DLR-EQUATIONS FOR LOCAL FUNCTIONS AND QUASILOCALITY HYPOTHESIS

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ABSTRACT. We shown that the validity of the DLR equations for bounded local functions can extended for any bounded measurable functions (with respect to the σ-algebra generated by the cylinder sets) for Gibbs measures specified by *Quasilocal Specifications* with parameter set $\mathbb{Z}^d$ and finite state space E.

This is an expository article and we remark that non content here is new.

1. INTRODUCTION

The aim of this short note is to present, in details, a particular case of the Theorem 4.17 and Remark 4.21 of the reference [1]. Here we work on compact configuration spaces, more precisely, those given by an infinite cartesian product of a fixed finite set. Such spaces are chosen in order to simplify the argument and avoid the use of topological nets in the proof.

The paper has three sections and the main result is presented in the last one. In the next section we introduce the basic definitions about specifications and then we present a classical characterization of the so callled DLR Gibbs Measures. In the third section, after a brief discussion about quasilocality for functions and specifications we prove the Theorem 3.4, which is the main result of this work.

We refer the reader to [1] for a comprehensive exposition of the Theory of Gibbs Measures.

2. SPECIFICATIONS AND DLR GIBBS MEASURES

Let $E \subset \mathbb{R}$ be a finite set and $\Omega = E^{\mathbb{Z}^d} \equiv \{(\omega_i)_{i\in\mathbb{Z}^d} : \omega_i \in E\ \forall i \in \mathbb{Z}^d\}$. For a subset $\Lambda \subset \mathbb{Z}^d$ we use the notation $|\Lambda|$ to denote its cardinality. In order to lightening the notation, from now on, we use the Greek letters Λ and Γ exclusively to denote **finite** subsets of $\mathbb{Z}^d$. For a fixed $i \in \mathbb{Z}^d$ consider the coordinate function $X_i : \Omega \to E$ given by $X_i(\omega) = \omega_i$ and for any non-empty set Λ let $\mathcal{F}_\Lambda = \sigma(X_i : i \in \Lambda)$ the σ-algebra generated by the collection $\{X_i\}_{i\in\Lambda}$. We define $\mathcal{F}_{\Lambda^c}$ being $\sigma(\cup_\Gamma \mathcal{F}_\Gamma : \Gamma \subset \Lambda^c)$ and finally we define $\mathcal{F} = \sigma(\cup_\Lambda \mathcal{F}_\Lambda)$.

A function $\gamma_\Lambda : \mathcal{F} \times \Omega \to [0, 1]$ is called a **proper probability kernel** from $\mathcal{F}_{\Lambda^c}$ to $\mathcal{F}$, if the following conditions are satisfied:

- $\gamma_\Lambda(\cdot|\omega)$ is a measure on $(\Omega, \mathcal{F})$ for any fixed $\omega \in \Omega$;
- $\gamma_\Lambda(A|\cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable for any fixed $A \in \mathcal{F}$.
- $\gamma_\Lambda(A \cap B|\omega) = 1_B(\omega)\gamma_\Lambda(A|\omega)$, for any $A \in \mathcal{F}$, $B \in \mathcal{F}_{\Lambda^c}$ and $\omega \in \Omega$.

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We say that a family $\gamma = (\gamma)_{\Lambda\subset\mathbb{Z}^d}$ is **consistent** if for all $A \in \mathcal{F}$ and $\omega \in \Omega$, we have

$$\int_{\Omega} \gamma_\Lambda(A|\cdot)d\gamma_\Gamma(\cdot|\omega) = \gamma_\Gamma(A|\omega), \text{ whenever } \emptyset \subsetneq \Lambda \subset \Gamma.$$

The lhs above is denoted in [1] simply by $\gamma_\Gamma\gamma_\Lambda(A|\omega)$. In this notation the consistency defined above reads $\gamma_\Gamma\gamma_\Lambda = \gamma_\Lambda$, for any pair of non-empty sets $\Lambda \subset \Gamma$.

Definition 2.1. A **specification** with parameter set $\mathbb{Z}^d$ and state space E is a family $\gamma = (\gamma)_{\Lambda\subset\mathbb{Z}^d}$ of proper probability kernels γ_Λ from $\mathcal{F}_{\Lambda^c}$ to $\mathcal{F}$ which satisfy the consistence condition $\gamma_\Gamma\gamma_\Lambda = \gamma_\Gamma$, when $\emptyset \subsetneq \Lambda \subset \Gamma$.

We denote the set of all probability measures defined on $(\Omega, \mathcal{F})$ by $\mathscr{P}(\Omega, \mathcal{F})$. Now we are ready to define the set of the Gibbs Measures.

Definition 2.2 (Gibbs Measures)**.** Given a specification γ with parameter set $\mathbb{Z}^d$ and state space E. The set of all probability measures defined by

$$\mathscr{G}(\gamma) := \left\{ \mu \in \mathscr{P}(\Omega, \mathcal{F}) : \begin{array}{c} \mu(A|\mathcal{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(A|\omega)\ \mu - a.s. \\ \text{for all } A \in \mathcal{F} \text{ and } \Lambda \subset \mathbb{Z}^d. \end{array} \right\}$$

is called the set of the Gibbs Measures determined by the specification γ. Each element $\mu \in \mathscr{G}(\gamma)$ is called a Gibbs measure.

Theorem 2.3. *Suppose that $\gamma = (\gamma_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ is a specification with parameter set $\mathbb{Z}^d$ and state space E and $\mu \in \mathscr{P}(\Omega, \mathcal{F})$. Then the following statements are equivalent:*

(1) $\mu \in \mathscr{G}(\gamma)$;
(2) for all $A \in \mathcal{F}$ and $\Lambda \subset \mathbb{Z}^d$, we have $\mu(A) = \int_\Omega \gamma_\Lambda(A|\omega)\, d\mu(\omega) := \mu\gamma_\Lambda(A)$;
(3) There is a cofinal collection [1] $\{\Gamma_\alpha : |\Gamma_\alpha| < +\infty,\ \forall\, \alpha \in I\}$, (i.e., directed by inclusion and for any finite $\Lambda \subset \mathbb{Z}^d$ there is an index $\alpha \in I$ so that $\Lambda \subset \Gamma_\alpha$) satisfying:

$$\mu(A) = \int_\Omega \gamma_{\Gamma_\alpha}(A|\omega)\, d\mu(\omega) := \mu\gamma_{\Gamma_\alpha}(A).$$

Proof. **1) implies 2)**. If $\mu \in \mathscr{G}(\gamma)$ then follows from the definition of $\mathscr{G}(\gamma)$ and the basic property of the conditional expectation that

$$\mu(A) = \int_\Omega \mu(A|\mathcal{F}_{\Lambda^c})(\omega)\ d\mu(\omega) = \int_\Omega \gamma_\Lambda(A|\omega)\, d\mu(\omega) := \mu\gamma_\Lambda(A),$$

for any $A \in \mathcal{F}$. Now we prove that **2) implies 1)**. Let $A \in \mathcal{F}$ and $B \in \mathcal{F}_{\Lambda^c}$. Using the hypothesis and that γ_Λ is a proper probability kernel we get that

$$\mu(A \cap B) = \int_\Omega \pi_\Lambda(A \cap B|\omega)\, d\mu(\omega) = \int_\Omega 1_B(\omega)\pi_\Lambda(A|\omega)\, d\mu(\omega).$$

From the basic properties of the conditional probability and the previous equality we get

$$\int_\Omega 1_B(\omega)\mu(A|\mathcal{F}_{\Lambda^c})(\omega)\, d\mu(\omega) = \mu(A \cap B) = \int_\Omega 1_B(\omega)\pi_\Lambda(A|\omega)\, d\mu(\omega).$$

From the above equation we have for any $B \in \mathcal{F}_{\Lambda^c}$ that

$$\int_\Omega 1_B(\omega)\big[\mu(A|\mathcal{F}_{\Lambda^c})(\omega) - \pi_\Lambda(A|\omega)\big]\, d\mu(\omega) = 0.$$

[1]most used cofinal collection in this context is $\{[-n, n]^d \cap \mathbb{Z}^d : n \geq 1\}$.

Since the two functions in the brackets are $\mathcal{F}_{\Lambda^c}$-measurable their difference is also $\mathcal{F}_{\Lambda^c}$-measurable so we can take B in the above equation as being $B = \{\omega \in \Omega : \mu(A|\mathcal{F}_{\Lambda^c})(\omega) - \pi_\Lambda(A|\omega) > 0\}$. With this choice of B it follows from the above equation that $\mu(B) = 0$. Analogously the set where this difference is negative has also μ measure zero. Therefore $\mu(A|\mathcal{F}_{\Lambda^c})(\omega) = \pi_\Lambda(A|\omega)\ \mu - a.s.$.

The statement **2) implies 3)** is obvious. We proceed to **3) implies 2)**. Since $\{\Gamma_\alpha : |\Gamma_\alpha| < +\infty,\ \forall\ \alpha \in I\}$ is a cofinal sequence, for any given Λ there is an index $\alpha \in I$ such that $\Lambda \subset \Gamma_\alpha := \Gamma$. From the hypothesis we have $\mu = \mu\gamma_\Gamma$. By integrating the kernel γ_Λ with respect to this measure we obtain the measure $\mu\gamma_\Lambda = (\mu\gamma_\Gamma)\gamma_\Lambda$. We claim that $(\mu\gamma_\Gamma)\gamma_\Lambda = \mu\gamma_\Gamma$. Let $A \in \mathcal{F}$. By definition

$$(\mu\gamma_\Gamma)\gamma_\Lambda(A) = \int_\Omega \gamma_\Lambda(A|\omega)\, d(\mu\gamma_\Gamma)(\omega). \tag{2.1}$$

By standard arguments of the Measure Theory, we know that there exist a sequence of $\mathcal{F}_{\Lambda^c}$-measurable simple functions φ_n such that $\varphi_n(\omega) \uparrow \gamma_\Lambda(A|\omega)$ for all $\omega \in \Omega$. By using several times the Monotone Convergence Theorem in the equation (2.1) and the consistency of the specification γ we obtain

$$\begin{aligned}
(\mu\gamma_\Gamma)\gamma_\Lambda(A) = \int_\Omega \gamma_\Lambda(A|\omega)\, d(\mu\gamma_\Gamma)(\omega) &= \lim_{n\to\infty} \int_\Omega \varphi_n(\omega)\, d(\mu\gamma_\Gamma)(\omega) \\
&= \lim_{n\to\infty} \mu\gamma_\Gamma(\varphi_n) \\
&= \lim_{n\to\infty} \int_\Omega \gamma_\Gamma(\varphi_n|\omega)\, d\mu(\omega) \\
&= \int_\Omega \gamma_\Gamma(\gamma_\Lambda(A|\cdot)|\omega)\, d\mu(\omega) \\
&= \int_\Omega \gamma_\Gamma(A|\eta)\, d\mu(\omega) \\
&= \mu\gamma_\Gamma(A).
\end{aligned}$$

Piecing together the equations obtained above and use the hypothesis we arrive at $\mu\gamma_\Lambda = (\mu\gamma_\Gamma)\gamma_\Lambda = \mu\gamma_\Gamma = \mu$. □

3. Quasilocality and the Main Result

Definition 3.1. A real function $f : \Omega \to \mathbb{R}$ is called a **local function** if f is $\mathcal{F}_\Lambda$-measurable for some finite Λ. For each Λ we denote by $\mathscr{L}_\Lambda$ the space of all **bounded** $\mathcal{F}_\Lambda$-measurable local functions. Let $\mathscr{L} = \cup_\Lambda \mathscr{L}_\Lambda$ denote the set of all bounded local functions.

Definition 3.2. A function $f : \Omega \to \mathbb{R}$ is said to be **quasilocal** if there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{L}$ such that $\|f - f_n\|_\infty \to 0$, when $n \to \infty$. Here $\|\cdot\|_\infty$ is the sup-norm. We write $\overline{\mathscr{L}}$ for the space of all bounded quasilocal functions.

Definition 3.3. We say that a given specification $\gamma = (\gamma_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ is quasilocal if, for each finite $\Lambda \subset \mathbb{Z}^d$ and $f \in \overline{\mathscr{L}}$ the mapping

$$\omega \mapsto \int_\Omega f(\eta)\, d\gamma_\Lambda(\eta|\omega),$$

is quasilocal. This mapping will be denoted simply by $\gamma_\Lambda f$.

Theorem 3.4. *Suppose that γ is a quasilocal specification with parameter set $\mathbb{Z}^d$ and state space E. Then $\mu \in \mathscr{G}(\gamma)$ if and only if*

$$\mu(A) = \int_\Omega \gamma_\Lambda(A|\omega)\, d\mu(\omega), \quad \forall\ A \in \mathcal{F}_\Lambda \text{ and all } \Lambda \subset \mathbb{Z}^d \text{ with } |\Lambda| < +\infty.$$

Proof. We first assume that $\mu \in \mathscr{G}(\gamma)$. Then it follows from the elementary properties of the conditional expectation that

$$\mu(A) = \int_\Omega \gamma_\Lambda(A|\omega)\, d\mu(\omega), \quad \forall\ A \in \mathcal{F}_\Lambda \text{ and all } \Lambda \subset \mathbb{Z}^d \text{ with } |\Lambda| < +\infty.$$

Conversely, suppose that the above equality holds true. Let $\Lambda_n = \{[-n,n] \cap \mathbb{Z}^d : n \geq 1\}$. We first prove that, when $n \to \infty$, we have $\mu\gamma_{\Lambda_n} \rightharpoonup \mu$, where the convergence is in the weak sense. To prove this weak convergence we fix a continuous function $f : \Omega \to \mathbb{R}$. Since the state space E is finite we can assure that any local function is continuous and therefore any function in $\overline{\mathscr{L}}$ is continuous. A stronger result holds $C(\Omega) = \overline{\mathscr{L}}$. Therefore there is a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{L}$ so that $\|f_k - f\|_\infty \to 0$. Let $n_k \in \mathbb{N}$ the smaller integer for which $f_k \in \mathscr{L}_{\Lambda_{n_k}}$. Given $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ so that for any $k \geq k_0$ we have $\|f_k - f\|_\infty < \varepsilon$. On the other hand, for any $n \in \mathbb{N}$ we get from the triangular inequality that

$$|\mu\gamma_{\Lambda_n}(f) - \mu(f)| \leq \mu\gamma_{\Lambda_n}(|f - f_k|) + |\mu\gamma_{\Lambda_n}(f_k) - \mu(f)|.$$

The first term on rhs is bounded by ε for any $n \in \mathbb{N}$ and $k = k_0$. If $n \geq n_{k_0}$ then follows from the hypothesis that $\mu\gamma_{\Lambda_n}(f_k) = \mu(f_k)$. So the second term in rhs in the above inequality is also smaller than ε as long as $n \geq n_{k_0}$. Since ε is arbitrary we have that $\mu\gamma_{\Lambda_n}(f) \to \mu(f)$, $\forall f \in C(\Omega)$.

The next step is to prove that DLR equations are satisfied, i.e., $\mu(A) = \mu\gamma_\Lambda(A)$ for all Λ and $A \in \mathcal{F}$. First let us fix Λ and $f \in \mathscr{L}$. Using the **quasilocality** of the specification γ we can assure that the function $\gamma_\Lambda(f)$ is quasilocal and therefore continuous, so it follows from the weak convergence established above that

$$|\mu\gamma_\Lambda(f) - \mu(f)| = \lim_{n\to\infty} |(\mu\gamma_{\Lambda_n})\gamma_\Lambda(f) - (\mu\gamma_{\Lambda_n})(f)|.$$

The **consistency** of the specification, implies that the second term on rhs above (for large enough n, so that $\Lambda \subset \Lambda_n$) satisfies the following equality $(\mu\gamma_{\Lambda_n})(f) = (\mu\gamma_{\Lambda_n})\gamma_\Lambda(f)$ which in turn implies that $|\mu\gamma_\Lambda(f) - \mu(f)| = 0$.

By taking $f = 1_C$, where $C \subset \Omega$ is a cylinder event, we have from the previous result that $\mu\gamma_\Lambda(C) = \mu(C)$. In other words, the restriction of both measures $\mu\gamma_\Lambda$ and μ to the algebra of the cylinder sets coincide. By the Carathéodry Extension Theorem both measures have an unique extension to the σ-algebra generated by the cylinder sets and this conclude the proof.

□

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References

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