Boundedness of solutions of measure differential equations and dynamic equations on time scales

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Abstract

In this paper, we investigate the boundedness results for measure differential equations. In order to obtain our results, we use the correspondence between these equations and generalized ODEs. Furthermore, we prove our results concerning boundedness of solutions for dynamic equations on time scales, using the fact that these equations represent a particular case of measure differential equations.

Keywords: Measure differential equations, generalized ordinary differential equations, dynamic equations on time scales, boundedness, Kurzweil-Henstock-Stieltjes integral, Lyapunov functionals.

1 Introduction

In this paper, we are interested to investigate the boundedness of solutions of the measure differential equations given by

$$Dx = f(x,t)Dg,$$

where Dx and Dg are the distributional derivative in the sense of L. Schwartz, of x and g, respectively. In [6], the authors proved that this equation has the following integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\mathrm{d}g(s), \qquad t \geq t_0, \tag{1.1}$$

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where the integral in the right-hand side is understood in the sense of Lebesgue-Stieltjes. Here, we are interested in the formulation (1.1), but in the case when the integral in the right-hand side of equation (1.1) is in the sense of Kurzweil-Henstock-Stieltjes.

We start by introducing new concepts concerning boundedness of the solutions of the measure differential equations and then we obtain results concerning boundedness of solutions using the correspondence between the solutions of these equations and the solutions of the generalized ODEs. We point out that the proved theorems do not require the Lipschitz condition on the Lyapunov functional and, thus they are more general than the ones found in the literature. See, for instance, [1, 5, 9].

More precisely, we ensure that the measure differential equation is uniformly bounded, only requiring the following conditions concerning the Lyapunov functional

1. For each left-continuous function $z : [\alpha, \beta] \to \mathbb{R}^n$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]$.

2. There are two increasing monotone functions $p,\ b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(0) = b(0) = 0$, $\lim_{s\to+\infty} b(s) = +\infty$ and $b(\|z\|) \leq U(t, z) \leq p(\|z\|)$ for every pair $(t, z) \in [t_0, +\infty) \times \mathbb{R}^n$.

3. For every solution $z : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, of the measure differential equation (1.1), we have
$$U(s, z(s)) - U(t, z(t)) \leq 0,$$
for every $s_0 \leq t < s < +\infty$.

Also, our conditions concerning the functions f and g are more general than the classical conditions which appears in the classical results on boundedness of solutions. It is due to the fact that we require the condition in the integral of the function f, not in the proper function f. These types of conditions allow that our function f may be highly oscillating and have many discontinuities, since the integral which we are considering is in the sense of Kurzweil-Henstock-Stieltjes.

Furthermore, we prove that the measure differential equation is uniformly ultimately bounded, only requiring the conditions 1, 2 and the following two conditions concerning the functional

- For every $x,\ y : [\alpha, \beta] \to \mathbb{R}^n$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation on $[\alpha, \beta]$, we have
$$\|U(t, x(t)) - U(s, x(s)) - U(t, y(t)) + U(s, y(s))\| \leq$$
$$\leq \left(\int_s^t K(\tau) du(\tau)\right) \sup_{\xi\in[\alpha,\beta]} \|x(\xi) - y(\xi)\|$$
for every $\alpha \leq s < t \leq \beta$, where $u : [t_0, +\infty) \to \mathbb{R}$ is nondecreasing and left-continuous function and $K : [t_0, +\infty) \to \mathbb{R}$ is a locally Kurzweil-Henstock Δ-integrable function with respect to u.

- There exists a continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with $\Phi(0) = 0$ and $\Phi(x) > 0$, $x \neq 0$, such that for every solution $z : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, of the measure differential equation, we have

$$U(s, z(s)) - U(t, z(t)) \leq (s - t)\Big(- \Phi(z(t))\Big)$$

for every $s_0 \leq t < s < +\infty$.

After that, we extend our results concerning boundedness of the solutions for dynamic equations on time scales, using the fact that a dynamic equation on time scales is a special case of a measure differential equation. See [17].

We recall that an initial value problem for a dynamic equation on time scales is given by

$$\begin{cases} x^{\Delta}(t) = f(x^*, t), \quad t \in J \cap \mathbb{T}, \\ x(t_0) = x_0, \end{cases} \tag{1.2}$$

where $J \subset [t_0, +\infty)$ is a nondegenerated interval such that $J \cap \mathbb{T}$ is nonempty.

Then, we introduce new concepts of boundedness of the solutions of the equation (1.2) and we prove the results concerning boundedness of the solutions of the dynamic equations on time scales. We point out that these equations play an important role for applications, since they encompass the differential equations (when $\mathbb{T} = \mathbb{R}$), difference equations (when $\mathbb{T} = \mathbb{Z}$) and the cases in "between". See, for instance, [3, 4].

The present paper is organized as follows. In the second section, we recall some basic definitions and results about Kurzweil integration and generalized ODEs. The third section is devoted to recall the boundedness concepts for generalized ODEs and to prove new results about boundedness of solutions of generalized ODEs. In the fourth section, we recall some basic definitions and properties about measure differential equations. Also, we present boundedness concepts for measure differential equations and using the correspondence between generalized ODEs and measure differential equations, we prove the results concerning boundedness of solutions for these last equations. Finally, in the fifth section, we remember some basic definitions about dynamic equations on time scales and present the boundedness concepts of solutions of dynamic equations on time scales. Therefore, using the correspondence between the solutions of dynamic equations on time scales and measure differential equations, we extend the results on the boundedness to the dynamic equations on time scales.

2 Generalized ODEs

In this section, we will recall the basic concepts and properties of the Kurzweil integral which plays an important role in the study of the generalized ODEs. For more details, see [16].

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on a set $E \subset [a,b]$ is any function $\delta : E \to (0, \infty)$. Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ is *δ-fine* if, for every i, we have

$$[s_{i-1}, s_i] \subset \{t \in [a,b]\, ;\, |t - \tau_i| < \delta(\tau_i)\}.$$

We denote such tagged division by $d = (\tau_i, [s_{i-1}, s_i])$.

Throughout this section, let us assume that X is a Banach space, $O \subset X$ is an open subset and $t_0 \in \mathbb{R}$.

Definition 2.1. A function $U : [a,b] \times [a,b] \to X$ is called *Kurzweil integrable* over $[a,b]$, if there is a unique element $I \in X$ such that given $\epsilon > 0$, there is a gauge δ on $[a,b]$ such that

$$\left\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \epsilon$$

for every δ-fine tagged division $(\tau_i, [s_{i-1}, s_i])$ of $[a,b]$. In this case, I is called the Kurzweil integral of U over $[a,b]$ and it is denoted by $\int_a^b DU(\tau, t)$.

The following result is an immediate consequence of the Saks-Henstock Lemma. See [16, Lemma 1.13].

Corollary 2.2. *Let $U : [a,b] \times [a,b] \to X$ be Kurzweil integrable over $[a,b]$. Given $\epsilon > 0$, let δ be a gauge of $[a,b]$ corresponding to ϵ in the definition of the Kurzweil integral and $[\gamma, v] \subset [a,b]$. Then, we have*

(i) $(v - \gamma) < \delta(\gamma)$ *implies*

$$\left\| U(\gamma, v) - U(\gamma, \gamma) - \int_\gamma^v DU(\tau, t) \right\| < \epsilon;$$

(ii) $(v - \gamma) < \delta(v)$ *implies*

$$\left\| U(v, v) - U(v, \gamma) - \int_\gamma^v DU(\tau, t) \right\| < \epsilon.$$

Suppose $F : \Omega \to X$ is defined for each $(x,t) \in \Omega$, where $\Omega = O \times [t_0, +\infty)$.

Definition 2.3. A function $x : [\alpha, \beta] \to X$ is called a *solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t) \tag{2.1}$$

on the interval $[\alpha, \beta] \subset [t_0, +\infty)$, if $(x(t), t) \in \Omega$ for every $t \in [\alpha, \beta]$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t), \tag{2.2}$$

for every $s_1, s_2 \in [\alpha, \beta]$.

The integral on the right-hand side of (2.2) can be understood as the Kurzweil integral, according to the Definition 2.1.

We can also define the solution of the generalized ODE (2.1) with an initial condition. It is the content of the next definition.

Definition 2.4. The function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ODE* (2.1) *with initial condition* $x(s_0) = z_0$ on the interval $[\alpha, \beta] \subset [t_0, +\infty)$, if $s_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for every $t \in [\alpha, \beta]$ and

$$x(s) - z_0 = \int_{s_0}^{s} DF(x(\tau), t)$$

for every $s \in [\alpha, \beta]$.

Remark 2.5. Although the solution x of the generalized ODE (2.1) is defined in an interval $[\alpha, \beta]$ in the Definition 2.3, it is possible to extend this definition when the solution of the generalized ODE (2.1) is defined in a nondegenerated interval I, not necessarily of the form $[a, b]$. See the next definition.

Definition 2.6. We say that $x : I \to X$, where I is a nondegenerated subinterval of $[t_0, +\infty)$, is a *solution* of the generalized ODE(2.1) on I, if $(x(t), t) \in \Omega$ for every $t \in I$ and if the following equality

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t),$$

is satisfied for every $s_1,\ s_2 \in I$.

In what follows, we present a class of functions $F : \Omega \to X$. This class will give us important properties concerning the solutions of the generalized ODE (2.1).

Definition 2.7. Given a nondecreasing function $h : [t_0, +\infty) \to \mathbb{R}$, we say that a function $F : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, if

$$\|F(x, t) - F(x, s)\| \leq |h(t) - h(s)| \tag{2.3}$$

for all $(x, t),\ (x, s) \in \Omega$ and if

$$\|F(x, t) - F(x, s) - F(y, t) + F(y, s)\| \leq \|x - y\|\, |h(t) - h(s)| \tag{2.4}$$

for all $(x, t),\ (x, s),\ (y, t),\ (y, s) \in \Omega$.

The following results give us important properties concerning the solutions of the generalized ODE (2.1), when F satisfies (2.3). They can be found in [16].

Lemma 2.8. *Let $F:\Omega\to X$ satisfy condition* (2.3). *If $[\alpha,\beta]\subset[t_0,+\infty)$ and $x:[\alpha,\beta]\to X$ is such that $(x(t),t)\in\Omega$ for every $t\in[\alpha,\beta]$ and if the Kurzweil integral $\int\limits_{\alpha}^{\beta}DF(x(\tau),t)$ exists, then for every $s_2,\ s_1\in[\alpha,\beta]$, we have*

$$\left\|\int\limits_{s_1}^{s_2}DF(x(\tau),t)\right\|\leq|h(s_2)-h(s_1)|\,.$$

Lemma 2.9. *Let $F:\Omega\to X$ satisfy the condition* (2.3). *If $[\alpha,\beta]\subset[t_0,+\infty)$ and $x:[\alpha,\beta]\to X$ is a solution of* (2.1), *then the following inequality*

$$\|x(t)-x(s)\|\leq|h(t)-h(s)| \tag{2.5}$$

holds for every $t,\ s\in[\alpha,\beta]$. Moreover, every point of $[\alpha,\beta]$, for which the function h is continuous, is a continuity point of x.

Corollary 2.10. *Let $F:\Omega\to X$ satisfy the condition* (2.3). *If $[\alpha,\beta]\subset[t_0,+\infty)$ and $x:[\alpha,\beta]\longrightarrow X$ is a solution of* (2.1), *then x is a function of bounded variation on $[\alpha,\beta]$ and*

$$\mathrm{var}_{[\alpha,\beta]}x\leq h(\beta)-h(\alpha)<+\infty. \tag{2.6}$$

Proposition 2.11. *Let $F\in\mathcal{F}(\Omega,h)$. If $[\alpha,\beta]\subset[t_0,+\infty)$ and $x:[\alpha,\beta]\to X$ is a function of bounded variation on $[\alpha,\beta]$ and $(x(s),s)\in\Omega$ for every $s\in[\alpha,\beta]$, then the Kurzweil integral $\int\limits_{\alpha}^{\beta}DF(x(\tau),t)$ exists and the function $s\mapsto\int\limits_{\alpha}^{s}DF(x(\tau),t)$ is of bounded variation on $[\alpha,\beta]$.*

Lemma 2.12. *Consider $[\alpha,\beta]\subset[t_0,+\infty)$, $x:[\alpha,\beta]\to X$ is a solution of* (2.1) *and $F:\Omega\to X$ satisfies the condition* (2.3). *Then, we have:*

$$x(t^+)-x(t)=F(x(t),t^+)-F(x(t),t),\ \textit{for every } t\in[\alpha,\beta)$$

and

$$x(t)-x(t^-)=F(x(t),t)-F(x(t),t^-),\ \textit{for every } t\in(\alpha,\beta]$$

where

$$F(x,t^+)=\lim_{s\to t^+}F(x,s),\ \textit{for every } t\in[\alpha,\beta),$$

and

$$F(x,t^-)=\lim_{s\to t^-}F(x,s),\ \textit{for every } t\in(\alpha,\beta].$$

The next result ensures us the existence and uniqueness of a maximal solution of the generalized ODE (2.1). It can be found in [10].

Theorem 2.13. *If $\Omega = X \times [t_0, +\infty)$ and $F \in \mathcal{F}(\Omega, h)$, where the function h is nondecreasing and left-continuous. Then for every $(z_0, s_0) \in \Omega$, there exists a unique maximal solution of the generalized ODE* (2.1)*, with $x(s_0) = z_0$ defined in $[s_0, +\infty)$.*

3 Boundedness of solutions of generalized ODEs

In this section, our goal is to prove some results concerning the boundedness of the solutions of the generalized ODEs using Lyapunov functionals.

Throughout this section, consider that X is a Banach space with the norm $\|\cdot\|$, $t_0 \geq 0$ and $\Omega = X \times [t_0, +\infty)$. Let $F : \Omega \to X$ be a function defined for every $(x, t) \in \Omega$ and taking values in the Banach space X. Also, suppose $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \to \mathbb{R}$ is left-continuous on $(t_0, +\infty)$.

Consider the following generalized ODE

$$\frac{dz}{d\tau} = DF(z, t) \tag{3.1}$$

with the initial condition

$$z(s_0) = z_0, \tag{3.2}$$

where $(z_0, s_0) \in \Omega$.

From now on, we assume that for every $(z_0, s_0) \in \Omega$, there exists a unique (maximal) solution $x : [s_0, +\infty) \to X$ of (3.1) with $x(s_0) = z_0$. The existence and uniqueness of such solution is ensured by Theorem 2.13.

Remark 3.1. In what follows, for every $(z_0, s_0) \in \Omega$, we denote by $x(s, s_0, z_0)$ the unique (maximal) solution of the generalized ODE (3.1) with $x(s_0) = z_0$.

Next, we recall the concepts of uniform boundedness for generalized ODEs. The basic reference for this subject is [1].

Definition 3.2. We say that the generalized ODE (3.1) is:

• **Uniformly bounded:** if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for every $s_0 \in [t_0, +\infty)$ and for all $z_0 \in X$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \geq s_0.$$

• **Quasi-uniformly ultimately bounded:** if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and for all $z_0 \in X$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \geq s_0 + T.$$

• **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

It is important to mention that the concepts, introduced in [1], were motivated by the definition of boundedness introduced by I. Stamova in [18].

The following result can be found in [16, Proposition 10.11]. It will help us to prove the next lemma, which is the key to prove our main results in this section.

Proposition 3.3. *Suppose $-\infty < a < b < +\infty$ and $f, g : [a, b] \to \mathbb{R}$ are left continuous functions on $(a, b]$. If for every $\sigma \in [a, b)$, there exists $\delta = \delta(\sigma) > 0$ such that for all $\eta \in (0, \delta)$, the following inequality holds*

$$f(\sigma + \eta) - f(\sigma) \leq g(\sigma + \eta) - g(\sigma),$$

then

$$f(s) - f(a) \leq g(s) - g(a),$$

for every $s \in [a, b]$.

The next auxiliary result will be crucial to prove our main results.

Lemma 3.4. *Let $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \to \mathbb{R}$ is nondecreasing and left-continuous. Also, suppose $V : [t_0, +\infty) \times X \to \mathbb{R}$ is such that for each left-continuous function $z : [\alpha, \beta] \to X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose V satisfies the following conditions.*

(V1) For all functions $x,\ y : [\alpha, \beta] \to X$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation on $[\alpha, \beta]$, the following condition

$$|V(t, x(t)) - V(t, y(t)) - V(s, x(s)) + V(s, y(s))|$$
$$\leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|$$

holds for every $\alpha \leq s < t \leq \beta$, where $h_1 : [t_0, , +\infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function.

(V2) There exists a function $\Phi : X \to \mathbb{R}$ such that for every solution $z : [s_0, +\infty) \to X$, $s_0 \geq t_0$, of (3.1), we have

$$V(s, z(s)) - V(t, z(t)) \leq (s - t)\Phi(z(t)),$$

for every $s_0 \leq t < s < +\infty$.

If $\overline{x} : [\gamma, v] \to X$, $t_0 \leq \gamma < v < \infty$, is left-continuous on $(\gamma, v]$ and of bounded variation on $[\gamma, v]$, then

$$V(v, \overline{x}(v)) - V(\gamma, \overline{x}(\gamma)) \leq (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| + (v - \gamma)K,$$

where $K = \sup\{\Phi(\overline{x}(t)) : t \in [\gamma, v]\}$.

Proof. Let $\overline{x} : [\gamma, v] \to X$, $[\gamma, v] \subset [t_0, +\infty)$, be a left-continuous function on $(\gamma, v]$ and of bounded variation on $[\gamma, v] \subset [t_0, +\infty)$ and $K := \sup\{\Phi(\overline{x}(t)) : t \in [\gamma, v]\}$. If $K = +\infty$, then clearly the desired inequality is trivially satisfied. Therefore, the statement of the theorem holds.

Now, let us assume that $K < +\infty$. Note that Proposition 2.11 implies the existence of the integral $\int_{\gamma}^{v} DF(\overline{x}(\tau), t)$.

Take $\sigma \in [\gamma, v]$. Since $(\overline{x}(\sigma), \sigma) \in \Omega = X \times [t_0, +\infty)$, by Theorem 2.13, there exists a unique maximal solution $x : [\sigma, +\infty) \to X$ of the generalized ODE (3.1) on $[\sigma, +\infty)$, satisfying the initial condition $x(\sigma) = \overline{x}(\sigma)$.

Let $\eta_1 > 0$ be fixed. Then, $x|_{[\sigma, \sigma+\eta_1]}$ is also a solution of the generalized ODE (3.1). Thus, by Corollary 2.10 and Proposition 2.11, the integral $\int_{\sigma}^{\sigma+\eta_1} DF(x(\tau), t)$ exists.

Consider $\eta_2 > 0$ such that $\eta_2 \leq \eta_1$ and $\sigma + \eta_2 \leq v$. Then, the integral $\int_{\sigma}^{\sigma+\eta_2} DF(x(\tau), t)$ exists and the integral $\int_{\sigma}^{\sigma+\eta_2} D[F(\overline{x}(\tau), t) - F(x(\tau), t)]$ also exists by the property of integrability on subintervals of the Kurzweil integral. Therefore, given $\epsilon > 0$, there exists a gauge δ of the interval $[\sigma, \sigma + \eta_2]$ corresponding to ϵ in the definition of the last integral. We can assume, without loss of generality, that $\eta_2 < \delta(\sigma)$. By hypothesis (V2), there exists a function $\Phi : X \to \mathbb{R}$ such that

$$V(s, x(s)) - V(t, x(t)) \leq (s - t)\Phi(x(t)),$$

for every $\sigma \leq t < s < +\infty$. In particular, for every $0 < \eta < \eta_2$, we have

$$V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \leq \eta \Phi(x(\sigma)), \tag{3.3}$$

where $s = \sigma + \eta$ and $t = \sigma$. By Corollary 2.2, we have

$$\left\| F(\overline{x}(\sigma), s) - F(\overline{x}(\sigma), \sigma) - \int_{\sigma}^{s} DF(\overline{x}(\tau), t) \right\| < \frac{\eta \epsilon}{2(h_1(\sigma + \eta) - h_1(\sigma))} \tag{3.4}$$

and

$$\left\| F(x(\sigma), s) - F(x(\sigma), \sigma) - \int_{\sigma}^{s} DF(x(\tau), t) \right\| < \frac{\eta \epsilon}{2(h_1(\sigma + \eta) - h_1(\sigma))}, \tag{3.5}$$

for every $s \in [\sigma, \sigma + \eta]$. Notice that

$$\sup_{s \in [\sigma, \sigma+\eta]} \left\| \int_{\sigma}^{s} D[F(\overline{x}(\tau), t) - F(x(\tau), t)] \right\|$$

$$-\sup_{s\in[\sigma,\sigma+\eta]}\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma)\|$$

$$\leq \sup_{s\in[\sigma,\sigma+\eta]}\left\|\int_\sigma^s D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\right.$$

$$\left.-(F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma))\right\|$$

$$\leq \sup_{s\in[\sigma,\sigma+\eta]}\left\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-\int_\sigma^s DF(\overline{x}(\tau),t)\right\|$$

$$+\sup_{s\in[\sigma,\sigma+\eta]}\left\|F(x(\sigma),s)-F(x(\sigma),\sigma)-\int_\sigma^s DF(x(\tau),t)\right\|. \tag{3.6}$$

On the other hand,

$$\sup_{s\in[\sigma,\sigma+\eta]}\|F(\overline{x}(\sigma),s)-F(\overline{x}(\sigma),\sigma)-F(x(\sigma),s)+F(x(\sigma),\sigma)\|$$

$$\leq \|\overline{x}(\sigma)-x(\sigma)\|\sup_{s\in[\sigma,\sigma+\eta]}|h(s)-h(\sigma)|=0, \tag{3.7}$$

where the first inequality follows from the fact that $F\in\mathcal{F}(\Omega,h)$, while the second equality follows by $x(\sigma)=\overline{x}(\sigma)$. Therefore, by (3.4), (3.5), (3.6) and (3.7), we have

$$\sup_{s\in[\sigma,\sigma+\eta]}\left\|\int_\sigma^s D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\right\|\leq\frac{\eta\epsilon}{(h_1(\sigma+\eta)-h_1(\sigma))}. \tag{3.8}$$

Since $F\in\mathcal{F}(\Omega,h)$ and the function h is nondecreasing, by Corollary 2.10, x is of bounded variation on $[\gamma,v]$ and, therefore, in $[\sigma,\sigma+\eta]\subset[\gamma,v]$. Thus, by hypothesis (V1) and by the relation $\overline{x}(\sigma)=x(\sigma)$, we get

$$V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))$$

$$=V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))-V(\sigma,\overline{x}(\sigma))+V(\sigma,x(\sigma))$$

$$\leq|V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))-V(\sigma,\overline{x}(\sigma))+V(\sigma,x(\sigma))|$$

$$\leq(h_1(\sigma+\eta)-h_1(\sigma))\sup_{s\in[\sigma,\sigma+\eta]}\|\overline{x}(s)-x(s)\|$$

$$=(h_1(\sigma+\eta)-h_1(\sigma))\sup_{s\in[\sigma,\sigma+\eta]}\|\overline{x}(s)-\overline{x}(\sigma)+x(\sigma)-x(s)\|$$

$$=(h_1(\sigma+\eta)-h_1(\sigma))\sup_{s\in[\sigma,\sigma+\eta]}\|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(x(\tau),t)\|,$$

which implies

$$V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))$$

$$\leq (h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(x(\tau),t)\|. \tag{3.9}$$

Therefore, by (3.3) and (3.9), we obtain

$$\begin{aligned}
&V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma,\overline{x}(\sigma))\\
&= V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))+V(\sigma+\eta,x(\sigma+\eta))-V(\sigma,x(\sigma))\\
&\leq (h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \left\|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(x(\tau),t)\right\| + \eta\Phi(x(\sigma))\\
&\leq (h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \left\|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(x(\tau),t)\right\| + \eta K\\
&\leq (h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \left\|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(\overline{x}(\tau),t)\right\|\\
&\quad +(h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \left\|\int_\sigma^s D[F(\overline{x}(\tau),t)-F(x(\tau),t)]\right\| + \eta K\\
&\leq (h_1(\sigma+\eta)-h_1(\sigma)) \sup_{s\in[\sigma,\sigma+\eta]} \left\|\overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(\overline{x}(\tau),t)\right\| + \eta\epsilon+\eta K, \qquad (3.10)
\end{aligned}$$

where the last inequality follows from (3.8). Given $s\in[\gamma,v]$, define

$$P(s) := \overline{x}(s)-\int_\gamma^s DF(\overline{x}(\tau),t).$$

Since $\overline{x}$ is a function of bounded variation on $[\gamma,v]$ and $(\overline{x}(s),s)\in\Omega$ for every $s\in[\gamma,v]$, by Proposition 2.11, it follows that the Kurzweil integral $\int_\gamma^v DF(\overline{x}(\tau),t)$ exists and the function $s\mapsto \int_\gamma^s DF(\overline{x}(\tau),t)$ is of bounded variation on $[\gamma,v]$. Hence, for each $s\in[\gamma,v]$, the Kurzweil integral $\int_\gamma^s DF(\overline{x}(\tau),t)$ also exists by the property of integrability on subintervals of the Kurzweil integral. Then, the function P is well-defined and is of bounded variation on $[\gamma,v]$. Moreover, by Lemma 2.8, P is left-continuous on $(\gamma,v]$, since $\bar{x}$ and h are left-continuous on $(\gamma,v]$.

On the other hand, for $s\in[\gamma,v]$, we have

$$\begin{aligned}
P(s)-P(\sigma) &= \overline{x}(s)-\overline{x}(\sigma)-\int_\gamma^s DF(\overline{x}(\tau),t)+\int_\gamma^\sigma DF(\overline{x}(\tau),t)\\
&= \overline{x}(s)-\overline{x}(\sigma)-\int_\sigma^s DF(\overline{x}(\tau),t). \qquad (3.11)
\end{aligned}$$

Now, we define the function $f : [\gamma, v] \to \mathbb{R}$ by

$$f(t) = \begin{cases} (h_1(t) - h_1(\sigma)) \sup\limits_{s \in [\gamma, t]} \|P(s) - P(\sigma)\| + \epsilon t + Kt, & t \in [\gamma, \sigma] \\ (h_1(t) - h_1(\sigma)) \sup\limits_{s \in [\sigma, t]} \|P(s) - P(\sigma)\| + \epsilon t + Kt, & t \in [\sigma, v]. \end{cases}$$

Clearly, f is well-defined. Moreover, by the left continuity of the functions h_1 and P, f is left-continuous on $(\gamma, v]$. Also, since $\overline{x} : [\gamma, v] \to X$ is left-continuous, it follows from the hypotheses that the function $[\gamma, v] \ni t \mapsto V(t, \overline{x}(t))$ is left-continuous on $(\gamma, v]$.

On the other hand, by (3.10) and (3.11), we have

$$\begin{aligned} V(\sigma + \eta, \overline{x}(\sigma + \eta)) - V(\sigma, \overline{x}(\sigma)) \;\; &\leq \;\; (h_1(\sigma + \eta) - h_1(\sigma)) \sup_{s \in [\sigma, \sigma + \eta]} \|P(s) - P(\sigma)\| \\ &+ \;\; \eta\varepsilon + \eta K = f(\sigma + \eta) - f(\sigma). \end{aligned}$$

Thus, the functions $[\gamma, v] \ni t \mapsto V(t, \overline{x}(t))$ and $[\gamma, v] \ni t \mapsto f(t)$ satisfy all the hypotheses of Proposition 3.3. Hence

$$\begin{aligned} V(v, \overline{x}(v)) - V(\gamma, \overline{x}(\gamma)) \;\; &\leq \;\; f(v) - f(\gamma) \\ &= \;\; (h_1(v) - h_1(\sigma)) \sup_{s \in [\sigma, v]} \|P(s) - P(\sigma)\| + \epsilon v + Kv \\ &- \;\; (h_1(\gamma) - h_1(\sigma)) \sup_{s \in [\gamma, \gamma]} \|P(s) - P(\sigma)\| - \epsilon\gamma - K\gamma \\ &= \;\; (h_1(v) - h_1(\sigma)) \sup_{s \in [\sigma, v]} \|P(s) - P(\sigma)\| + \epsilon v + Kv \\ &+ \;\; (h_1(\sigma) - h_1(\gamma)) \sup_{s \in [\gamma, \gamma]} \|P(s) - P(\sigma)\| - \epsilon\gamma - K\gamma \\ &\leq \;\; (h_1(v) - h_1(\sigma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| + \epsilon v + Kv \\ &+ \;\; (h_1(\sigma) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| - \epsilon\gamma - K\gamma \\ &= \;\; (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \|P(s) - P(\sigma)\| + \epsilon(v - \gamma) + K(v - \gamma) \\ &= \;\; (h_1(v) - h_1(\gamma)) \sup_{s \in [\gamma, v]} \left\| \overline{x}(s) - \overline{x}(\sigma) - \int_\sigma^s DF(\overline{x}(\tau), t) \right\| \\ &+ \;\; K(v - \gamma) + \epsilon(v - \gamma). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

The next theorem ensures us that the generalized ODE (3.1) is uniformly bounded. Our result generalizes the one found in [1, Theorem 4.3].

Theorem 3.5. *Let $V : [t_0, +\infty) \times X \to \mathbb{R}$ be a function such that, for each left-continuous function $z : [\alpha, \beta] \to X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose V satisfies the following conditions* *(i) There are two monotone increasing functions p, $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(0) = b(0) = 0$,*

$$\lim_{s \to +\infty} b(s) = +\infty \tag{3.12}$$

and

$$b(\|z\|) \leq V(t, z) \leq p(\|z\|), \tag{3.13}$$

for every pair $(t, z) \in [t_0, +\infty) \times X$.

(ii) For every solution of type $z : [s_0, +\infty) \to X$, $s_0 \geq t_0$, of the generalized ODE (3.1), we have

$$V(s, z(s)) - V(t, z(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$

Then, the generalized ODE (3.1) is uniformly bounded.

Proof. Let $\alpha > 0$ be fixed. Since $p(\alpha) > 0$, by (3.12), there exists $M = M(\alpha) > 0$ such that

$$p(\alpha) < b(s), \text{ for all } s \geq M.$$

In particular, for $s = M$, we obtain

$$p(\alpha) < b(M). \tag{3.14}$$

Now, let $s_0 \in [t_0, +\infty)$, $z_0 \in X$ and $x(\cdot) = x(\cdot, s_0, z_0) : [s_0, +\infty) \to X$ be the solution of the generalized ODE (3.1) with initial condition $x(s_0) = z_0$, where $\|z_0\| < \alpha$. We will show that

$$\|x(t)\| < M, \text{ for all } s \geq s_0.$$

Indeed, by hypothesis (ii) and condition (3.13), for each $s \geq s_0$, we have

$$\begin{aligned} V(s, x(s)) &\leq V(s_0, x(s_0)) = V(s_0, z_0) \\ &\leq p(\|z_0\|) \\ &\leq p(\alpha) \\ &< b(M), \end{aligned} \tag{3.15}$$

that is,

$$V(s, x(s)) < b(M), \text{ for all } s \geq s_0. \tag{3.16}$$

Finally, we will show that $\|x(s, s_0, z_0)\| = \|x(s)\| < M$, for all $s \geq s_0$. Suppose the contrary, that is, suppose there exists $\bar{s} \in [s_0, +\infty)$ such that $\|x(\bar{s})\| \geq M$. Then, by hypothesis (3.13) and using the fact that b is an increasing function, we have

$$V(\bar{s}, x(\bar{s})) \geq b(\|x(\bar{s})\|) \geq b(M),$$

which contradicts (3.16). Therefore, $\|x(s)\| < M$ for all $s \geq s_0$, and the result follows. $\square$

The next result gives us sufficient conditions to guarantee that the generalized ODE (3.1) is uniformly ultimately bounded.

Theorem 3.6. *Let $V : [t_0, \infty) \times X \to \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \to X$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto V(t, z(t))$ is left-continuous on $(\alpha, \beta]$ and satisfies condition (i) from Theorem 3.5. Moreover, suppose V satisfies the following conditions*

(V1) For every $x,\ y : [\alpha, \beta] \to X$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation on $[\alpha, \beta]$, we have
$$|V(t, x(t) - V(t, y(t)) - V(s, x(s)) + V(s, y(s))|$$
$$\leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha,\beta]} \|x(\xi) - y(\xi)\|,$$
for every $\alpha \leq s < t \leq \beta$, where $h_1 : [t_0, , +\infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function.

(V2) There exists a continuous function $\Phi : X \longrightarrow \mathbb{R}$, with $\Phi(0) = 0$ and $\Phi(x) > 0$, $x \neq 0$, such that for every solution $z : [s_0, +\infty) \to X$, $s_0 \geq t_0$, of (3.1), we have
$$V(s, z(s)) - V(t, z(t)) \leq (s - t)\Big(-\Phi(z(t))\Big),$$
for every $s_0 \leq t < s < +\infty$.

Then, the generalized ODE (3.1) is uniformly ultimately bounded.

Proof. At first, by hypothesis $(V2)$, we have
$$V(s, z(s)) - V(t, z(t)) \leq (s - t)\Big(-\Phi(z(t))\Big) \leq 0,$$
for every solution $z : [s_0, +\infty) \to X$, $s_0 \in [t_0, +\infty)$, of the generalized ODE (3.1), with $s_0 \leq t < s < +\infty$ Hence, all the hypotheses of the Theorem 3.5 are satisfied and, consequently, the generalized ODE (3.1) is uniformly bounded. It remains to show that equation (3.1) is quasi-uniformly ultimately bounded.

By the uniform boundedness of the generalized ODE (3.1), there exists $B = B(t_0+1) > 0$ such that, for every $\bar{t} \in [t_0, +\infty)$ and for every $\bar{x} \in X$ with $\|\bar{x}\| < t_0 + 1$, we have
$$\big\|x(s, \bar{t}, \bar{x})\big\| < B, \qquad \text{for all } s \geq \bar{t}, \tag{3.17}$$
where $x(s, \bar{t}, \bar{x})$ is the maximal solution of the generalized ODE (3.1) with $x(\bar{t}) = x(\bar{t}, \bar{t}, \bar{x}) = \bar{x}$. Without loss of generality, we can take $B \in (t_0 + 1, +\infty)$ (because, otherwise, we can take $B > B'$ such that $B' \in (t_0 + 1, +\infty)$) and we have
$$\big\|x(s, \bar{t}, \bar{x})\big\| < B, \qquad \text{for all } s \geq \bar{t}.$$

Let $\alpha > 0$, $s_0 \in [t_0, +\infty)$, $z_0 \in X$ and $x(\cdot) = x(\cdot, s_0, z_0) : [s_0, +\infty) \to X$ be the solution of the generalized ODE (3.1) with initial condition (3.2), where $\|z_0\| < \alpha$. Since (3.1) is uniformly bounded, there exists a positive number $M_1 = M_1(\alpha)$ (we can take $M_1 > \max\{\alpha, t_0 + 1\}$) such that

$$\|x(s, s_0, z_0)\| < M_1, \quad \text{for all } s \geq s_0.$$

On the other hand, using the same argument as in (3.14) from the proof of Theorem 3.5, there exists $M_2 = M_2(\alpha) > 0$ such that $p(\alpha) < b(M_2)$.

Now, let $M = M(\alpha) := \max\{M_1(\alpha), M_2(\alpha)\}$. Notice that

$$\|x(s, s_0, z_0)\| < M, \qquad \text{for all } s \geq s_0. \tag{3.18}$$

and

$$p(\alpha) < b(M). \tag{3.19}$$

Define

$$N := \sup\{-\Phi(z) : \ t_0 + 1 \leq \|z\| < M\} < 0$$

and

$$T(\alpha) := -\frac{2b(M)}{N} > 0.$$

We want to show that $\|x(s, s_0, z_0)\| < B$, for all $s \geq s_0 + T(\alpha)$. Suppose the contrary, that is, there exists $\overline{s} > s_0 + T(\alpha)$ such that

$$\|x(\overline{s}, s_0, z_0)\| \geq B > t_0 + 1. \tag{3.20}$$

Assertion 1. The following inequality holds

$$\|x(s, s_0, z_0)\| \geq t_0 + 1, \qquad \text{for all } s \in [s_0, \overline{s}].$$

Suppose the assertion is false, that is, there exists $\overline{t} \in [s_0, \overline{s}]$ such that

$$\left\|x(\overline{t}, s_0, z_0)\right\| < t_0 + 1.$$

On the other hand, by (3.17) (with $\overline{x} = x(\overline{t}, s_0, z_0)$)

$$\left\|x(s, \overline{t}, \overline{x})\right\| < B, \qquad \text{for all } s \geq \overline{t}. \tag{3.21}$$

Also, we know that $x(s, \overline{t}, \overline{x})$, $s \in [\overline{t}, +\infty)$, is the unique solution of the initial value problem

$$\begin{cases} \dfrac{dz}{d\tau} = DF(z, t) \\ z(\overline{t}) = \overline{x} = x(\overline{t}, s_0, z_0). \end{cases} \tag{3.22}$$

But $x(\cdot, s_0, z_0)|_{[\bar{t},+\infty)}$ is also a solution of the generalized ODE (3.22). Therefore, we have

$$x(s, s_0, z_0) = x(s, \bar{t}, \bar{x}), \qquad \text{for all } s \in [\bar{t}, +\infty).$$

In particular, since $\bar{s} \in [\bar{t}, +\infty)$, we obtain

$$x(\bar{s}, s_0, z_0) = x(\bar{s}, \bar{t}, \bar{x}). \tag{3.23}$$

Thus, (3.21) and (3.23) imply that $\|x(\bar{s}, s_0, z_0)\| < B$, which contradicts (3.20). Thus, Assertion 1 follows.

By Assertion 1, we have

$$\|x(s, s_0, z_0)\| \geq t_0 + 1, \qquad \text{for all } s \in \left[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)\right]. \tag{3.24}$$

since $[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)] \subset [s_0, \bar{s}]$.

However, since $x(\cdot) := x(\cdot, s_0, z_0)\Big|_{[s_0+\frac{T(\alpha)}{2}, s_0+T(\alpha)]}$ is a solution of the generalized ODE (3.1) and $F \in \mathcal{F}(\Omega, h)$, where the function h is left-continuous and nondecreasing, the function $x(\cdot, s_0, z_0)\Big|_{[s_0+\frac{T(\alpha)}{2}, s_0+T(\alpha)]}$ is left-continuous on $(s_0 + T(\alpha)/2, s_0 + T(\alpha)]$ and of bounded variation on $I_\alpha := [s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]$ by Lemma 2.9 and Corollary 2.10. Thus, by Lemma 3.4, it follows that

$$\begin{aligned} V(s_0 + T(\alpha), x(s_0 + T(\alpha))) \leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) + \\ + \left(h_1(s_0 + T(\alpha)) - h_1\left(s_0 + \frac{T(\alpha)}{2}\right)\right). \\ \underbrace{\sup_{s \in I_\alpha} \left\| x(s) - x\left(s_0 + \frac{T(\alpha)}{2}\right) - \int_{s_0+\frac{T(\alpha)}{2}}^{s} DF(x(\tau, s_0, z_0), t) \right\|}_{0} + \\ + \frac{T(\alpha)}{2} \cdot \sup\left\{-\Phi(x(s)) : s \in [s_0 + T(\alpha)/2, s_0 + T(\alpha)]\right\} \end{aligned}$$

which implies

$$\begin{aligned} V(s_0 + T(\alpha), x(s_0 + T(\alpha))) &\leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) \\ &+ \frac{T(\alpha)}{2} . \sup\left\{-\Phi(x(s)) : \ s \in \left[s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)\right]\right\} \\ &\leq V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) \\ &+ \frac{T(\alpha)}{2} . \sup\{-\Phi(z) : \ t_0 + 1 \leq \|z\| < M\}, \end{aligned} \tag{3.25}$$

where the last inequality follows from the relation

$$t_0 + 1 \leq \|x(s)\| = \|x(s, s_0, z_0)\| < M,$$

for all $s \in [s_0 + \frac{T(\alpha)}{2}, s_0 + T(\alpha)]$. Also, by (3.19) and using the same argument as in (3.15) from the proof of Theorem 3.5, we get

$$V\left(s_0 + \frac{T(\alpha)}{2}, x\left(s_0 + \frac{T(\alpha)}{2}\right)\right) < b(M).$$

Therefore, by (3.25), we have

$$\begin{aligned} V(s_0 + T(\alpha), x(s_0 + T(\alpha))) &< b(M) + \frac{T(\alpha)}{2}. \sup\{-\Phi(z) : \ t_0 + 1 \leq \|z\| < M\} \\ &= b(M) + \frac{T(\alpha)}{2}.N \\ &= b(M) - \frac{2b(M)}{2N}.N \\ &= 0, \end{aligned}$$

which implies that

$$V(s_0 + T(\alpha), x(s_0 + T(\alpha))) < 0. \tag{3.26}$$

On the other hand, by condition (3.13) and from the inequality (3.24), we have

$$V(s_0 + T(\alpha), x(t_0 + T(\alpha))) \geq b(\|x(s_0 + T(\alpha))\|) \geq b(t_0 + 1) > 0,$$

which contradicts (3.26). Therefore, $\|x(s, s_0, z_0)\| < B$, for all $s \geq s_0 + T(\alpha)$. Then the generalized ODE (3.1) is quasi-uniformly ultimately bounded and the proof is complete. □

4 Boundedness of solutions of measure differential equations

In this section, our goal is to present results concerning boundedness of solutions of measure differential equations.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$. Consider the integral form of a measure differential equation (MDE, for short) of type

$$x(t) = x(\tau_0) + \int_{\tau_0}^{t} f(x(s), s)\mathrm{d}g(s), \qquad t \geq \tau_0, \tag{4.1}$$

where $\tau_0 \geq t_0$, $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$, and the integral in the right-hand side is understood in the sense of a Kurzweil-Henstock-Stieltjes.

We recall the reader that $G([t_0,+\infty),\mathbb{R}^n)$ denotes the vector space of functions $x : [t_0,+\infty) \to \mathbb{R}^n$ such that $x|_{[\alpha,\beta]}$ belongs to the space $G([\alpha,\beta],\mathbb{R}^n)$, for all $[\alpha,\beta] \subset [t_0,+\infty)$. We use the symbol $G_0([t_0,+\infty),\mathbb{R}^n)$ to denote the vector space of all functions $x \in G([t_0,+\infty),\mathbb{R}^n)$ such that $\sup\limits_{s\in[t_0,+\infty)} e^{-(s-t_0)}|x(s)| < +\infty$. This space is endowed with the norm

$$\|x\|_{[t_0,+\infty)} = \sup_{s\in[t_0,+\infty)} e^{-(s-t_0)}|x(s)|, \qquad x \in G_0([t_0,+\infty),\mathbb{R}^n),$$

and it becomes a Banach space (see [14]).

In what follows, we say that a function $M : [t_0,+\infty) \to \mathbb{R}^+$ is locally Kurzweil-Henstock-Stieltjes integrable with respect to g if, and only if, the function $[s_1,s_2] \ni t \mapsto M(t)$ is Kurzweil-Henstock-Stieltjes integrable with respect to g, for every $s_1\ s_2 \in [t_0,+\infty)$.

Let us assume the following conditions concerning the functions $f : \mathbb{R}^n \times [t_0,+\infty) \to \mathbb{R}^n$ and $g : [t_0,+\infty) \to \mathbb{R}$.

(A1) The function $g : [t_0,+\infty) \to \mathbb{R}$ is nondecreasing and left-continuous on $(t_0,+\infty)$.

(A2) The Kurzweil-Henstock-Stieltjes integral

$$\int_{s_1}^{s_2} f(x(t),t)\mathrm{d}g(t)$$

exists, for all $x \in G([t_0,+\infty),\mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0,+\infty)$.

(A3) There exists a locally Kurzweil-Henstock-Stieltjes integrable function $M : [t_0,+\infty) \to \mathbb{R}^+$ with respect to g such that

$$\left|\int_{s_1}^{s_2} f(x(s),s)\mathrm{d}g(s)\right| \leq \int_{s_1}^{s_2} M(s)\mathrm{d}g(s),$$

for all $x \in G([t_0,+\infty),\mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0,+\infty),\ s_1 \leq s_2$.

(A4) There exists a locally Kurzweil-Henstock-Stieltjes integrable function $L : [t_0,+\infty) \to \mathbb{R}^+$ with respect to g such that

$$\left|\int_{s_1}^{s_2} [f(x(s),s) - f(z(s),s)]\mathrm{d}g(s)\right| \leq \|x-z\|_{[t_0,+\infty)} \int_{s_1}^{s_2} L(s)\mathrm{d}g(s),$$

for all $x,\ z \in G_0([t_0,+\infty),\mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0,+\infty),\ s_1 \leq s_2$.

The next result ensures that if the function $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies the condition (A1), then the function F given by $F(x,t) = \int_{\tau_0}^{t} f(x,s)\mathrm{d}g(s)$, for $(x,t) \in \mathbb{R}^n \times [t_0, +\infty)$ belongs to the class $\mathcal{F}(\mathbb{R}^n \times [t_0, +\infty), h)$, where $h(t) = \int_{\tau_0}^{t} (M(s) + L(s))\mathrm{d}g(s)$, $t \in [t_0, +\infty)$. See [10] for a proof of this result.

Theorem 4.1. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies the condition (A1). Choose an arbitrary $\tau_0 \in [t_0, +\infty)$ and define $F : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ by*

$$F(x,t) = \int_{\tau_0}^{t} f(x,s)\mathrm{d}g(s), \qquad (x,t) \in \mathbb{R}^n \times [t_0, +\infty). \tag{4.2}$$

Then $F \in \mathcal{F}(\Omega, h)$, where $\Omega = \mathbb{R}^n \times [t_0, +\infty)$, and $h : [t_0, +\infty) \to \mathbb{R}$ given by

$$h(t) = \int_{\tau_0}^{t} (M(s) + L(s))\mathrm{d}g(s), \qquad t \in [t_0, +\infty) \tag{4.3}$$

is a nondecreasing and left-continuous function.

The next result describes the relation between the Kurzweil-Henstock-Stieltjes integral and the Kurzweil integral. It can be found in [10].

Theorem 4.2. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1). Let $F : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ be defined by* (4.2)*. If $[a,b] \subset [t_0, +\infty)$ and $x : [a,b] \to \mathbb{R}^n$ is a regulated function (in particular, a function of bounded variation) on $[a,b]$, then both the Kurzweil integral $\int_a^b DF(x(\tau), t)$ and the Kurzweil-Henstock-Stieltjes integral $\int_a^b f(x(s), s)\mathrm{d}g(s)$ exists and have the same value.*

The following result describes a correspondence between the solutions of the MDE (4.1) and the solutions of the generalized ODE given by

$$\frac{dx}{d\tau} = DF(x,t),$$

where F is given by $F(x,t) = \int\limits_{t_0}^{t} f(x,s)\mathrm{d}g(s)$. This result can be found in [10].

Theorem 4.3. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies the condition (A1). Then, the function $x : I \to \mathbb{R}^n$ is a solution of the MDE* (4.1) *on $I \subset [t_0, +\infty)$ if, and only if, x is a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t)$$

on I with the function F given by (4.2)*.*

From now on, we assume that for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)$, there exists a unique (maximal) solution $x : [s_0, +\infty) \to \mathbb{R}^n$ of the MDE (4.1) with $x(s_0) = z_0$. The next result, which can be found in [10], ensures the existence and uniqueness of such solution.

Theorem 4.4. *Suppose $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies the condition (A1). Then for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0, +\infty)$, there exists a unique maximal solution of the MDE* (4.1) *defined in $[s_0, +\infty)$ with $x(s_0) = z_0$.*

In the sequel, we present the concepts of uniform boundedness for measure differential equations.

Definition 4.5. We say that the MDE (4.1) is:

• **Uniformly bounded:** if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that, for every $s_0 \in [t_0, +\infty)$ and for all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \geq s_0.$$

• **Quasi-uniformly ultimately bounded:** if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0, +\infty)$ and for all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \geq s_0 + T.$$

• **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

The next result ensures us that the MDE (4.1) is uniformly bounded.

Theorem 4.6. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) and the function $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1). Let $U : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \to \mathbb{R}^n$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]$. Moreover, suppose U satisfies the following conditions*

(i) *There are two monotone increasing functions $p,\ b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(0) = b(0) = 0$,*

$$\lim_{s \to +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq U(t, z) \leq p(\|z\|)$$

for every pair $(t, z) \in [t_0, +\infty) \times \mathbb{R}^n$.

(ii) *For every solution $z : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, of the MDE* (4.1)*, we have*

$$U(s, z(s)) - U(t, z(t)) \leq 0,$$

for every $s_0 \leq t < s < +\infty$.

Then, the MDE (4.1) *is uniformly bounded.*

Proof. Define a function $F : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^{t} f(x, s)\mathrm{d}g(s) \tag{4.4}$$

for all $(x, t) \in \mathbb{R}^n \times [t_0, +\infty)$. Since f satisfies conditions (A2), (A3) and (A4) and g satisfies condition (A1), by Theorem 4.1, $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \to \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^{t} (M(s) + L(s))\mathrm{d}g(s).$$

Now, by Theorem 4.3 and from the hypotheses, it is not difficult to see that $U : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfies all the hypotheses from Theorem 3.5. Hence, the generalized ODE (3.1) is uniformly bounded, where F is given by (4.4).

Again, by Theorem 4.3, it follows that the MDE (4.1) is also uniformly bounded, obtaining the desired result. □

Finally, we present the last result of this section. It ensures that the MDE (4.1) is uniformly ultimately bounded.

Theorem 4.7. *Assume $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies conditions (A2), (A3) and (A4) and the function $g : [t_0, +\infty) \to \mathbb{R}$ satisfies condition (A1). Let $U : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha, \beta] \to \mathbb{R}^n$ on $(\alpha, \beta]$, the function $[\alpha, \beta] \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha, \beta]$ and satisfies condition (i) from Theorem 4.6. Moreover, suppose U satisfies the following conditions*

(U1) *For every $x,\ y : [\alpha, \beta] \to \mathbb{R}^n$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation on $[\alpha, \beta]$, we have*

$$|U(t, x(t)) - U(t, y(t)) - U(s, x(s)) + U(s, y(s))| \leq \left(\int_s^t K(\tau)du(\tau)\right) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|,$$

for every $\alpha \leq s < t \leq \beta$, where $u : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function and $K : [t_0, +\infty) \to \mathbb{R}$ is a locally Kurzweil-Henstock-Stieltjes integrable function with respect to u.

(U2) *There exists a continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$, with $\phi(0) = 0$ and $\phi(x) > 0$, $x \neq 0$, such that for every solution $z : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, of the MDE* (4.1)*, we have*

$$U(s, z(s)) - U(t, z(t)) \leq (s - t)\Big(- \phi(z(t))\Big),$$

for every $s_0 \leq t < s < +\infty$.

Then, the MDE (4.1) *is uniformly ultimately bounded.*

Proof. Define a function $F : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ by

$$F(x, t) = \int_{t_0}^{t} f(x, s)\mathrm{d}g(s), \tag{4.5}$$

for all $(x, t) \in \mathbb{R}^n \times [t_0, +\infty)$. Since f satisfies conditions (A2), (A3) and (A4) and g satisfies condition (A1), by Theorem 4.1, $F \in \mathcal{F}(\Omega, h)$, where the function $h : [t_0, +\infty) \to \mathbb{R}$ is given by

$$h(t) = \int_{t_0}^{t} (M(s) + L(s))\mathrm{d}g(s).$$

By Theorem 4.3, U satisfies hypothesis (i) from Theorem 3.5. Also, defining the function $h_1 : [t_0, +\infty) \to \mathbb{R}$ by

$$h_1(t) = \int_{t_0}^{t} K(\tau)\mathrm{d}u(\tau),$$

for every $t \in [t_0, +\infty)$, it follows that h_1 is a nondecreasing and left-continuous function. Moreover, by (U1), U satisfies the following condition

$$|U(t, x(t)) - U(t, y(t)) - U(s, x(s)) + U(s, y(s))| \leq (h_1(t) - h_1(s)) \sup_{\xi \in [\alpha, \beta]} \|x(\xi) - y(\xi)\|,$$

for every $\alpha \leq s < t \leq \beta$, and for every $x,\ y : [\alpha, \beta] \to \mathbb{R}^n$, $[\alpha, \beta] \subset [t_0, +\infty)$, of bounded variation on $[\alpha, \beta]$.

Also, by Theorem 4.3 and by hypothesis (U2), it is clear that U satisfies the hypothesis $(V2)$ from Theorem 3.6. Therefore, all the hypotheses from Theorem 3.6 are fulfilled and the generalized ODE (3.1) is uniformly ultimately bounded. By Theorem 4.3, the MDE (4.1) is also uniformly ultimately bounded, obtaining the desired result. □

5 Boundedness of solutions of dynamic equations on time scales

In this section, our goal is to prove the results concerning boundedness of solutions of dynamic equations on time scales, via Lyapunov functionals.

We will start with an overview of the theory of time scales. For more details, the reader may consult [3, 4].

A time scale $\mathbb{T}$ is a closed and nonempty subset of $\mathbb{R}$. Given a pair of numbers $a,\ b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will denote a closed interval in $\mathbb{T}$. Similarly, we define the open half-open intervals.

For all $t \in \mathbb{T}$, we define the *forward jump operator* by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the *backward jump operator* by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$; and use the convention that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The *graininess* function $\mu : \mathbb{T} \to [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$.

If $\sigma(t) > t$, we say that t is a right-scattered point; if $\rho(t) = t$, then t is right-dense. Similarly, we define left-scattered and left-dense points, depending on ρ.

To be able to relate dynamic equations on time scales and measure differential equations, we need some important concepts, which can be found in [17].

Given a real number $t \leq \sup \mathbb{T}$, define

$$t^* = \inf\{s \in \mathbb{T} : s \geq t\}.$$

Since $\mathbb{T}$ is a closed set, we have $t^* \in \mathbb{T}$. Also, define the extension of the time scale $\mathbb{T}$ as follows

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise.} \end{cases} \tag{5.1}$$

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, its extension $f^* : \mathbb{T}^* \to \mathbb{R}^n$ is defined by

$$f^*(t) = f(t^*),\ t \in \mathbb{T}^*.$$

Remark 5.1. Let $a,\ b \in \mathbb{T}$, $a \leq b$, and $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be a function. Notice that if $t \in [a, b]$, then t^* is well defined and $t^* \in [a, b]_{\mathbb{T}}$, because $a^* = a \leq t \leq b = b^* \leq \sup \mathbb{T}$. Thus, $f^* : [a, b] \to \mathbb{R}^n$ give by $f^*(t) = f(t^*)$ for every $t \in [a, b]$, is well defined.

On the other hand, given a real number $t_0 \in \mathbb{T}$ and $f : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ a function, it is clear that we can have points $t \in [t_0, +\infty)$ such that $t > \sup \mathbb{T}$. Therefore, t^* is not well-defined. Thus, it is necessary to require that $\sup \mathbb{T} = +\infty$ so that t^* is well-defined

Lemma 5.2. *Let* $\mathbb{T}$ *be a time scale such that* $\sup \mathbb{T} = +\infty$, *and* $t_0 \in \mathbb{T}$. *Let* $g : [t_0, +\infty) \to \mathbb{R}$ *be given by* $g(t) = t^*$, *for all* $t \in [t_0, +\infty)$. *Then,* g *satisfies the following conditions*

(i) g *is a nondecreasing function;*

(ii) g *is left-continuous on* $(t_0, +\infty)$.

The next result can be found in [12, Theorem 4.2]. It ensures that the Kurzweil-Henstock Δ-integral and Kurzweil-Henstock-Stieltjes integral coincide when $g(t) = t^*$.

Theorem 5.3. *Let $\mathbb{T}$ be a time scale such that $a,\ b \in \mathbb{T}$ and $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be a function. Define $g(t) = t^*$, for every $t \in [a,b]$. Then the Kurzweil-Henstock Δ-integral $\int_a^b f(t)\Delta t$ exists if, and only if, the Kurzweil-Henstock-Stieltjes integral $\int_a^b f^*(t)dg(t)$ exists. In this case, both integrals have the same value.*

The next three results will be crucial to prove our main results and they can be found in [10].

Theorem 5.4. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $g(t) = t^*$ for every $t \in [t_0, +\infty)$. If $f : [t_0, +\infty) \to \mathbb{R}^n$ is such that the integral $\int_c^d f(t)\mathrm{d}g(t)$ exists for every $c,\ d \in [t_0, +\infty)$, then*

$$\int_c^d f(t)\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\mathrm{d}g(t),$$

for every $t_0 \leq c < d < \infty$.

Theorem 5.5. *Let $f : \mathbb{T} \to \mathbb{R}^n$ be a function such that the Δ-Kurzweil-Henstock integral*

$$\int_a^b f(s)\Delta s$$

exists for every $a,\ b \in \mathbb{T}$, $a < b$. Choose an arbitrary $a \in \mathbb{T}$ and define

$$F_1(t) = \int\limits_a^t f(s)\Delta s,\ t \in \mathbb{T},$$

$$F_2(t) = \int\limits_a^t f^*(s)dg(s),\ t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$. In particular, $F_2(t) = F_1(t)$ for all $t \in \mathbb{T}$.

Theorem 5.6. *Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$ and $[a,b] \subset \mathbb{T}^*$. Consider a pair of functions $f_1,\ f_2 : [a,b] \to \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [a,b] \cap \mathbb{T}$. If $\int_a^b f_1(s)\mathrm{d}g(s)$ exists, then $\int_a^b f_2(s)\mathrm{d}g(s)$ exists as well, and both integrals have the same value.*

In the sequel, we present a correspondence between the solutions of the dynamic equations on time scales and the solutions of the measure differential equations. For more details see [10].

Theorem 5.7. *Let $\mathbb{T}$ be a time scale such that* $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$, *and* $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. *Assume that for every* $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, *the function* $t \mapsto f(x(t), t)$ *is Kurzweil-Henstock* Δ*-integrable on* $[s_1, s_2]_{\mathbb{T}}$, *for every* $s_1, \; s_2 \in [t_0, +\infty)_{\mathbb{T}}$. *Define* $g : [t_0, +\infty) \to \mathbb{R}$ *by* $g(s) = s^*$, *for every* $s \in [t_0, +\infty)$. *Also, let* $J \subset [t_0, +\infty)$ *be a nondegenerate interval such that* $J \cap \mathbb{T}$ *is nonempty and for each* $t \in J$, *we have* $t^* \in J \cap \mathbb{T}$. *If* $x : J \cap \mathbb{T} \to \mathbb{R}^n$ *is a solution of the initial value problem given by*

$$\begin{cases} x^{\Delta}(t) = f(x^*, t), & t \in J \cap \mathbb{T}, \\ x(s_0) = x_0, \end{cases} \tag{5.2}$$

where $x_0 \in \mathbb{R}^n$ *and* $s_0 \in J \cap \mathbb{T}$, *then* $x^* : J \to \mathbb{R}^n$ *is a solution of the initial value problem*

$$y(t) = x_0 + \int_{s_0}^{t} f^*(y(s), s)\mathrm{d}g(s) = x_0 + \int_{s_0}^{t} f(y(s), s^*)\mathrm{d}g(s) \tag{5.3}$$

Conversely, if $y : J \to \mathbb{R}^n$ *satisfies the initial value problem* (5.3), *then it must have the form* $y = x^*$, *where* $x : J \cap \mathbb{T} \to \mathbb{R}^n$ *is a solution of the initial value problem* (5.2).

As an immediate consequence of Theorem 5.7, we have the following result.

Corollary 5.8. *Let $\mathbb{T}$ be a time scale such that* $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$, *and* $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. *Assume that for every* $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, *the function* $t \mapsto f(x(t), t)$ *is Kurzweil-Henstock* Δ*-integrable on* $[s_1, s_2]_{\mathbb{T}}$, *for every* $s_1, \; s_2 \in [t_0, +\infty)_{\mathbb{T}}$. *Define* $g : [t_0, +\infty) \to \mathbb{R}$ *by* $g(s) = s^*$, *for every* $s \in [t_0, +\infty)$. *Also, let* $J \subset [t_0, +\infty)$ *be a nondegenerate interval such that* $J \cap \mathbb{T}$ *is nonempty and for each* $t \in J$, *we have* $t^* \in J \cap \mathbb{T}$. *If* $x : J \cap \mathbb{T} \to \mathbb{R}^n$ *is a solution of the dynamic equation on time scales given by*

$$x^{\Delta}(t) = f(x^*, t), \qquad t \in J \cap \mathbb{T}. \tag{5.4}$$

then $x^* : J \to \mathbb{R}^n$ *is a solution of the MDE*

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} f^*(y, s)\mathrm{d}g(s), \qquad t_1, \; t_2 \in J. \tag{5.5}$$

Conversely, if $y : J \to \mathbb{R}^n$ *satisfies the MDE* (5.5), *then it must have the form* $y = x^*$, *where* $x : J \cap \mathbb{T} \to \mathbb{R}^n$ *is a solution of* (5.4).

The symbol $G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ will be used to denote the set of all regulated functions $x : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. Also, the symbol $G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ will be used to denote the set of all functions $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ such that $\sup\limits_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-(s-t_0)} |x(s)|$ is finite. This space is endowed with the norm

$$\|x\|_{[t_0, +\infty)_{\mathbb{T}}} = \sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-(s-t_0)} |x(s)|, \qquad x \in G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$$

is a Banach Space.

In what follows, we say that $M : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^+$ is locally Kurzweil-Henstock Δ-integrable if, and only if, the function $[s_1, s_2]_{\mathbb{T}} \ni t \mapsto M(t)$ is Kurzweil-Henstock Δ-integrable for every $s_1\ s_2 \in [t_0, +\infty)_{\mathbb{T}}$.

From now on, we will assume the following conditions concerning the function $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$.

(B1) The Kurzweil-Henstock Δ-integral $\displaystyle\int_{s_1}^{s_2} f(y(t), t)\Delta t$ exists, for all $y \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0, +\infty)_{\mathbb{T}}$.

(B2) There exists a locally Kurzweil-Henstock Δ-integrable function $M : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left|\int_{s_1}^{s_2} f(y(t), t)\Delta t\right| \leq \int_{s_1}^{s_2} M(t)\Delta t,$$

for all $y \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0, +\infty)_{\mathbb{T}}$, $s_1 \leq s_2$.

(B3) There exists a locally Kurzweil-Henstock Δ-integrable function $L : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left|\int_{s_1}^{s_2} [f(y(t), t) - f(w(t), t)]\Delta t\right| \leq \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{s_1}^{s_2} L(t)\Delta t,$$

for all $y,\ w \in G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ and all $s_1,\ s_2 \in [t_0, +\infty)_{\mathbb{T}}$, $s_1 \leq s_2$.

The next result describes a relation between the conditions of the function f with the analogues to its extension f^*. It can be found in [10].

Theorem 5.9. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$, and $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ be a function. Define $g(t) = t^*$ and $f^*(y, t) = f(y, t^*)$ for every $y \in \mathbb{R}^n$ and $t \in [t_0, +\infty)$.*

1. *If $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the condition (B1), then the integral $\displaystyle\int_{s_1}^{s_2} f^*(x(t), t)\mathrm{d}g(t)$ exists, for all $x \in G([t_0, +\infty), \mathbb{R}^n)$ and for all $s_1,\ s_2 \in [t_0, +\infty)$.*

2. *If $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the condition (B2), then $f^* : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the following condition*

$$\left|\int_{s_1}^{s_2} f^*(x(t), t)\mathrm{d}g(t)\right| \leq \int_{s_1}^{s_2} M^*(t)\mathrm{d}g(t),$$

for all $s_1,\ s_2 \in [t_0, +\infty)$, $s_1 \leq s_2$, and for all $x \in G([t_0, +\infty), \mathbb{R}^n)$, where $g(t) = t^*$.

3. *If $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the condition (B3), then $f^* : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the condition*

$$\left|\int_{s_1}^{s_2} [f^*(x(t), t) - f^*(z(t), t)]\mathrm{d}g(t)\right| \leq \|x - z\|_{[t_0, +\infty)} \int_{s_1}^{s_2} L^*(t)\mathrm{d}g(s),$$

for all s_1, $s_2 \in [t_0,+\infty)$, $s_1 \le s_2$, and for all x, $z \in G([t_0,+\infty),\mathbb{R}^n)$, where $g(t) = t^*$.

Consider the dynamic equation on time scales given by

$$x^{\Delta}(t) = f(x^*, t). \tag{5.6}$$

From now on, we assume that for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}}$, there exists a unique maximal solution $x : [s_0,+\infty) \to X$ of (3.1) with $x(s_0) = z_0$. The existence of such solution is ensured by the following result, which can be found in [10].

Theorem 5.10. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0,+\infty)_{\mathbb{T}}$ be a time scale interval. Assume that $f : \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the conditions (B1), (B2) and (B3). Then, for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}}$, there exists a unique maximal solution of (5.6) with $x(s_0) = z_0$ is defined in $[s_0,+\infty)_{\mathbb{T}}$.*

In what follows, for every $(z_0, s_0) \in \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}}$, we denote by $x(s, s_0, z_0)$ the unique maximal solution of the dynamic equation on time scales (5.6) with $x(s_0) = z_0$ defined in $[s_0,+\infty)_{\mathbb{T}}$.

In the sequel, we present the concepts concerning the boundedness of the solutions of the dynamic equation on time scales.

Definition 5.11. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$. We say that the dynamic equation on time scales (5.6) is:

• **Uniformly bounded:** if for every $\alpha > 0$, there exists a $M = M(\alpha) > 0$ such that, for every $s_0 \in [t_0,+\infty)_{\mathbb{T}}$ and for all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < M, \quad \text{for all } s \in [s_0,+\infty)_{\mathbb{T}}.$$

• **Quasi-uniformly ultimately bounded:** if there exists $B > 0$ such that for every $\alpha > 0$, there exists $T = T(\alpha) > 0$, such that for all $s_0 \in [t_0,+\infty)_{\mathbb{T}}$ and for all $z_0 \in \mathbb{R}^n$, where $\|z_0\| < \alpha$, we have

$$\|x(s, s_0, z_0)\| < B, \quad \text{for all } s \in [s_0 + T,+\infty) \cap \mathbb{T}.$$

• **Uniformly ultimately bounded:** if it is uniformly bounded and quasi-uniformly ultimately bounded.

In what follows, we will prove a result which ensures that the dynamic equation on time scales (5.6) is uniformly bounded.

Theorem 5.12. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $[t_0,+\infty)_{\mathbb{T}}$ be a time scale interval. Suppose $f : \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}}$ satisfies conditions (B1), (B2) and (B3), and $U : [t_0,+\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for each left-continuous function $z : [\alpha,\beta]_{\mathbb{T}} \to \mathbb{R}^n$ on $(\alpha,\beta]_{\mathbb{T}}$, the function $[\alpha,\beta]_{\mathbb{T}} \ni t \mapsto U(t, z(t))$ is left-continuous on $(\alpha,\beta]_{\mathbb{T}}$. Moreover, suppose the following conditions concerning U are satisfied*

(i) *There are two monotone increasing functions $p,\ b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(0) = 0$,*

$$\lim_{s \to +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq U(t, z) \leq p(\|z\|),$$

for every pair $(t, z) \in [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n$.

(ii) *For every solution $z : [s_0, +\infty) \cap \mathbb{T} \to \mathbb{R}^n$, $s_0 \geq t_0$, of the dynamic equation on time scales* (5.6)*, we have*

$$U(s, z(s)) - U(t, z(t)) \leq 0,$$

for every $s,\ t \in [s_0, +\infty) \cap \mathbb{T}$ with $t \leq s$.

Then, the dynamic equation on time scales (5.6) *is uniformly bounded.*

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 5.9, f^* satisfies conditions (A2), (A3) and (A4), and by Lemma 5.2, $g(t) = t^*$ is a nondecreasing and left-continuous function.

Define a function $U^* : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ by

$$U^*(t, x) = U(t^*, x), \qquad t \in [t_0, +\infty),\ x \in \mathbb{R}^n.$$

Then, by hypothesis (i), there are two monotone increasing functions $p,\ b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p(0) = b(0) = 0$,

$$\lim_{s \to +\infty} b(s) = +\infty$$

and

$$b(\|z\|) \leq \underbrace{U^*(t, z)}_{=U(t^*, z)} \leq p(\|z\|)$$

for every pair $(t, z) \in [t_0, +\infty) \times \mathbb{R}^n$, which implies that U^* satisfies the hypothesis (i) from Theorem 4.6.

Now, let $y : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, be a solution of the measure differential equation

$$x(t) = x(s_0) + \int_{s_0}^{t} f^*(x, s) \mathrm{d}g(s), \qquad t \geq s_0. \tag{5.7}$$

By Corollary 5.8, $y : [s_0, +\infty) \to \mathbb{R}^n$ must have the form $y = z^*$, where $z : [s_0, +\infty) \cap \mathbb{T} \to \mathbb{R}^n$ is a solution of the dynamic equation on time scales (5.6). Hence, for each $s_0 \leq t < s < +\infty$, we have

$$\begin{aligned} U^*(s, y(s)) - U^*(t, y(t)) &= U^*(s, z^*(s)) - U^*(t, z^*(t)) \\ = U(s^*, z^*(s)) - U(t^*, z^*(t)) &= U(s^*, z(s^*)) - U(t^*, z(t^*)) \leq 0, \end{aligned}$$

by hypothesis (ii). Therefore, all the hypotheses from Theorem 4.6 are satisfied. Then the measure differential equation (5.7) is uniformly bounded. By Corollary 5.8, the dynamic equation on time scales (5.6) is uniformly bounded, obtaining the desired result. □

Finally, we present the last result of our paper. It ensures that the dynamic equation on time scales (5.6) is uniformly ultimately bounded.

Theorem 5.13. *Let* $\mathbb{T}$ *be a time scale such that* $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ *and* $t_0 \geq 0$. *Suppose* $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$ *satisfies conditions (B1), (B2) and (B3), and* $U : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ *be a function such that for each left-continuous function* $z : [\alpha, \beta]_{\mathbb{T}} \to \mathbb{R}^n$ *on* $(\alpha, \beta]_{\mathbb{T}}$, *the function* $[\alpha, \beta]_{\mathbb{T}} \ni t \mapsto U(t, z(t))$ *is left-continuous on* $(\alpha, \beta]_{\mathbb{T}}$ *and satisfies condition (i) from Theorem 5.12. Moreover, suppose* U *satisfies the following conditions*

(I) *For all* $x,\ y,\ z,\ w \in \mathbb{R}^n$ *and for all* $\alpha,\ \beta \in [t_0, +\infty)_{\mathbb{T}}$ *with* $\alpha < \beta$, *we have*

$$|U(\beta, x) - U(\beta, y) - U(\alpha, z) + U(\alpha, w)| \leq$$

$$\left(\int_{\alpha}^{\beta} K(\tau)\Delta\tau\right) \max\left\{\|x - y\|, \|z - w\|\right\},$$

where $K : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ *is a locally Kurzweil-Henstock* Δ*-integrable function.*

(II) *There exists a continuous function* $\phi : \mathbb{R}^n \to \mathbb{R}$, *with* $\phi(0) = 0$ *and* $\phi(x) > 0$, $x \neq 0$ *such that for every solution* $z : [s_0, +\infty) \cap \mathbb{T} \to \mathbb{R}^n$, $s_0 \geq t_0$, *of the dynamic equation on time scales* (5.6), *we have*

$$U(s^*, z(s^*)) - U(t^*, z(t^*)) \leq (s - t)^*\Big(-\phi(z(t^*))\Big),$$

for every $t,\ s \in [s_0, +\infty) \cap \mathbb{T}$ *with* $t \leq s$.

Then, the dynamic equation on time scales (5.6) *is uniformly ultimately bounded.*

Proof. Since f satisfies conditions (B1), (B2) and (B3), by Theorem 5.9, f^* satisfies conditions (A2), (A3) and (A4), and by Lemma 5.2, $g(t) = t^*$ is a nondecreasing and left-continuous function.

Also, define the functional $U^* : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ by

$$U^*(t, x) = U(t^*, x),$$

for every pair $(t, x) \in [t_0, +\infty) \times \mathbb{R}^n$.

Since K is a locally Kurzweil-Henstock Δ-integrable function on $[t_0, +\infty)_{\mathbb{T}}$, it follows from Theorem 5.3 that the function $K^* : [t_0, +\infty) \to \mathbb{R}$ is locally Kurzweil-Henstock integrable with respect to the nondecreasing function $u : \mathbb{T}^* \to \mathbb{R}$, given by $u(t) = t^*$ and

$$\int_{\alpha}^{\beta} K(\tau)\Delta\tau = \int_{\alpha}^{\beta} K^*(\tau)du(\tau),$$

for all $\alpha,\ \beta \in [t_0, +\infty)_{\mathbb{T}}$. Thus, for all $x,\ y : [\upsilon, \gamma] \to \mathbb{R}^n$, $[\upsilon, \gamma] \subset [t_0, +\infty)$, of bounded variation on $[\upsilon, \gamma]$ and all $\upsilon \leq s < t \leq \gamma$, we have

$$|U^*(t, x(t)) - U^*(t, y(t)) - U^*(s, x(s)) + U^*(s, y(s))|$$

$$
\begin{aligned}
&= |U(t^*, x(t)) - U(t^*, y(t)) - U(s^*, x(s)) + U(s^*, y(s))| \\
&\leq \left(\int_{s^*}^{t^*} K(\tau)\Delta\tau\right) \max\{\|x(t)-y(t)\|, \|x(s)-y(s)\|\} \\
&= \left(\int_{s^*}^{t^*} K^*(\tau)du(\tau)\right) \max\{\|x(t)-y(t)\|, \|x(s)-y(s)\|\} \\
&\leq \left(\int_{s^*}^{t^*} K^*(\tau)du(\tau)\right) \sup_{\xi\in[\upsilon,\gamma]} \|x(\xi)-y(\xi)\| \\
&= \left(\int_{s}^{t} K^*(\tau)du(\tau)\right) \sup_{\xi\in[\upsilon,\gamma]} \|x(\xi)-y(\xi)\|,
\end{aligned}
$$

where Theorem 5.4 is used to establish the last equality. Thus, the hypothesis (U1) from Theorem 4.7 is fulfilled.

Now, let $y : [s_0, +\infty) \to \mathbb{R}^n$, $s_0 \geq t_0$, be a solution of the measure differential equation (5.7). Then, by Corollary 5.8, $y : [s_0, +\infty) \to \mathbb{R}^n$ must have the form $y = z^*$, where $z : [s_0, +\infty) \cap \mathbb{T} \to \mathbb{R}^n$ is a solution of the dynamic equation on time scales (5.6). Hence, we get

$$
\begin{aligned}
&U^*(s, y(s)) - U^*(t, y(t)) = U^*(s, z^*(s)) - U^*(t, z^*(t)) \\
&= U(s^*, z^*(s)) - U(t^*, z^*(t)) = U(s^*, z(s^*)) - U(t^*, z(t^*)) \\
&\leq (s-t)^*(-\phi(z(t^*))) = (s-t)^*(-\phi(y(t))) \\
&\leq (s-t)(-\phi(y(t))),
\end{aligned}
$$

for every $s_0 \leq t < s < +\infty$. Thus, condition (U2) from the Theorem 4.7 is fullfiled. Therefore, all the hypotheses from Theorem 4.7 hold and the measure differential equation (5.7) is uniformly ultimately bounded. By Corollary 5.8, it follows that the dynamic equation on time scales (5.6) is uniformly ultimately bounded, obtaining the desired result. □

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