 2.3 (Variable discount function). *A function $\delta : D \subset \mathbb{R} \to \mathbb{R}$ will be called a variable discount function if it has a generalized modulus of contraction $\gamma : [0, \infty) \to [0, \infty)$. A variable discount function δ is called*

a) idempotent if $\delta = \gamma$, for some generalized modulus of contraction γ;

b) subadditive if $\delta(t_1 + t_2) \leq \delta(t_1) + \delta(t_2)$*, for any* $t_1, t_2 \in D$ *such that* $t_1 + t_2 \in D$.

Proposition 2.4. *Let* $\delta : [0, \infty) \to [0, \infty)$ *be a continuous and increasing function satisfying* $\delta^n(t) \to 0$*, when* $n \to \infty$*. If* δ *is subadditive then it is idempotent.*

Proof. From subadditivity we get, for any pair $x, y \in [0, \infty)$ satisfying $x \geq y$, the following inequality

$$\delta(x) - \delta(y) = \delta(x - y + y) - \delta(y) \leq \delta(x - y) + \delta(y) - \delta(y) \leq \delta(x - y).$$

Similarly, we obtain $\delta(y) - \delta(x) \leq \delta(y - x)$, for $y \geq x$. Since δ is an increasing function we get that $|\delta(x) - \delta(y)| \leq \delta(|x - y|)$. By taking $\gamma = \delta$, it follows from the hypothesis that δ is itself a modulus of contraction for δ. □

Example 2.5. *Given* $\beta \in (0, 1)$*, the function* $\delta(t) := \beta t$ *for* $t \in \mathbb{R}$ *is a idempotent discount function, because it is linear. This is the canonical discount function used in dynamic programming.*

Example 2.6. *The function* $\delta(t) := \ln(1+t)$ *for* $t \geq 0$ *is a nonlinear idempotent discounted function. Indeed,*

$$|\delta(t_1) - \delta(t_2)| \leq |\ln\left(\frac{1 + t_1}{1 + t_2}\right)| \leq \ln\left(1 + \frac{|t_1 - t_2|}{1 + t_2}\right) \leq \ln(1 + |t_1 - t_2|)$$

and $0 < \gamma'(t) = 1/(1 + t) < 1$*, for all* $t > 0$ *so* $\gamma^n(t) \to 0$*, when* $n \to \infty$*. Therefore* $\gamma(t) = \delta(t)$ *is a generalized modulus of contraction. Note that* δ *is also subadditive. Indeed,* $\delta(t_1+t_2) \leq \ln(1+(t_1+t_2)) \leq \ln(1+(t_1+t_2)+(t_1 \cdot t_2)) = \ln((1 + t_1)(1 + t_2)) = \delta(t_1) + \delta(t_2)$.

Example 2.7. *A piecewise linear function* $\delta_1 : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\delta_1(t) := \begin{cases} \beta t, & t \leq 1; \\ \frac{\beta}{2}t + \frac{\beta}{2}, & t > 1, \end{cases}$$

where $\beta \in (0, 1)$ *is also a variable discount function with the same generalized contraction modulus as* $\delta(t) := \beta t$*. Additionally,* δ_1 *is an example of subadditive but not idempotent variable discount function.*

Example 2.8. *Consider the function* $\delta : [0, \infty) \to [0, \infty)$ *defined by*

$$\delta(t) = -1 + \sqrt{t + 1}.$$

We have that $\delta(0) = 0$ *and* δ *is an increasing function and for all* $t_1, t_2 \in [0, \infty)$*, we have*

$$|\delta(t_1) - \delta(t_2)| = \left|\sqrt{t_1 + 1} - \sqrt{t_2 + 1}\right| = \left|\frac{(t_1 + 1) - (t_2 + 1)}{\sqrt{t_1 + 1} + \sqrt{t_2 + 1}}\right| \leq \frac{1}{2}|t_1 - t_2|.$$

By taking $\beta = 1/2 \in (0, 1)$*, we can show that* δ *is a variable discount function with the same generalized contraction modulus as* $\gamma(t) := \beta t$*.*

Example 2.9. *If $\delta : [0, \infty) \to [0, \infty)$ is a C^2-function such that:*

a) $\delta(0) = 0$*;*

b) $\delta'(0^+) := \lim_{t \downarrow 0} \delta'(t) = \beta \in (0, 1)$ *and* $\delta'(t) > 0$*;*

c) $\delta''(t) \leq 0$*.*

Then δ is increasing and $|\delta(t_1) - \delta(t_2)| = \delta'(t_0)|t_1 - t_2| \leq \delta'(0^+)|t_1 - t_2|$, since $\delta'(t_0) \leq \delta'(0^+)$. Thus $\gamma(t) := \beta t$ is a generalized contraction modulus for δ. Note that for every fixed $p > 1$ the function $\delta(t) = -1 + (t + 1)^{1/p}$, satisfies conditions a)–c) with $\delta'(0^+) = 1/p$. This generalizes the Example 2.8, when $p = 2$.

The main reason to consider such general variable discounts is to develop a perturbation theory. The idea is to consider a parametric family of discounts $\delta_n : [0, +\infty) \to \mathbb{R}$, where $\delta_n(t) \to I(t) = t$, in the pointwise topology, and then to study the properties of possible limits, when $n \to \infty$, of the fixed points $v_n(x)$ and $w_n(x)$. In this regard, we consider sequences of variable discount decision-making process $(S_n)_{n \in \mathbb{N}}$, where $S_n = \{X, A, \Psi, f, u, \delta_n\}$ is defined by a continuous and bounded immediate reward $u : X \times A \to \mathbb{R}$ and sequence of discounts $(\delta_n)_{n \geq 0}$, satisfy some admissibility conditions:

a) the contraction modulus γ_n of the variable discount δ_n is also a variable discount function;

b) $\delta_n(0) = 0$ and $\delta_n(t) \leq t$, for $t \geq 0$.

c) $\delta_n(t + \alpha) - \delta_n(t) \to \alpha$, when $n \to \infty$, uniformly in $t > 0$, for any fixed constant $\alpha \geq 0$.

In Section 3.3 we prove two of the main results of this paper which are Theorem 3.24, ensuring the existence of a value $\bar{u} \in [0, \|u\|_\infty]$ and a function h such that

$$h(x) = \max_{a \in \Psi(x)} u(x, a) - \bar{u} + h(f(x, a)),$$

and Theorem 3.28 which guarantees the existence of a value $k \in [0, \|u\|_\infty]$ and a function h given by

$$h(x) = \ln \int_{a \in \Psi(x)} e^{u(x,a) + h(f(x,a)) - k} \, d\nu_x(a),$$

such that $\rho := e^k$ and $\varphi := e^{h(x)}$ are the maximal eigenvalue and eigenfunction of the Ruelle operator, that is,

$$e^k e^{h(x)} = \int_{a \in \Psi(x)} e^{u(x,a)} e^{h(f(x,a))} d\nu_x(a).$$

For both results the key hypothesis in u are uniformly δ-boundedness and uniformly δ-domination, see Definition 3.19.

Regarding this hypothesis on u, we want to stress that we prove in Theorem 3.20 that if f is a contractive dynamics , that is,

$$\sup_{a\in A} d_X(f(x,a), f(y,a)) \leq \lambda d_X(x,y)$$

and $u(\cdot, a)$ is C-Lipschitz (or α-Hölder) then u is uniformly δ-dominated. If additionally, $\mathrm{diam}(X) < \infty$, then u is uniformly δ-bounded. In particular, if v_n and w_n are respectively the solutions of Bellman's equation and the transfer discounted operator equation, they are uniformly $C(1-\lambda)^{-1}$-Lipschitz (or α-Hölder, with $\mathrm{Hol}_\alpha(v_n) = \mathrm{Hol}_\alpha(w_n) = \mathrm{Hol}_\alpha(u)(1-\lambda^\alpha)^{-1}$). This shows that most of the previous results in the literature for IFS or expanding maps, with either Lipschitz or Hölder weights are particular cases of our theorems, with constant discounts satisfying $\delta_n(t) = \beta_n t$, where $0 < \beta_n < 1$ and $\beta_n \to 1$.

2.2 Generalized Matkowski Contraction Theorem

In 1975, Janusz Matkowski [Mat75], obtained a generalization of Banach's contraction theorem for a variable contraction map. Before state this result we need one more definition.

Definition 2.10. *Let (X, d) be a complete metric space and $T : X \to X$ a map. We say that T is a generalized Matkowski contraction, if there exists a witness function for T, that is, a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi^n(t) \to 0$, when $n \to \infty$ and*

$$d(T(x), T(y)) \leq \varphi(d(x,y))$$

for any $x, y \in X$.

When the contraction is not fixed, e.g. $d(T(x), T(y)) \leq \lambda d(x,y)$, the function φ witness the fact that T is a generalized contraction, e.g. $d(T(x), T(y)) \leq \varphi(d(x,y))$. In other words, it is not enough to say that T is a generalized contraction, we need a witness φ.

Theorem 2.11 ([Mat75]). *If (X, d) is a complete metric space and $T : X \to X$ a generalized Matkowski contraction, then there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$ and $d(T^n(x), x_0) \to 0$ for all $x_0 \in X$.*

A weaker version of this theorem was known. It required the witness function φ to be right USC (instead of non-decreasing) and $\varphi(t) < t$, for all $t > 0$ (instead of $\varphi^n(t) \to 0$, when $n \to \infty$). However, the set of all contractions where Theorem 2.11 works is wider than this one, as pointed by Matkowski, we may apply the theorem for a map T, having a witness function

$$\varphi(t) := \begin{cases} 1 & , t > 1; \\ \frac{1}{n+1} & , \frac{1}{n+1} < t \leq \frac{1}{n}; \\ 0 & , t = 0, \end{cases}$$

which is not a right USC function.

2.3 Variable Discounting in Dynamic Programming

This section is devoted to present some results of the recent theory developed by Jaśkiewicz, Matkowski and Nowak [JMN13, JMN14a, JMN14b]. The applications in these works focused on Markov decision processes, and the theory of optimal economic growth and resource extraction models, but as will be explained below it has far-reaching consequences.

We shall consider a sequential decision-making process $S = \{X, A, \Psi, f, u, \delta\}$ as a dynamical system specified as follows: at the state x_0 we take an action a_0, and receive an immediate return $u(x_0, a_0)$ and go forward to the new state $x_1 = f(x_0, a_0)$. Based on it, one decides to take a new action $a_1 \in \Psi(x_1)$ and so on. In this way we obtain a feasible sequence $(x_0, a_0, x_1, a_1, \ldots) \in (X \times A)^{\mathbb{N}}$ which is a orbit of the dynamical system S.

Definition 2.12. *The set of all the feasible sequences of a sequential decision-making process $S = \{X, A, \Psi, f, u, \delta\}$ is given by*

$$\Omega := \{(x_0, a_0, x_1, a_1, \ldots) \in (X \times A)^{\mathbb{N}} |\, x_{i+1} = f(x_i, a_i),\ a_i \in \Psi(x_i)\}.$$

Typically, the above defined set is strictly contained in the Cartesian product, that is, $\Omega \subsetneq (X \times A)^{\mathbb{N}}$, unless $\Psi(x) = A$, for all $x \in X$ and f is surjective. It is useful to define the set of all feasible action sequences starting from x_0,

$$\Pi(x_0) = \{\bar{a} = (a_i) \in A^{\mathbb{N}} \mid (x_0, a_0, x_1, a_1, \ldots) \in \Omega\}.$$

We point out that an element in Ω depends only on the initial point x_0 and on a feasible action sequence $\bar{a} \in \Pi(x_0)$, so we can use a concise notation:

$$h_{x_0}(\bar{a}) = (x_0, a_0, x_1, a_1, \ldots) \in \Omega.$$

Proposition 2.13. *The set $\Omega \subset (X \times A)^{\mathbb{N}}$ is closed relative to the product topology on $(X \times A)^{\mathbb{N}}$.*

Proof. Since X is complete and $\Psi(x)$ is compact, for all $x \in X$, we can obtain, by an inductive argument, a feasible sequence in $\Pi(\lim_i x_0^i)$ for any Cauchy sequence $(h_{x_0^i}(\bar{a}^i))_{i \geq 0}$. □

Remark 2.14. *An alternative way to define the space Ω is to introduce it as the set $\Omega' := \{(x, \Pi(x)) \mid x \in X\}$. The set Ω' is like a fiber bundle and has a natural structure of metric space*

$$d_{\Omega'}((x, \bar{a}), (y, \bar{b})) := d_X(x, y) + d_A(\bar{a}, \bar{b})$$

Thus $(\Omega', d_{\Omega'})$ is a complete metric space and the topology is equivalent to the product topology.

Definition 2.15. *Let $u : X \times A \to \mathbb{R}$ be a bounded from above function. A recursive utility associated to the immediate rewards $u(x, a)$ with a discount function δ is a function $U : \Omega \to \mathbb{R} \cup \{-\infty\}$, such that*

$$U(h_{x_0}(\bar{a})) = u(x_0, a_0) + \delta(U(h_{x_1}\sigma\bar{a}))$$

for any history $h_{x_0}(\bar{a})$.

Definition 2.16. *Let $u(x,a)$ be an immediate rewards and δ a discount function. We define, for any history $h_{x_0}(\bar{a})$, the associated inductive limit*

$$\sum_i^\star u(x_i,a_i) = \lim_{n\to\infty} \sum_{i\in[n]}^\star u(x_i,a_i),$$

where

$$\sum_{i\in[n]}^\star u(x_i,a_i) = u(x_0,a_0) + \delta\big(u(x_1,a_1) + \delta(u(x_2,a_2) + \ldots + \delta(u(x_n,a_n)))\big)$$

and the notation $[n]$ stands for the interval $\{0,1,\ldots,n\}$ in the set of integers numbers.

Corollary 2.22 provides necessary conditions to ensure the existence of the above limit.

The connection between Definitions 2.15 and 2.16 is given by the next proposition.

Proposition 2.17. *Let $u(x,a)$ be a bounded from above immediate rewards and δ a continuous discount function. If $\sum_i^\star u(x_i,a_i)$ converges then the function $V:\Omega\to\mathbb{R}\cup\{-\infty\}$ defined by $V(h_{x_0}(\bar{a})) = \sum_i^\star u(x_i,a_i)$ is a recursive utility. Reciprocally, if $U:\Omega\to\mathbb{R}\cup\{-\infty\}$ is a bounded from above recursive utility then U is represented by $U(h_{x_0}(\bar{a})) = \sum_i^\star u(x_i,a_i)$.*

Proof. For the first part we define $V(h_{x_0}(\bar{a})) = \sum_i^\star u(x_i,a_i)$. A simple computation shows that

$$\sum_{i\in[n]}^\star u(x_i,a_i) = u(x_0,a_0) + \delta\Big(\sum_{i\in[n-1]}^\star u(x_{i+1},a_{i+1})\Big).$$

Using the continuity of δ and taking the limit we obtain

$$V(h_{x_0}(\bar{a})) = u(x_0,a_0) + \delta(V(h_{x_1}\sigma\bar{a})).$$

Therefore, V is a recursive utility.

Reciprocally, if U is a bounded from above recursive utility, then we have $U\leq K$ for some $K>0$ and

$$\begin{aligned} U(h_{x_0}(a)) &= u(x_0,a_0) + \delta(U(h_{x_1}\sigma\bar{a})) \\ &= u(x_0,a_0) + \delta(u(x_1,a_1) + \delta(U(h_{x_2}\sigma^2\bar{a}))) \\ &= u(x_0,a_0) + \delta(u(x_1,a_1) + \delta(\cdots u(x_{n-1},a_{n-1}) + \delta(U(h_{x_n}\sigma^n\bar{a})))). \end{aligned}$$

By using repeatedly the inequality $|\delta(t_2)-\delta(t_1)|\leq\gamma(|t_2-t_1|)$ we have

$$|\sum_{i\in[n]}^\star u(x_i,a_i) - U(h_{x_0}(a))| \leq \gamma^n(U(h_{x_n}\sigma^n\bar{a})) \leq \gamma^n(K)\to 0,$$

proving that $U(h_{x_0}(\bar{a})) = \sum_i^\star u(x_i,a_i)$. □

Definition 2.18. *Given $U : \Omega \to \mathbb{R}$ a function, $\hat{V}(x) = \sup_{h_x(\bar{a})\in\Omega} U(h_x(\bar{a}))$ is called an optimal return. An element $a^* \in \Pi(x)$ (sometimes called plan) is said to be optimal if $\hat{V}(x) = U(h_x(a^*))$.*

Definition 2.19. *A function $W : X \times A \times D \to \mathbb{R}$ given by*

$$W(x, a, r) := u(x, a) + \delta(r),$$

where $a \in \Psi(x)$ is called an aggregator function.

In dynamic programming we can always assume that $\delta(0) = 0$, otherwise we can redefine $\tilde{u}(x, a) = u(x, a) + \delta(0)$ and $\tilde{\delta}(t) = \delta(t) - \delta(0)$ without changing the aggregator function value neither the solutions of some problems associated to it.

Now we introduce some dynamics on Ω, by considering the maps

a) $\sigma : \Pi(x) \to \Pi(f(x, \cdot))$ the left shift given by $\sigma(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$. Note that this mapping is well-defined since for any $(a_0, a_1, \ldots) \in \Pi(x)$ we have that $(a_1, a_2, \ldots) \in \Pi(f(x, a_0))$;

b) $\phi : X \times \Pi(\cdot) \to X$ the skew map

$$\phi_a x = f(x, a),\ a \in \Psi(x);$$

c) $\hat{\sigma} : \Omega \to \Omega$ the "double left shift" operator given by

$$\hat{\sigma}(h_x(\bar{a})) := h_{\phi_{a_0}(x)}(\sigma(\bar{a})) := (x_1, a_1, x_2, a_2, \ldots) \in \Omega.$$

Definition 2.20. *Given a bounded and continuous immediate reward u and a variable discount function δ, satisfying $\delta(0) = 0$, the Koopman operator $K := K_{u,\delta} : C_b(\Omega, \mathbb{R}) \to C_b(\Omega, \mathbb{R})$ is defined by*

$$K(U)(h_x(\bar{a})) = W(x_0, a_0, U(\hat{\sigma}(h_x(\bar{a})))).$$

Note that a fixed point for the Koopman operator, that is, $K(U) = U$ is a recursive utility, in the sense of Definition 2.15.

Theorem 2.21 ([JMN14b]). *Let u be a bounded and continuous immediate reward, and δ a variable discount, satisfying $\delta(0) = 0$. Then there exists a unique fixed point $U \in C_b(\Omega, \mathbb{R})$, for the Koopman operator and moreover*

$$\|K^n(Q) - U\|_\infty \to 0$$

for any $Q \in C_b(\Omega, \mathbb{R})$.

Proof. Since the function u, δ and f are continuous and u is bounded we have that $K(C_b(\Omega, \mathbb{R})) \subseteq C_b(\Omega, \mathbb{R})$. The result is a consequence of Theorem 2.11 because K is a generalized Matkowski contraction with the witness function $\varphi(t) := \gamma(t)$, where γ is the contraction modulus of δ and the metric space $(C_b(\Omega, \mathbb{R}), \|\cdot\|_\infty)$ is complete. □

As a corollary we obtain sufficient conditions for the existence of the inductive limits.

Corollary 2.22. *Under the assumptions of Theorem2.21 there exists the inductive limit*

$$\sum_i{}^{\star} u(x_i, a_i) = \lim_{n\to\infty} \sum_{i\in[n]}{}^{\star} u(x_i, a_i),$$

where the convergence is in the uniform topology. In particular,

$$\sum_i{}^{\star} u(x_i, a_i) = \lim_{n\to\infty} K^n(0)(h_x(\bar{a})) = U(h_x(\bar{a})),$$

is the unique bounded continuous recursive utility.

3 Bellman and Discounted Transfer Operators

Note that until now, we have only assumed that u is a bounded continuous function. In the sequel, we add an extra assumption which is $u \geq 0$. This technical assumption is convenient when considering iterates of K, since δ is only defined on $D := [0, +\infty)$. This is actually not a restrictive assumption since in the bounded continuous case, we can always replace u by $u - \min u \geq 0$. See Remark 3.29 for further details on this issue.

Definition 3.1. *Given a non-negative bounded and continuous immediate reward u and a discount function δ, satisfying $\delta(0) = 0$, the Bellman operator $B := B_{u,\delta} : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$ applied to v and evaluated at x is defined by*

$$B(v)(x) := \sup_{a\in\Psi(x)} W(x, a, v(f(x, a))).$$

Definition 3.2. *Let u and δ be as in Definition 3.1. The discounted transfer operator, $P := P_{u,\delta} : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$, applied to v and evaluated at x is defined by*

$$P(v)(x) := \ln \int_{a\in\Psi(x)} e^{W(x,a,v(f(x,a)))} d\nu_x(a),$$

where ν_x is a Borel probability measure on A, satisfying $\nu_x(\Psi(x)) = 1$, for all $x \in X$.

The transfer operator, or Ruelle operator, is the linear operator on $C_b(X, \mathbb{R})$ defined by

$$L(v)(x) := \int_{a\in\Psi(x)} e^{u(x,a)} v(f(x, a)) d\nu_x(a).$$

Before proceed, we recall a basic fact from general topology. For more details, see reference [Ber97], page 115, Theorems 1 and 2.

Theorem 3.3. *Let X, Y be topological spaces $F : X \times Y \to \mathbb{R}$ a USC (resp. LSC) mapping and $\Gamma : X \to 2^Y$ a USC (resp. LSC) set valued map, such that $\Gamma(x) \neq \varnothing$, for all $x \in X$. Then the function*

$$M(x) := \sup_{y \in \Gamma(x)} F(x, y),$$

is a USC (resp. LSC). In particular, if F and Γ are continuous, then M is continuous.

Lemma 3.4. *The Bellman and discount transfer operators, defined above, send the space $C_b(X, \mathbb{R})$ to itself.*

Proof. The prove that $B(C_b(X, \mathbb{R})) \subset C_b(X, \mathbb{R})$ it is enough to apply Theorem 3.3, with $X = X$, $Y = A$, $\Gamma = \Psi$ and $F(a) := u(x, a) + \delta(v(f(x, a))$, which is clearly continuous, thus showing that

$$B(v)(x) := \sup_{a \in \Psi(x)} F(a) = \max_{a \in \Psi(x)} u(x, a) + \delta(v(f(x, a))$$

is a continuous and bounded function.

For the discount transfer operator the proof is similar. We keep the above setting and consider the continuous functions

$$M(x) := \sup_{a \in \Psi(x)} F(a) \quad \text{and} \quad N(x) := \inf_{a \in \Psi(x)} F(a).$$

From definition of P, we have $N(x) \leq P(v)(x) \leq M(x)$, for all $x \in X$. Therefore, $-(M(y) - N(x)) \leq P(v)(x) - P(v)(y) \leq M(x) - N(y)$ and the continuity and boundedness of M and N imply that $x \longmapsto P(v)(x)$ is continuous and bounded function. □

Theorem 3.5 ([JMN14b]). *Let u and δ be as in Definition 3.1 and $B : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$, the Bellman operator associated to u and δ. Then*

a) There is a unique $v^* \in C_b(X, \mathbb{R})$ such that $B(v^*) = v^*$. Moreover, v^* is an optimal return and satisfies the δ-discounted Bellman equation

$$v^*(x) := \max_{a \in \Psi(x)} u(x, a) + \delta(v^*(f(x, a)).$$

b) A plan $a^* \in \Pi(x)$ attaining the maximum $v^*(x_n) = u(x_n, a^*_n) + \delta(v^*(x_{n+1}))$ for all $n \in \mathbb{N}$ is optimal. In particular, there exists $a^* \in \Pi(x)$ such that $v^*(x) := U(h_x(a^*))$

Sketch of the proof. We provide here, for the reader's convenience, some of key steps of this proof.

a) The existence of v^* is a consequence of Theorem 2.11, because B is a generalized Matkowski contraction and the metric space $(C_b(X, \mathbb{R}), \| \cdot \|_\infty)$ is complete. Indeed, one can show that $\|B(v) - B(v')\|_\infty \leq \gamma(\|v - v'\|_\infty)$.

b) To show that v^* is optimal, we consider any $h_x(\bar{a}) \in \Omega$. From the fixed point equation we obtain $v^*(x_0) \geq u(x_0, a_0^*) + \delta(v^*(x_1))$, where $x_0 = x$ and $x_1 = f(x_0, a_0)$. By iterating this equality we get

$$v^*(x_0) \geq u(x_0, a_0^*) + \delta(u(x_1, a_1^*) + \delta(v^*(x_2))),$$
$$v^*(x_0) \geq u(x_0, a_0^*) + \delta(u(x_1, a_1^*) + \delta(u(x_2, a_2^*) + \delta(v^*(x_3))))$$

and so on. If $\zeta : \Omega \to \mathbb{R}$ is a function given by $\zeta(h_x\bar{a}) = v^*(x)$, then $K(\zeta)(h_x\bar{a}) = u(x_0, a_0^*) + \delta(v^*(x_1)), \ldots, K^3(\zeta)(h_x\bar{a}) = u(x_0, a_0^*) + \delta(u(x_1, a_1^*) + \delta(u(x_2, a_2^*) + \delta(v^*(x_3))))$, and so on. Therefore $v^*(x_0) \geq K^n(\zeta)(h_x\bar{a}) \to U(h_x(\bar{a}))$, where U is the recursive utility given by the associated Koopman operator. Thus showing that

$$v^*(x) \geq \sup_{h_x(\bar{a}) \in \Omega} U(h_x(\bar{a})).$$

To show the equality, we use the continuity of u, δ and v^*. The compactness of $\Psi(\cdot)$ allow us to choose, from the fixed point equation, a sequence $a^* \in \Pi(x)$ attaining the maximum $v^*(x_n) = u(x_n, a_n^*) + \delta(v^*(x_{n+1}))$, for all $n \in \mathbb{N}$. Proceeding as before, we obtain $v^*(x) = U(h_x(a^*))$. So v^* is optimal and there exists $a^* \in \Pi(x)$ such that

$$v^*(x) = U(h_x(a^*)) = \sup_{h_x(\bar{a}) \in \Omega} U(h_x(\bar{a})).$$

Theorem 3.6. *Let u and δ be as in Definition 3.1, and $B : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$ the Bellman operator, associated to this pair. Then*

a) there is a unique $w^* \in C_b(X, \mathbb{R})$ such that $P(w^*) = w^*$;

b) $w^* \leq v^*$ where v^* is the unique solution of the Bellman equation

$$v^*(x) = \max_{a \in \Psi(x)} u(x, a) + \delta(v^*(f(x, a)));$$

c) if the family of measures ν_x can be chosen in such way that $\nu_x = \delta_{a_0}(x)$ where $a_0 \in \operatorname{argmax}\{u(x, a) + \delta(v^*(f(x, a)))\}$, then $w^* = v^*$.

Proof. *a)* It is easy to see that

$$\begin{aligned}\|P(w_1) - P(w_2)\|_\infty &\leq \max_a |\delta(w_1(f(x, a))) - \delta(w_2(f(x, a)))| \\ &\leq \max_a \gamma(|w_1(f(x, a)) - w_2(f(x, a))|) \\ &\leq \gamma(\|w_1 - w_2\|_\infty)\end{aligned}$$

and so P is a generalized Matkowski contraction in the complete metric space $(C_b(X, \mathbb{R}), \|\cdot\|_\infty)$. By Theorem 2.11 there is a unique $w^* \in C_b(X, \mathbb{R})$ such that $P(w^*) = w^*$ and $\|P^n(w) - w^*\|_\infty \to 0$, when $n \to \infty$, for any $w \in C_b(X, \mathbb{R})$.

b) To see that $w^* \leq v^*$ where v^* is the unique solution of the δ- discounted Bellman equation $v^*(x) := \max_{a\in\Psi(x)} u(x,a) + \delta(v^*(f(x,a))$, we recall that

$$\begin{aligned} P(v^*)(x) &= \ln \int_{a\in\Psi(x)} \exp\left(u(x,a) + \delta(v^*(f(x,a)))\right) d\nu_x(a) \\ &\leq \ln \int_{a\in\Psi(x)} \exp\Big(\max_{a\in\Psi(x)} u(x,a) + \delta(v^*(f(x,a)))\Big) d\nu_x(a) \\ &= v^*(x). \end{aligned}$$

Since δ is an increasing function it follows that $P(v^*) \leq v^*$, $P^2(v^*) \leq v^*$ and so on. Since $P^n(v^*) \to w^*$, when $n \to \infty$, we get from the previous inequality that $w^* \leq v^*$.

c) Suppose that $\nu_x = \delta_{a_0}(x)$, where $a_0 \in \operatorname{argmax}\{u(x,a) + \delta(v^*(f(x,a))\}$. Then

$$\begin{aligned} P(v^*)(x) &= \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta(v^*(f(x,a)))}\, d\nu_x(a) \\ &\leq w^*(x) \\ &= \ln\left(e^{u(x,a_0)+\delta(v^*(f(x,a_0)))}\right) \delta_{a_0}(\Psi(x)) \\ &= \max_{a\in\Psi(x)} u(x,a) + \delta(v^*(f(x,a))) \\ &= v^*(x), \end{aligned}$$

which implies that $w^* = v^*$. □

3.1 Monotone Convergence Principles

In this section we investigate the ordering and the minimality of the convergence of the iterations to the fixed points. This topic is closely related to the theory of viscosity solutions of Hamilton-Jacobi equations, where the subsolutions (supersolutions) characterizes the original one.

Lemma 3.7 (Monotonicity on δ). *Let $\delta_1 \leq \delta_2$ be discount functions. If v_1, v_2 are solutions of Bellman's equation $v_j(x) = \max_{a\in\Psi(x)} u(x,a) + \delta_j(v_j(f(x,a)))$, $j = 1, 2$, then $v_1 \leq v_2$. The same is true for the discounted transfer operator.*

Proof. Since $\delta_1 \leq \delta_2$ we have $u(x,a)+\delta_1(v_1(f(x,a))) \leq u(x,a)+\delta_2(v_1(f(x,a)))$. By taking the maximum over $\Psi(x)$ we obtain

$$v_1(x) \leq \max_{a\in\Psi(x)} u(x,a) + \delta_2(v_1(f(x,a))) = B_{\delta_2}(v_1)(x).$$

Iterating this inequality and using the fact that $B^n_{\delta_2}(v_1)(x) \to v_2$, when $n \to \infty$, we get $v_1 \leq v_2$. □

Lemma 3.8 (Monotonicity on the operator)**.** *Let v_1 and v_2 be bounded functions, such that $v_1 \leq v_2$. Consider the Bellman operator*

$$B(v)(x) = \max_{a \in \Psi(x)} u(x,a) + \delta(v(f(x,a))).$$

Then $B(v_1) \leq B(v_2)$. In particular,

a) if $B(v) \leq v$ and $B(v^*) = v^*$ then $v^* \leq v$;

b) if $B(v) \geq v$ and $B(v^*) = v^*$ then $v^* \geq v$.

The same is true for the discounted transfer operator.

Proof. Since δ is an increasing function it follows that

$$\begin{aligned} B(v_1)(x) &= \max_{a \in \Psi(x)} u(x,a) + \delta(v_1(f(x,a))) \\ &\leq \max_{a \in \Psi(x)} u(x,a) + \delta(v_2(f(x,a))) \\ &= B(v_2)(x). \end{aligned}$$

The statements *a)* and *b)* are proved in the same way. Using the fact that δ is increasing we obtain, from the first part, $B(v) \leq v$, $B^2(v) \leq B(v) \leq v$, etc. Recalling that the iterates $B^n(v) \to v^*$, when $n \to \infty$, for any initial v, we obtain $v^* \leq v$. □

Remark 3.9. *The actual solution v^* is minimal with respect to the set of all subsolutions, that is, $v^* \leq v$ for all v satisfying $B(v) \leq v$.*

3.2 Regularity

In this section we will establish the regularity of the fixed points of the Koopman, Bellman and Discounted Transfer operators. Such regularity properties will be proved under the following assumption.

Assumption 3.10. *The contraction modulus γ of the variable discount δ is also a variable discount function, and $\Psi(x) = \Psi(y)$, $\forall x, y \in X$.*

A particular case is when $\gamma = \delta$ (but they can be different, see Example 2.7) and $\Psi(x) = A$, for all $x \in X$.

Definition 3.11 (Joint sequential decision-making process)**.** *Let $S = \{X, A, \Psi, f, u, \delta\}$ be a sequential decision-making process satisfying Assumption 3.10. The joint sequential decision-making process associated to S is the decision-making process $\hat{S} = \{X^2, A, \hat{\Psi}, \hat{f}, \hat{u}, \gamma\}$, where*

- *$\hat{\Psi} : X^2 \to A$ given by $\hat{\Psi}(x,y) = \Psi(x) \subseteq A$ is the set of all feasible actions for a agent x.*
- *$\hat{f} : X^2 \times A \to X^2$ is given by $\hat{f}(x,y,a) = (f(x,a), f(y,a))$.*

- $\hat{u} : X^2 \times A \to \mathbb{R}$ *is the immediate reward* $\hat{u}(x,y,a) = |u(x,a) - u(y,a)|$;

Definition 3.12. *Let* $W : X \times A \times D \to \mathbb{R}$ *be an aggregator function of the form* $W(x,a,r) := u(x,a) + \delta(r)$*, where* $a \in \Psi(x)$*. We define a new aggregator function* $\hat{W} : X^2 \times A \times D \to \mathbb{R}$ *given by*

$$\hat{W}(x,y,a,r) := \hat{u}(x,y,a) + \gamma(r) = |u(x,a) - u(y,a)| + \gamma(r),$$

where γ *is a contraction modulus for the variable discount* δ*.*

Lemma 3.13. *Let* $v^* \in C_b(X,\mathbb{R})$ *be the unique solution of the Bellman equation* $v^*(x) = \max_{a\in\Psi(x)} u(x,a) + \delta(v^*(f(x,a)))$*, provided by Theorem 3.5. Let* $w^* \in C_b(X,\mathbb{R})$ *be the unique solution of the discounted transfer equation*

$$w^*(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta(w^*(f(x,a)))} \, d\nu_x(a),$$

given by Theorem 3.6. Then,

$$|v^*(x) - v^*(y)| \leq \hat{V}^*(x,y) \text{ and } |w^*(x) - w^*(y)| \leq \hat{V}^*(x,y), \ \forall x,y \in X,$$

where $\hat{V}^*$ *is the unique fixed point of the Bellman operator*

$$\hat{B}(\hat{V})(x,y) := \sup_{a\in\hat{\Psi}(x,y)} \hat{u}(x,y,a) + \gamma(\hat{V}(\hat{f}(x,y,a))).$$

Proof. From the definition and triangular inequality, we get

$$\begin{aligned}
v^*(x) &= \max_{a\in\Psi(x)} u(x,a) + \delta(v^*(f(x,a)) \\
&\leq \max_{a\in\Psi(x)} \{u(x,a) - u(y,a) + u(y,a) + \delta(v^*(f(x,a)) \\
&\qquad\qquad -\delta(v^*(f(y,a)) + \delta(v^*(f(x,a))\} \\
&\leq \max_{a\in\Psi(y)} \{u(y,a) + \delta(v^*(f(y,a)))\} + |u(x,a) - u(y,a)| \\
&\qquad\qquad + |\delta(v^*(f(x,a)) - \delta(v^*(f(y,a))| \\
&\leq v^*(y) + \hat{u}(x,y,a) + \gamma(|v^*(f(x,a) - v^*(f(y,a)|).
\end{aligned}$$

By a similar reasoning, replacing x by y, we obtain

$$|v^*(x) - v^*(y)| \leq \hat{u}(x,y,a) + \gamma(|v^*(f(x,a)) - v^*(f(y,a))|).$$

Analogously, $|w^*(x) - w^*(y)| \leq \hat{u}(x,y,a) + \gamma(|w^*(f(x,a) - w^*(f(y,a))|)$, since

$$\begin{aligned}
|w^*(x) - w^*(y)| &= \ln \frac{\int_{a\in\Psi(x)} e^{u(x,a)+\delta(w^*(f(x,a)))} \, d\nu_x(a)}{\int_{a\in\Psi(y)} e^{u(y,a)+\delta(w^*(f(y,a)))} \, d\nu_y(a)} \\
&\leq \sup_{a\in\Psi(x,y)} |u(x,a) - u(y,a)| + |\delta(w^*(f(x,a))) - \delta(w^*(f(y,a)))|.
\end{aligned}$$

In both cases, where $\zeta(x,y) = |v^*(x) - v^*(y)|$ or $\zeta(x,y) = |w^*(x) - w^*(y)|$ [1] we obtain $\hat{B}(\zeta) \leq \zeta$, where

$$\hat{B}(\hat{V})(x,y) = \sup_{a \in \hat{\Psi}(x,y)} \hat{u}(x,y,a) + \gamma(\hat{V}(\hat{f}(x,y,a)).$$

From Lemma 3.8 follows that $\zeta \leq \hat{V}^*$, where $\hat{V}^*$ is the unique solution of the Bellman operator $\hat{B}$. By Assumption 3.10 and Theorem 3.5 there exists a unique $\hat{U}$ solving the Koopman equation

$$\hat{U}(\hat{h}_{(x,y)}\bar{a})) = \hat{K}(\hat{U})(\hat{h}_{(x,y)}\bar{a}))) := \hat{u}(x,y,a) + \gamma(\hat{U}(\hat{\sigma}(\hat{h}_{(x,y)}\bar{a}))),$$

such that

$$\hat{V}^*(x,y) = \hat{U}(\hat{h}_{(x,y)}\bar{a}) = \sum_i^{\star} \hat{u}(x_i, y_i, a_i^*),$$

for some optimal plan $a^* \in \Pi(x,y)$[2]. □

Lemma 3.14. *Let $\hat{V}^*$ be the unique fixed point of the Bellman operator*

$$\hat{B}(\hat{V})(x,y) = \sup_{a \in \hat{\Psi}(x,y)} \hat{u}(x,y,a) + \gamma(\hat{V}(\hat{f}(x,y,a)).$$

Then

a) $\hat{V}^*(x,y) \geq 0$ *and* $\hat{V}^*(x,x) = 0$*;*

b) $\hat{V}^*(x,y) = \hat{V}^*(y,x)$*;*

That is, $V^* : X^2 \to \mathbb{R}$ *is a symmetric and nonnegative function. In particular, from optimality of the solutions of Bellman's equation we have*

$$\hat{V}^*(x,y) := \sup_{\hat{h}_{(x,y)}\bar{a} \in \hat{\Omega}} \hat{U}(\hat{h}_{(x,y)}\bar{a}) = \sum_i^{\star} \hat{u}(x_i, y_i, a_i^*),$$

for some optimal plan $a^* \in \Pi(x,y)$*.*

Proof. a) We recall that $\hat{V}^*(x,y) := \hat{U}(\hat{h}_{(x,y)}\bar{a}) = \sum_i^{\star} \hat{u}(x_i, y_i, a_i^*)$, for some optimal plan $a^* \in \Pi(x,y)$. Since γ is assumed to be a discounting function therefore increasing, and $\hat{u} \geq 0$, we have immediately

$$\hat{V}^*(x,y) = \hat{u}(x_0, y_0, a_0^*) + \gamma\Big(\sum_i^{\star} \hat{u}(x_{i+1}, y_{i+1}, a_{i+1}^*)\Big) \geq 0.$$

[1] the remaining of the argument works for both choices, because it depends only on the monotonicity properties, so all this formalism works equally to both families of fixed points v^* and w^*. Since $|v^*(x) - v^*(y)| \leq \hat{V}^*(x,y)$ and $|w^*(x) - w^*(y)| \leq \hat{V}^*(x,y)$, $\forall x,y \in X$. Note that u is the same in both cases.

[2] is the set of feasible action sequences for the joint sequential decision making process $\hat{S} = \{X^2, A, \hat{\Psi}, \hat{f}, \hat{u}, \gamma\}$

By definition $\hat{u}(x,x,a) = |u(x,a) - u(x,a)| = 0$ so $\hat{V}^*(x,x) = 0$.

b) By definition $\hat{u}(x,y) = |u(x,a) - u(y,a)| = |u(y,a) - u(x,a)| = \hat{u}(y,x)$. Let us define U' as the unique solution of the Koopman equation

$$U'(\hat{h}_{(x,y)}\bar{a})) = \hat{u}(x,y,a) + \gamma(U'(\hat{\sigma}(\hat{h}_{(x,y)}\bar{a}))),$$

and $U''(\hat{h}_{(x,y)}\bar{a})) = U'(\hat{h}_{(y,x)}\bar{a}))$. Then it satisfies

$$\begin{aligned} U''(\hat{h}_{(x,y)}\bar{a})) &= \hat{u}(y,x,a) + \gamma(U'(\hat{\sigma}(\hat{h}_{(y,x)}\bar{a}))) \\ &= \hat{u}(x,y,a) + \gamma(U''(\hat{\sigma}(\hat{h}_{(x,y)}\bar{a}))). \end{aligned}$$

By the uniqueness we obtain $U'' = U$ thus, $U'(\hat{h}_{(x,y)}\bar{a})) = U'(\hat{h}_{(y,x)}\bar{a}))$, which is equivalent to $\hat{V}^*(x,y) = \hat{V}^*(y,x)$. □

Definition 3.15. *We say that $u : X \times A \to \mathbb{R}$ is non-degenerated if for any $x \neq y$ in X there exists $n \in \mathbb{N}$ and $\bar{a} \in \Pi(x,y)$ such that*

$$u(x_n, a_n) \neq u(y_n, a_n)$$

where $(x_i, a_i)_{i\in\mathbb{N}}, (y_i, a_i)_{i\in\mathbb{N}} \in \Omega$.

Of course, if for any fixed $a \in A$, the function $x \longmapsto u(x,a)$ is strictly increasing (or decreasing) then u is non-degenerated. Therefore there is at least one very natural sufficient condition to non-degeneration.

Theorem 3.16. *If u is non-degenerated then $\hat{V}^*(x,y)$ is separating, that is, if $\hat{V}^*(x,y) = 0$ then $x = y$.*

Proof. Suppose that $x \neq y$, but $\hat{V}^*(x,y) = 0$. Then, by optimality we have

$$0 = \hat{V}^*(x,y) := \sup_{\hat{h}_{(x,y)}\bar{a}\in\hat{\Omega}} \hat{U}(\hat{h}_{(x,y)}\bar{a})$$

so $\hat{U}(\hat{h}_{(x,y)}\bar{a}) = 0$, that is, $\sum_i^\star \hat{u}(x_i, y_i, a_i^*) = 0$, where $\bar{a} = (a_0, a_1, \ldots)$. Since γ is an increasing function, with $\gamma(0) = 0$, and $\hat{u} \geq 0$, we have $\hat{u}(x_i, y_i, a_i) = 0$ for all $i \geq 0$, thus contradicting the non-degeneration property of u. □

3.3 Discounted Limits

In this section we consider the limits of fixed points of a variable discount decision-making process defined by a continuous and bounded immediate reward $u : X \times A \to \mathbb{R}$ and a sequence $(\delta_n)_{n\geq 0}$ of discounts $\delta_n : [0, +\infty) \to \mathbb{R}$, satisfying $\delta_n(t) \to I(t) = t$, when $n \to \infty$, in the pointwise topology. For instance, $\delta_n(t) = t(n-1)/n + (1/n)\ln(1+t)$ is a nonlinear, idempotent ($\gamma_n(t) = \delta_n(t)$) and subadditive discount function. It is easy to see that $\delta_n(t) \to t$, when $n \to \infty$, for all $t \geq 0$.

Under these assumptions we want to study the sequences

$$v_n^*(x) := \max_{a\in\Psi(x)} u(x,a) + \delta_n(v_n^*(f(x,a)))$$

and

$$w_n^*(x) := \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n^*(f(x,a)))} d\nu_x(a),$$

and investigate whether their normalizations

$$(v_n^*(x) - \sup_x v_n^*(x))_{n\geq 0} \quad \text{and} \quad (w_n^*(x) - \sup_x w_n^*(x))_{n\geq 0}$$

have some cluster points $v_\infty, w_\infty \in C_b(X,\mathbb{R})$, solving the equations

$$v_\infty(x) := \max_{a\in\Psi(x)} (u(x,a) - \alpha) + v_\infty(f(x,a))$$

and

$$e^k e^{w_\infty(x)} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{w_\infty(f(x,a))} d\nu_x(a),$$

for some $\alpha, k \in \mathbb{R}$. The first one is the subaction equation in ergodic optimization and the second is the eigenfunction equation for the Ruelle operator.

Assumption 3.17. *We assume that $\delta_n(t) \leq t$, for all $n \geq 0$.*

Since $\delta_n(0) = 0$, we can construct examples satisfying the above condition by requiring that $q_n(t) = t - \delta_n(t)$, for all $n \geq 0$, is not decreasing.

Lemma 3.18. *Under Assumption 3.17 we have*

$$0 \leq M_n - \delta_n(M_n) \leq \|u\|_\infty,$$

where

$$M_n = \sup_{x\in X} v_n(x) \qquad \text{and} \qquad v_n(x) = \max_{a\in\Psi(x)} u(x,a) + \delta_n(v_n(f(x,a)))$$

or

$$M_n = \sup_{x\in X} w_n(x) \qquad \text{and} \qquad w_n(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))} d\nu_x(a).$$

Proof. **Case** $M_n = \sup_{x\in X} v_n(x)$**:** by using Bellman's equation we obtain

$$\begin{aligned} u(x,a) + \delta_n(v_n(f(x,a))) &\leq \|u\|_\infty + \delta_n(M_n) \\ v_n(x) &\leq \|u\|_\infty + \delta_n(M_n) \\ M_n &\leq \|u\|_\infty + \delta_n(M_n) \\ M_n - \delta_n(M_n) &\leq \|u\|_\infty, \end{aligned}$$

By hypothesis we have $0 \leq M_n - \delta_n(M_n)$ and so follows from previous inequality that $0 \leq M_n - \delta_n(M_n) \leq \|u\|_\infty$.

Case $M_n = \sup_{x \in X} w_n(x)$**:** by using discounted transfer operator equation we obtain

$$
\begin{aligned}
u(x,a) + \delta_n(w_n(f(x,a))) &\leq \|u\|_\infty + \delta_n(M_n) \\
e^{u(x,a)+\delta_n(w_n(f(x,a)))} &\leq e^{\|u\|_\infty + \delta_n(M_n)} \\
w_n(x) &= \ln \int_{a \in \Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))} d\nu_x(a) \\
&\leq \ln \int_{a \in \Psi(x)} e^{\|u\|_\infty + \delta_n(M_n)} d\nu_x(a) \\
&= \|u\|_\infty + \delta_n(M_n) \\
M_n &\leq \|u\|_\infty + \delta_n(M_n) \\
M_n - \delta_n(M_n) &\leq \|u\|_\infty .
\end{aligned}
$$

By using the above inequality and proceeding as in the previous case we get $0 \leq M_n - \delta_n(M_n) \leq \|u\|_\infty$. □

We point out that $\delta(t) = \beta t$, $\beta \in (0,1)$, $\delta(t) = \ln(1+t)$, and $\delta(t) = \sum_{i=0}^{\infty} (\beta_i t + \alpha_i) \chi_{[i,i+1)}$ with $\beta_i \searrow 0$ satisfies the condition that $q(t) = t - \delta(t)$ is not a decreasing function.

Definition 3.19. *Given a discount δ, the return function u is called*

a) δ-bounded if $\sum_i^{\star} \hat{u}(x_i, y_i, a_i) \leq C_\delta$;

b) δ-dominated if

$$\lim_{\theta \to 0} \sup_{d_X(x,y) \leq \theta} \sup_{\bar{a} \in \Pi(x,y)} \sum_i^{\star} \hat{u}(x_i, y_i, a_i) = 0.$$

Given a family of discount functions $(\delta_n)_{n \geq 0}$ we say that u is

a) uniformly δ-bounded if u is δ_n-bounded for all n and $\sup_n C_{\delta_n} = C < +\infty$.

b) uniformly δ-dominated if u is δ_n-dominated for all n and

$$\lim_{\theta \to 0} \sup_{n \in \mathbb{N}} \sup_{d_X(x,y) \leq \theta} \sup_{\bar{a} \in \Pi(x,y)} \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i) = 0$$

where $\sum_i^{(\star,\gamma_n)}$ is $\sum_i^{\star}$ with the discount variable δ_n.

The next theorem shows that the class of uniformly δ-dominated contains the class of Lipschitz or α-Hölder potentials, when the dynamics of the decision process is uniformly contractive.

Theorem 3.20. *Suppose that the dynamics f is contractive, that is,*

$$\sup_{a \in A} d_X(f(x,a), f(y,a)) \le \lambda d_X(x,y).$$

If $u(\cdot, a)$ is C-Lipschitz (or α-Hölder) then u is uniformly δ-dominated. In addition, if $\operatorname{diam}(X) < \infty$ then u is uniformly δ-bounded. In particular, if v_n and w_n are respectively the solutions of Bellman's equation and the transfer discounted operator equation, they are uniformly $C(1-\lambda)^{-1}$-Lipschitz (or α-Hölder, with $\operatorname{Hol}_\alpha(v_n) = \operatorname{Hol}_\alpha(w_n) = \operatorname{Hol}_\alpha(u)(1-\lambda^\alpha)^{-1}$).

Proof. **Case 1:** $u(\cdot, a)$ is C-Lipschitz, that is, $|u(x,a) - u(y,a)| \le C d_X(x,y)$. In this case for any pair $x, y \in X$ satisfying $d_X(x,y) \le \theta$, we have the following estimate $\hat{u}(x_i, y_i, a_i) = |u(x_i, a_i) - u(y_i, a_i)| \le C d_X(x_i, y_i) \le C d_X(x,y)\lambda^i \le C\theta\lambda^i$, which immediately implies

$$\sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i) \le \sum_{i=0}^{\infty} C\lambda^i\theta = \frac{C\theta}{1-\lambda}$$

because $\gamma_n(x) < x$, for all $n \ge 0$. Thus,

$$\lim_{\theta \to 0} \sup_{n \in \mathbb{N}} \sup_{d_X(x,y) \le \theta} \sup_{\bar{a} \in \Pi(x,y)} \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i) \le \lim_{\theta \to 0} \frac{C\theta}{1-\lambda} = 0.$$

Case 2: $u(\cdot, a)$ is α-Hölder, that is, $|u(x,a) - u(y,a)| \le \operatorname{Hol}_\alpha(u) d_X(x,y)^\alpha$, for $0 < \alpha < 1$. A similar reasoning shows that $\hat{u}(x_i, y_i, a_i) \le \operatorname{Hol}_\alpha(u) d_X(x_i, y_i)^\alpha \le \operatorname{Hol}_\alpha(u)\lambda^{\alpha i} d_X(x,y)^\alpha \le \operatorname{Hol}_\alpha(u)(\lambda^\alpha)^i\theta^\alpha$ and

$$\sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i) \le \sum_{i=0}^{\infty} \operatorname{Hol}_\alpha(u)(\lambda^\alpha)^i\theta^\alpha = \frac{\operatorname{Hol}_\alpha(u)}{1-\lambda^\alpha}\theta^\alpha \xrightarrow{\theta \to 0} 0.$$

Thus proving that u is uniformly δ-dominated. The uniform δ-boundedness is trivial from the above computations as long as $\operatorname{diam}(X) < \infty$.

To prove the last claim (assuming Lipschitz condition), we use Lemma 3.13 and the inequalities

$$|v_n(x) - v_n(y)| \le \hat{V}^*(x,y) \text{ and } |w_n(x) - w_n(y)| \le \hat{V}^*(x,y), \ \forall x, y \in X.$$

By similar computations, replacing θ by $d_X(x,y)$, we obtain

$$|v_n(x) - v_n(y)| \le \hat{V}^*(x,y) = \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i^*) \le \frac{C}{1-\lambda} d_X(x,y).$$

Analogously for the α-Hölder case. □

Lemma 3.21. *Let the contraction modulus γ_n of the variable discount δ_n be also a variable discount function, $\Psi(x) = \Psi(y)$, $\forall x, y \in X$ and u uniformly*

δ-dominated. Then $\bar{v}_n = v_n(x) - M_n$, where $v_n(x) = \max_{a\in\Psi(x)}\{u(x,a) + \delta_n(v_n(f(x,a))\}$ is uniformly bounded, that is,

$$-2C \leq \bar{v}_n \leq 0.$$

The same is true for $\bar{w}_n = w_n(x) - M_n$, where

$$w_n(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a))} d\nu_x(a).$$

Proof. We give the argument for v_n. The proof for w_n is similar. We already know that $|v_n(x) - v_n(y)| \leq \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i^*) \leq C$, for some optimal plan $a^* \in \Pi(x,y)$, uniformly in n.

Obviously $\bar{v}_n(x) = v_n(x) - M_n \leq 0$. On the other hand, we get from the hypothesis $-C \leq v_n(x) - v_n(y)$ and subtracting M_n we obtain $v_n(y) - M_n - C \leq v_n(x) - M_n$ or, $m_n - M_n - C \leq \bar{v}_n(x)$, where $m_n = \min v_n$. Since $|v_n(x) - v_n(y)| \leq C$, it follows that $M_n - m_n \leq C$ and so $m_n - M_n \geq -C$. Thus, $-C - C \leq \bar{v}_n(x)$, which implies $-2C \leq \bar{v}_n(x)$. □

Now we present a sufficient condition for both families of fixed points to be equicontinuous, under normalization.

Lemma 3.22. *Under the hypothesis of Lemma 3.21, if u is uniformly δ-dominated with respect to $(\delta_n)_{n\geq 0}$, then the families $\bar{v}_n(x) = v_n(x) - M_n$ and $\bar{w}_n(x) = w_n(x) - M_n$ are equicontinuous.*

Proof. From Lemma 3.13 we know that

$$\hat{V}_n^*(x,y) = \sup_{\bar{a}\in\Pi(x,y)} \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i^*) \geq |v_n(x) - v_n(y)| = |\bar{v}_n(x) - \bar{v}_n(y)|,$$

that is, the modulus of uniform continuity $\omega(\bar{v}_n, \theta)$ of $\bar{v}_n$ satisfies

$$\omega(\bar{v}_n, \theta) = \sup_{d_X(x,y)\leq\theta} |\bar{v}_n(x) - \bar{v}_n(y)| \leq \sup_{d_X(x,y)\leq\theta} \sup_{\bar{a}\in\Pi(x,y)} \sum_i^{(\star,\gamma_n)} \hat{u}(x_i, y_i, a_i^*).$$

Thus, for any $\varepsilon > 0$ there exists $\theta > 0$ such that, $|\bar{v}_n(x) - \bar{v}_n(y)| < \varepsilon$ provided that $d_X(x,y) \leq \theta$ and it is independent of n. □

Assumption 3.23. *For any fixed $\alpha > 0$ we have*

$$\lim_{n\to\infty} \delta_n(t+\alpha) - \delta_n(t) = \alpha,$$

uniformly for $t > 0$.

Examples where the above assumption is satisfied are given by

$$\delta_n(t) = \frac{n-1}{n}t + \frac{1}{n}\ln(1+t) \quad \text{and} \quad \delta_n(t) = \frac{n-1}{n}t + \frac{1}{n}(-1+\sqrt{1+t}).$$

Theorem 3.24. *If the assumptions of Lemmas 3.21 and 3.22, and Assumption 3.23 are assumed to hold. Then there exists a value $\bar{u} \in [0, \|u\|_\infty]$ and a function h such that $h(x) = \max_{a\in\Psi(x)} u(x,a) - \bar{u} + h(f(x,a))$.*

Proof. We consider the sequence of functions $\bar{v}_n(x) = v_n(x) - M_n$ and the discounted limit $\delta_n \to Id_D$. Since each v_n satisfy Bellman's equation we have

$$\begin{aligned} v_n(x) &= \max_{a\in\Psi(x)} u(x,a) + \delta_n(v_n(f(x,a)) \\ v_n(x) - M_n &= \max_{a\in\Psi(x)} u(x,a) + \delta_n(v_n(f(x,a)) - M_n \\ \bar{v}_n(x) &= \max_{a\in\Psi(x)} u(x,a) + \delta_n(v_n(f(x,a)) - \delta_n(M_n) + \delta_n(M_n) - M_n \\ \bar{v}_n(x) &= \max_{a\in\Psi(x)} u(x,a) - (M_n - \delta_n(M_n)) + \delta_n(v_n(f(x,a)) - \delta_n(M_n). \end{aligned}$$

From Lemma 3.18 we know that $0 \le M_n - \delta_n(M_n) \le \|u\|_\infty$ so, possibly choosing a subsequence we can find $\bar{u} \in [0, \|u\|_\infty]$ such that $M_n - \delta_n(M_n) \to \bar{u}$ when $n \to \infty$. From Lemma 3.21 and Lemma 3.22 the sequence $\bar{v}_n$ is uniformly bounded and equicontinuous. From Arzelà-Ascoli's theorem we obtain a subsequence (that we still calling $\bar{v}_n$ to avoid extra indexes) that converges to a continuous function h satisfying $h(x) = \max_{a\in\Psi(x)} u(x,a) - \bar{u} + h(f(x,a))$, if $\delta_n(v_n(f(x,a))) - \delta_n(M_n) \to h(f(x,a))$ when $v_n(x) - M_n \to h(x)$. To prove that we recall that, from the definition of variable discount function δ_n, it is increasing so we have

$$\delta_n(M_n) - \delta_n(v_n(x)) \le \gamma_n(M_n - v_n(x)).$$

Since $M_n - v_n(x) \to -h(x) \ge 0$, we can conclude that for n big enough that $-h(x) - \varepsilon \le M_n - v_n(x) \le -h(x) + \varepsilon$, or equivalently

$$v_n(x) - h(x) - \varepsilon \le M_n \le v_n(x) - h(x) + \varepsilon.$$

Using the fact that δ_n is increasing we obtain

$$\delta_n(v_n(x) - h(x) - \varepsilon) \le \delta_n(M_n) \le \delta_n(v_n(x) - h(x) + \varepsilon).$$

By adding $-\delta_n(v_n(x))$, we obtain

$$\begin{aligned} \delta_n(v_n(x) - h(x) - \varepsilon) - \delta_n(v_n(x) &\le \delta_n(M_n) - \delta_n(v_n(x)) \\ &\le \delta_n(v_n(x) - h(x) + \varepsilon) - \delta_n(v_n(x). \end{aligned}$$

Now, from Assumption 3.23, it follows that

$$\lim_{n\to\infty} \delta_n(M_n) - \delta_n(v_n(x)) = -h(x).$$ □

Remark 3.25. *We can consider other families of δ_n's assuming the same hypothesis except for Assumption 3.23. For example, the family $\delta_n(t) = (-1 + \sqrt{1+t})$ satisfies: for any fixed $\alpha > 0$, we have $\lim_{n\to\infty} \delta_n(t+\alpha) - \delta_n(t) = 0$, uniformly on $t > 0$. In this case, the discount limit will produce an equation $h(x) = \max_{a\in\Psi(x)} u(x,a) - \bar{u}$, having a very different meaning.*

Remark 3.26. *In ergodic optimization this function h is called a calibrated sub-action of u with respect to the dynamics f. In the theory of viscosity solutions of the Hamilton-Jacobi-Bellman equations, the equation $h(x) = \max_{a\in\Psi(x)} u(x,a) - \bar{u} + h(f(x,a))$ can be rewritten as*

$$\bar{u} = \max_{a\in\Psi(x)} u(x,a) + h(f(x,a)) - h(x) = \max_{a\in\Psi(x)} d_x h(a) + u(x,a) = H(x, d_x h),$$

where the discrete differential is $d_x h(a) = h(f(x,a)) - h(x)$ and the Hamiltonian H is the Legendre transform of u.

Recall that the set of holonomic probability measures is defined by

$$\mathcal{H} := \left\{ \mu \in \mathscr{P}(\Omega) \mid \int_\Omega d_x g(a)\, d\mu(x,a) = 0,\ \forall g \in C(A,\mathbb{R}) \right\}.$$

Theorem 3.27. *Assume that the hypothesis of Theorem 3.24 are satisfied and put*

$$\check{u} = \sup_{\mu\in\mathcal{H}} \int_\Omega u(x,a)\, d\mu(x,a).$$

If Ω is compact then $\check{u} = \bar{u}$, in particular, the number given by Theorem 3.24 is unique.

Proof. From Remark 3.26 we know that $\bar{u} = \max_{a\in\Psi(x)} d_x h(a) + u(x,a) \geq d_x h(a) + u(x,a)$ and integrating with respect to $\mu \in \mathcal{H}$ we obtain

$$\bar{u} \geq \int_\Omega d_x h(a) d\mu(x,a) + \int_\Omega u(x,a) d\mu(x,a) = \int_\Omega u(x,a) d\mu(x,a),$$

thus $\bar{u} \geq \check{u}$.

To show the equality we will built a holonomic maximizing probability. Inductively, we choose $a_0 \in \Psi(x)$ such that $\bar{u} = d_x h(a_0) + u(x, a_0)$, $a_1 \in \Psi(x_1)$ such that $\bar{u} = d_{x_1} h(a_1) + u(x_1, a_1)$, and so on. Notice that $x_0 = x$ and $x_{n+1} = f(x_n, a_n)$, for all $n \geq 0$. Define a probability measure μ_k by

$$\mu_k(g) := \frac{1}{k} \sum_{i=0}^{k-1} g(x_i, a_i)$$

then, adding the above equations we get $k\bar{u} = \sum_{i=0}^{k-1} d_{x_i} h(a_i) + u(x_i, a_i)$ or equivalently

$$\bar{u} = \frac{1}{k} \sum_{i=0}^{k-1} d_{x_i} h(a_i) + \frac{1}{k} \sum_{i=0}^{k-1} u(x_i, a_i)$$

$$= \frac{h(x_{k-1}) - h(x_0)}{k} + \int_\Omega u(x,a)\, d\mu_k(x,a).$$

Since h is bounded and Ω is compact, up to subsequence, we can assume that $\mu_k \rightharpoonup \mu$. A straightforward calculation shows that $\mu \in \mathcal{H}$ and

$$\bar{u} = \int_\Omega u(x,a)\, d\mu(x,a). \qquad \square$$

Theorem 3.28. *If the assumptions of Lemmas 3.21 and 3.22, and Assumption 3.23 are assumed to hold. Then there exists a value $k \in [0, \|u\|_\infty]$ and a function h given by $h(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+h(f(x,a))-k} d\nu_x(a)$, such that $\rho := e^k$ and $\varphi := e^{h(x)}$ are a positive eigenvalue and a positive and continuous eigenfunction, respectively, for Ruelle operator, i.e.,*

$$e^k e^{h(x)} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{h(f(x,a))}\, d\nu_x(a).$$

Proof. Consider

$$M_n = \sup_{x\in X} w_n(x) \quad \text{and} \quad w_n(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))}\, d\nu_x(a).$$

Take the sequence of functions $\bar{w}_n(x) = w_n(x) - M_n$ and analyze the discounted limit $\delta_n \to Id_D$. Since each w_n satisfies the discounted transfer operator equation we have

$$w_n(x) = \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))} d\nu_x(a).$$

From this equality follows that

$$\begin{aligned} w_n(x) - M_n &= \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))-M_n}\, d\nu_x(a) \\ &= \ln \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))-\delta_n(M_n)+\delta_n(M_n)-M_n}\, d\nu_x(a). \end{aligned}$$

By taking exponential on both sides we get

$$e^{w_n(x)-M_n} = \int_{a\in\Psi(x)} e^{u(x,a)+\delta_n(w_n(f(x,a)))-\delta_n(M_n)-(M_n-\delta_n(M_n))} d\nu_x(a)$$

which in turn implies

$$e^{M_n-\delta_n(M_n)} e^{w_n(x)-M_n} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{\delta_n(w_n(f(x,a)))-\delta_n(M_n)} d\nu_x(a).$$

From Lemma 3.18 we know that $0 \leq M_n - \delta_n(M_n) \leq \|u\|_\infty$ so, possibly assuming a subsequence we can find $k \in [0, \|u\|_\infty]$ such that $M_n - \delta_n(M_n) \to k$ when $n \to \infty$. From Lemma 3.21 and Lemma 3.22 the sequence $\bar{w}_n$ is uniformly bounded and equicontinuous. From Arzelà-Ascoli's theorem we obtain a subsequence (that we still calling $\bar{w}_n$ to avoid extra indexes) that converges to a continuous function h satisfying $e^k e^{h(x)} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{h(f(x,a))} d\nu_x(a)$ if $\delta_n(w_n(f(x,a))) - \delta_n(M_n) \to h(f(x,a))$ when $w_n(x) - M_n \to h(x)$. To prove this we recall that, from the definition of variable discount function δ_n, it is increasing so we have

$$\delta_n(M_n) - \delta_n(w_n(x)) \leq \gamma_n(M_n - w_n(x))$$

Since $M_n - w_n(x) \to -h(x) \geq 0$, we can conclude that for n big enough we have $-h(x) - \varepsilon \leq M_n - w_n(x) \leq -h(x) + \varepsilon$, or equivalently

$$w_n(x) - h(x) - \varepsilon \leq M_n \leq w_n(x) - h(x) + \varepsilon.$$

Using the fact that δ_n is increasing we obtain

$$\delta_n(w_n(x) - h(x) - \varepsilon) \leq \delta_n(M_n) \leq \delta_n(w_n(x) - h(x) + \varepsilon),$$

and by adding $-\delta_n(w_n(x))$, we obtain

$$\begin{aligned}\delta_n(w_n(x) - h(x) - \varepsilon) - \delta_n(w_n(x) &\leq \delta_n(M_n) - \delta_n(w_n(x)) \\ &\leq \delta_n(w_n(x) - h(x) + \varepsilon) - \delta_n(w_n(x).\end{aligned}$$

Now, from Assumption 3.23 it follows that

$$\delta_n(M_n) - \delta_n(w_n(x)) = -h(x).$$ □

Remark 3.29. *All the results of this section were obtained under the assumption that $u(x,a) \geq 0$, $\forall x, a$. If we start with a bounded u we can pick a constant c such that $u' = u+c \geq 0$. We claim that this hypothesis is actually not a restriction neither changes our results. In the regularity section, all the results depends on $\hat{u}'(x,y,a) = |(u+c)(x,a) - (u+c)(y,a)| = |u(x,a) - u(y,a)| = \hat{u}(x,y,a)$ and it does not changes under addition of a constant. In Theorem 3.24, we have $h(x) = \max_{a\in\Psi(x)} u'(x,a) - \bar{u}' + h(f(x,a))$, and $\bar{u}' = \sup_{\mu\in\mathcal{H}} \int_\Omega u'(x,a)d\mu(x,a)$, so $h(x) = \max_{a\in\Psi(x)} u(x,a) - (\bar{u}' - c) + h(f(x,a))$ and $\bar{u}' = \sup_{\mu\in\mathcal{H}} \int_\Omega u(x,a) + c d\mu(x,a)$ that is*

$$\bar{u}' - c = \sup_{\mu\in\mathcal{H}} \int_\Omega u(x,a)d\mu(x,a) = \bar{u},$$

thus the equation holds for u, $h(x) = \max_{a\in\Psi(x)} u(x,a) - \bar{u} + h(f(x,a))$ with the same solution h. Analogously, in Theorem 3.28, if we replace an initial u that can be negative by $u' = u + c$, we obtain

$$e^k e^{h(x)} = \int_{a\in\Psi(x)} e^{u'(x,a)} e^{h(f(x,a))} d\nu_x(a) = \int_{a\in\Psi(x)} e^{u(x,a)+c} e^{h(f(x,a))} d\nu_x(a)$$

or equivalently

$$e^{k-c} e^{h(x)} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{h(f(x,a))} d\nu_x(a)$$

which means that Theorem 3.28 holds, with the same eigenfunction $e^{h(x)}$ and a new eigenvalue e^{k-c}.

4 Applications to IFS and Related Problems

4.1 Subshifts of finite type

Let $A = \{0, 1, \ldots, m-1\}$ be an alphabet and $C = (c_{ij})_{m\times m}$ an adjacency matrix with entries in $\{0,1\}$. Let $X = \Sigma_C \subseteq A^{\mathbb{N}}$ be the set of all infinite admissible sequences. To get information about thermodynamic formalism in the setting of sequence decision-making processes, for each $x \in \Sigma_C$, we put $\Psi(x) = \{i \in A \mid c_{i,x_0} = 1\}$ and we recover the dynamics by considering the maps $f(x,a) = (a, x_0, x_1, \ldots)$, for each $a \in \Psi(x)$. Given a Hölder potential $g : X \to \mathbb{R}$, we define $u(x,a) = g(f(x,a))$. Considering a variable discount δ we obtain a sequential decision-making process $S = \{X, A, \Psi, f, u, \delta\}$.

4.2 Dynamics of expanding endomorphisms

Let (X,d) be a complete metric space and $T : X \to X$ a continuous expanding endomorphism. Suppose that for each point $x \in X$ there is finite set of injective domains $J_0, J_1, \ldots, J_{n-1}$ for T. Take $A := \{0, 1, \ldots, n-1\}$ and for each $x \in X$ define $\Psi(x) := \{a \in A \,|\, x \in T^{-1}|_{J_a}(X)\}$. The function f is defined as follows $f(x,a) = T^{-1}|_{J_a}(x)$, for each $a \in \Psi(x)$. Given a Hölder potential $\varphi : X \to \mathbb{R}$ we define $u(x,a) := \varphi(f(x,a))$ and $\delta(t) = \beta t$. Then $S = \{X, A, \Psi, f, u, \delta\}$ is a sequential decision-making process associated with the thermodynamical formalism for the endomorphism T with a potential φ.

Following the classical approach in thermodynamical formalism as in [Bou01] and [LMMS15] we have that the Bellman and the discounted transfer operators are given by

$$B(v)(x) := \sup_{T(y)=x} \varphi(y) + \beta v(y) \quad \text{and} \quad P(v)(x) := \ln \sum_{T(y)=x} e^{\varphi(y)+\beta v(y)}.$$

In [Bou01] and [LMMS15] the author shows that the discounted limit of the first one provides a calibrated subaction equation:

$$v_\infty(x) := \sup_{T(y)=x} \varphi(y) - m_\infty + v_\infty(y).$$

It is well known (see [Bou01] or [Gar17]) that a measure μ_{max} satisfying

$$m_\infty = \sup_{T^*\mu=\mu} \int_X g \, d\mu = \int_X \varphi \, d\mu_{max},$$

is supported in $\{y \in X \,|\, v_\infty(T(y)) = \varphi(y) - m_\infty + v_\infty(y)\}$. It is also well known (see [Bou01] or [LMMS15]) that the discounted limit of the second one gives a positive eigenfunction $e^{v_\infty(x)}$ and an maximal eigenvalue e^k (which is the spectral radius) of the Ruelle operator, that is,

$$e^k e^{v_\infty(x)} = \sum_{T(y)=x} e^{\varphi(y)} e^{v_\infty(y)}.$$

4.3 IFS with Weights and Thermodynamic Formalism

Let (X,d) be a complete metric space, and (A, d_A) an arbitrary metric space, indexing a family of continuous functions $\phi_a : X \to X$. Consider the IFS $(X, (\phi_a)_{a\in A})$. If in addition, a family of probability measures $p_a : X \to [0,1]$, indexed in A, is given one can construct an ordered triple $(X, (\phi_a)_{a\in A}, (p_a)_{a\in A})$ which is called an iterated function system with place dependent probabilities (IFSpdp). To view such IFSpdp as sequential decision-making process associate to this IFS, we take $\Psi(x) = A,\ \forall x \in X$, consider the immediate return $u(x, a) = \ln p_a(x)$, which is bounded from above if each p_a is so, and is bounded from below if each $p_a(x) > \alpha > 0$. If we consider the dynamics $f(x, a) = \phi_a(x)$ and a discount function δ then $S = \{X, A, \Psi, f, u, \delta\}$ is a sequential decision-making process associated with the thermodynamical formalism of the IFSpdp (X, ϕ_a, p_a).

Assuming the hypothesis of Lemmas 3.21 and 3.22, Assumption 3.23, Theorem 3.24, and Remark 3.26 we have that the equation $b(x) = \max_{a\in\Psi(x)} \ln p_a(x) - \bar{u} + b(f(x, a))$ can be rewritten as

$$\bar{u} = \max_{a\in\Psi(x)} \ln p_a(x) + b(f(x, a)) - b(x) = \max_{a\in\Psi(x)} d_x b(a) + u(x, a),$$

where the discrete differential is $d_x b(a) = b(f(x, a)) - b(x)$ and

$$\bar{u} = \sup_{\mu\in\mathcal{H}} \int_\Omega \ln p_a(x)\, d\mu(x, a)$$

where the set of holonomic probabilities is given by

$$\mathcal{H} := \left\{ \mu \in \mathscr{P}(\Omega) \mid \int_\Omega d_x g(a)\, d\mu(x, a) = 0,\ \forall g \in C(A, \mathbb{R}) \right\}.$$

From Theorem 3.28 there exists a value $k \in [-\|u\|_\infty, \|u\|_\infty]$ and a function h such that $\rho := e^k$ and $\varphi := e^{h(x)}$ are respectively a positive eigenvalue and a positive and continuous eigenfunction for Ruelle's Operator

$$e^k e^{h(x)} = \int_{a\in\Psi(x)} e^{u(x,a)} e^{h(f(x,a))} d\nu_x(a),$$

or equivalently

$$\int_{a\in\Psi(x)} p_a(x) e^{h(f(x,a))} d\nu_x(a) = e^k e^{h(x)}.$$

As a historical remark we shall mention that the first version of the Ruele-Perron-Frobenius theorem for contractive IFS, via shift conjugation, was obtained in [FL99].

Theorem 4.1. *The IFS case encompasses the expanding endomorphism case, if $A = \{0, 1, \ldots, n-1\}$, $\Psi(x) = \{j \in A \,|\, x \in T_j(X)\}$, $f(x, a) = T^{-1}|_{J_a}$ and $u(x, a) := \varphi(f(x, a))$, where $\varphi : X \to \mathbb{R}$ is a Hölder potential, similarly to Example 4.2. In this case the IFSpdp (X, ϕ_a, p_a) is such that $\phi_a(x) = f(x, a)$ and $p_a(x) = \exp(\varphi(f(x, a)))$.*

Proof. We notice that $T \circ f(x, a) = x,\ \forall a \in A$. The fact that T is uniformly expanding implies that $f(\cdot, a)$ is a uniform contraction and, the fact that φ is

Hölder implies bounded and domination conditions of Theorem 3.20 are in hold. Obviously, Bellman's equation is

$$b(x) = \max_{a\in\Psi(x)} \ln p_a(x) - \bar{u} + b(f(x,a)) = \max_{a\in\Psi(x)} \varphi(f(x,a)) - \bar{u} + b(f(x,a))$$

since $u(x,a) = \ln p_a(x) = \varphi(f(x,a))$. It remains to show that $\bar{u} = m_\infty$. Indeed, take any g and $\mu \in \mathcal{H}$ then

$$\begin{aligned}
0 &= \int_\Omega d_x(g\circ T)(a)\, d\mu(x,a) \\
&= \int_\Omega (g\circ T)(f(x,a)) - (g\circ T)(x)\, d\mu(x,a) \\
&= \int_X \int_A g(x) - g(T(x))\, d\mu_x(a)\, d\pi^*(\mu)(x) \\
&= \int_X g(x) - g(T(x))\, d\pi^*(\mu)(x),
\end{aligned}$$

where π^* is the push forward with respect to the projection in X. Thus, $\pi^*(\mu)$ is a T-invariant measure, that is $\pi^*(\mathcal{H}) \subseteq \{\eta \mid T\eta = \eta\}$. Moreover

$$\begin{aligned}
\int_\Omega u(x,a)\, d\mu(x,a) &= \int_\Omega \varphi(f(x,a))\, d\mu(x,a) \\
&= \int_\Omega \varphi(f(x,a)) - \varphi(x) + \varphi(x)\, d\mu(x,a) \\
&= \int_\Omega d_x(\varphi)(a) d\mu(x,a) + \int_X \varphi(x)\, d\pi^*(\mu)(x) \\
&= 0 + \int_X \varphi(x)\, d\pi^*(\mu)(x),
\end{aligned}$$

thus showing that $\bar{u} \leq m_\infty$. To obtain the equality we will construct a special holonomic measure $\mu \in \mathcal{H}$ satisfying

$$\int_\Omega u(x,a) d\mu(x,a) = m_\infty.$$

We first observe that from the calibrated subaction equation

$$v_\infty(x) = \sup_{T(y)=x} \varphi(y) - m_\infty + v_\infty(y)$$

which is equivalent to

$$m_\infty = \sup_{T(y)=x} \varphi(y) + d_x v_\infty(y)$$

we can obtain, proceeding similarly as in Theorem 3.27, an optimal holonomic measure $\mu \in \mathcal{H}$, such that

$$\int_\Omega u(x,a) d\mu(x,a) = m_\infty.$$

This shows that the calibrated subaction equation is equivalent to the associated Bellman's equation.

Finally, notice that the equation

$$\int_{a\in\Psi(x)} p_a(x)e^{h(f(x,a))}d\nu_x(a) = e^k e^{h(x)},$$

for the IFS is equivalent to,

$$\int_{a\in\Psi(x)} e^{\varphi(f(x,a))}e^{h(f(x,a))}d\nu_x(a) = e^k e^{h(x)}.$$

Recalling that $T \circ f(x,a) = x$, $\forall a \in A$, we obtain

$$\sum_{T(y)=x} e^{\varphi(y)-\ln n}e^{h(y)} = e^k e^{h(x)},$$

where $\nu_x(a) = (1/n)\sum_j \delta_j(a)$, which is the same operator as considered in the endomorphism case, up to the constant $(-\ln n)$. $\square$

Acknowledgments

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. L. Cioletti would like to acknowledge financial support by CNPq through project 310818/2015-0.

References

[BC88] P. C. Bhakta and S. R. Choudhury. Some existence theorems for functional equations arising in dynamic programming. II. *J. Math. Anal. Appl.*, 131(1):217–231, 1988.

[BCL+11] A. T. Baraviera, L. Cioletti, A. O. Lopes, J. Mohr, and R. R. Souza. On the general one-dimensional XY model: positive and zero temperature, selection and non-selection. *Rev. Math. Phys.*, 23(10):1063–1113, 2011.

[Bel57] R. Bellman. *Dynamic programming.* Princeton University Press, Princeton, N. J., 1957.

[Ber97] C. Berge. *Topological spaces.* Dover Publications, Inc., Mineola, NY, 1997. Including a treatment of multi-valued functions, vector spaces and convexity, Translated from the French original by E. M. Patterson, Reprint of the 1963 translation.

[Ber13] D. P. Bertsekas. *Abstract dynamic programming.* Athena Scientific, Belmont, MA, 2013.

[BG10] A. Biryuk and D. A. Gomes. An introduction to the Aubry-Mather theory. *São Paulo J. Math. Sci.*, 4(1):17–63, 2010.

[BKRLU06] R. Bamón, J. Kiwi, J. Rivera-Letelier, and R. Urzúa. On the topology of solenoidal attractors of the cylinder. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(2):209–236, 2006.

[BM84] P. C. Bhakta and S. Mitra. Some existence theorems for functional equations arising in dynamic programming. *J. Math. Anal. Appl.*, 98(2):348–362, 1984.

[Bou01] T. Bousch. La condition de Walters. *Ann. Sci. École Norm. Sup. (4)*, 34(2):287–311, 2001.

[CM91] S-s. Chang and Y. H. Ma. Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming. *J. Math. Anal. Appl.*, 160(2):468–479, 1991.

[CV87] F. H. Clarke and R. B. Vinter. The relationship between the maximum principle and dynamic programming. *SIAM J. Control Optim.*, 25(5):1291–1311, 1987.

[dVVM$^+$08] Y. del Valle, G. K. Venayagamoorthy, S. Mohagheghi, J. C. Hernandez, and R. G. Harley. Particle swarm optimization: Basic concepts, variants and applications in power systems. *IEEE Transactions on Evolutionary Computation*, 12(2):171–195, 2008.

[Fat97a] A. Fathi. Solutions KAM faibles conjuguées et barrières de Peierls. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(6):649–652, 1997.

[Fat97b] A. Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(9):1043–1046, 1997.

[Fat98a] A. Fathi. Orbites hétéroclines et ensemble de Peierls. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(10):1213–1216, 1998.

[Fat98b] A. Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(3):267–270, 1998.

[FL99] Ai Hua Fan and Ka-Sing Lau. Iterated function system and Ruelle operator. *J. Math. Anal. Appl.*, 231(2):319–344, 1999.

[GAGK12] M. Gheshlaghi Azar, V. Gómez, and H. J. Kappen. Dynamic policy programming. *J. Mach. Learn. Res.*, 13:3207–3245, 2012.

[Gar17] E. Garibaldi. *Ergodic optimization in the expanding case*. SpringerBriefs in Mathematics. Springer, Cham, 2017. Concepts, tools and applications.

[GO12] D. A. Gomes and E. R. Oliveira. Mather problem and viscosity solutions in the stationary setting. *São Paulo J. Math. Sci.*, 6(2):301–334, 2012.

[Gom05] D. A. Gomes. Viscosity solution methods and the discrete Aubry-Mather problem. *Discrete Contin. Dyn. Syst.*, 13(1):103–116, 2005.

[Gom08] D. A. Gomes. Generalized Mather problem and selection principles for viscosity solutions and Mather measures. *Adv. Calc. Var.*, 1(3):291–307, 2008.

[Jen18] O. Jenkinson. Ergodic optimization in dynamical systems. *Ergodic Theory and Dynamical Systems*, pages 1–26, 2018.

[JMN13] A. Jaśkiewicz, J. Matkowski, and A. S. Nowak. Persistently optimal policies in stochastic dynamic programming with generalized discounting. *Math. Oper. Res.*, 38(1):108–121, 2013.

[JMN14a] A. Jaśkiewicz, J. Matkowski, and A. S. Nowak. Generalised discounting in dynamic programming with unbounded returns. *Oper. Res. Lett.*, 42(3):231–233, 2014.

[JMN14b] A. Jaśkiewicz, J. Matkowski, and A. S. Nowak. On variable discounting in dynamic programming: applications to resource extraction and other economic models. *Ann. Oper. Res.*, 220:263–278, 2014.

[Liu01] Z. Liu. Existence theorems of solutions for certain classes of functional equations arising in dynamic programming. *J. Math. Anal. Appl.*, 262(2):529–553, 2001.

[LMMS15] A. O. Lopes, J. K. Mengue, J. Mohr, and R. R. Souza. Entropy and variational principle for one-dimensional lattice systems with a general *a priori* probability: positive and zero temperature. *Ergodic Theory Dynam. Systems*, 35(6):1925–1961, 2015.

[LO14] A. O. Lopes and E. R. Oliveira. On the thin boundary of the fat attractor. *Preprint arXiv:1402.7313*, 2014.

[Mat75] J. Matkowski. Integrable solutions of functional equations. *Dissertationes Math. (Rozprawy Mat.)*, 127:68, 1975.

[Mn92] R. Mañé. On the minimizing measures of Lagrangian dynamical systems. *Nonlinearity*, 5(3):623–638, 1992.

[Mn96] R. Mañé. Generic properties and problems of minimizing measures of Lagrangian systems. *Nonlinearity*, 9(2):273–310, 1996.

[Put94] M. L. Puterman. *Markov decision processes: discrete stochastic dynamic programming.* Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Inc., New York, 1994. A Wiley-Interscience Publication.

[Roc66] R. T. Rockafellar. Extension of Fenchel's duality theorem for convex functions. *Duke Math. J.*, 33:81–89, 1966.

[TBS10] E. A. Theodorou, J. Buchli, and S. Schaal. A generalized path integral control approach to reinforcement learning. *J. Mach. Learn. Res.*, 11:3137–3181, 2010.

[Tsu01] M. Tsujii. Fat solenoidal attractors. *Nonlinearity*, 14(5):1011–1027, 2001.