EXISTENCE OF A POSITIVE SOLUTION AND NUMERICAL SOLUTION FOR SOME ELLIPTIC SUPERLINEAR PROBLEM

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Abstract. In this paper, we prove the existence of positive solutions of an elliptic superlinear problem. Also, we are interested here in getting results concerning the existence of positive solutions for the discrete formulation of our problem. Therefore, in order to do it, we employ the radial solutions of the elliptic superlinear problem, obtaining a second-order dynamic equation on time scales, which encompasses discrete, continuous and hybrid formulations of our problem. This unified equation allows us to present numerical simulations, which give us a more precise analysis and description concerning the behavior of the solution according to the parameters.

1. Introduction

One of the most studied problems in the last decades in nonlinear PDEs concerns about the existence of positive solutions of the following elliptic superlinear problem

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is an increasing function and satisfies the following general conditions

(A1) $\displaystyle\lim_{u \to 0} \frac{f(x,u)}{u} = 0$;

(A2) $\displaystyle\lim_{u \to +\infty} \frac{f(x,u)}{u} = +\infty$.

Results on existence for this problem and with so much generality, can only be obtained to the one-dimensional case or to the radial case when Ω is some annulus centered at the origin. For non symmetric domains, the problem is more delicate, however some techniques can be used, for example variational methods, where the pioneer work was due to Ambrosetti and Rabinowitz [2]. More precisely, using the very well know Mountain Pass Theorem, they proved the existence of a positive solution assuming conditions (A1), (A2) and the following technical hypotheses:

(G) There are positive constants a and b such that

$$|f(x,t)| \leq a + b|t|^{q-1}, \ \forall t \in \mathbb{R},$$

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where $1 \leq q < 2^* = \frac{2N}{N-2}$.

(AR) For some $\theta > 2$, $C > 0$, we have

$$0 < \theta F(x,t) \leq f(x,t)t, \quad \forall |t| \geq C, \ x \in \Omega.$$

The first condition is some subcritical assumption and the (AR) is the famous Ambrosetti-Rabinowitz condition. These types of conditions appear in most of the studies of the existence of nontrivial solutions via variational methods. See, for instance, [11, 13, 16, 19, 20, 21, 22, 38, 44] and the references therein. In a recent work of Miyagaki and Souto [34] (see also [27, 31]), the authors introduced some monotonicity hypotheses which allow to consider some nonlinearities that do not satisfy (AR), for example when $f(x,t) = t(2\ln t + 1)$. More precisely, they assumed the following conditions:

(P1) $\dfrac{f(t)}{t}$ is increasing for $t \geq t_0$ and decreasing for $t \leq -t_0$ or

(P2) there exists $C > 0$ such that

$$tf(t) - 2F(t) \leq 2f(s) - 2F(s) + C$$

for all $0 < t < s$ or $s < t < 0$.

On the other hand, in [24], the authors ensured the existence of solution, assuming the conditions (A1) and (A2), $f(x,u) = a(x)g(u)$ and that is regularly varying (at infinity) of index 1, i.e.,

(RV) The function g satisfies

$$\lim_{t \to +\infty} \frac{g(\sigma t)}{g(t)} = \sigma \quad \text{for all} \quad \sigma > 0.$$

Note that in their case $a(x)$ may change sign. They also give some examples which do not verify (AR).

In this work, our goal is to investigate the existence of positive solutions of problem (1.1) when Ω is a ball and $f(x,u) = |x|^\alpha f(u)$, i.e., we will study the following problem

$$\begin{cases} -\Delta u = |x|^\alpha f(u), & x \in B, \\ u(x) = 0, & x \in \partial B \end{cases} \tag{1.2}$$

where $B = \left\{x \in \mathbb{R}^N : |x| < 1\right\}$, $\alpha > -2$ and $f : \mathbb{R} \to \mathbb{R}^+$ satisfies the conditions below

(B0) f is a continuous function such that $f(u) > 0$, $u > 0$;

(B1) $\displaystyle\lim_{u \to 0} \frac{f(u)}{u} = 0$;

(B2) $\displaystyle\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$;

(B3) There exist a continuous function $\varphi : [0,+\infty) \to [0,+\infty)$, $M > 0$ and $\bar{\tau} \in \mathbb{R}_+$ such that

$$\int_1^{+\infty} \varphi(\tau)\tau^{-a_N} d\tau < +\infty \quad \text{and} \quad \frac{f(u \cdot \tau)}{f(u)} \leq M\varphi(\tau)$$

for all $u > 0$ and for all $\tau > \bar{\tau}$, where $a_N = \dfrac{N+\alpha}{N-2}$.

We point out that another important technique to obtain positive solutions of problem (1.2) is by employing fixed point methods, for which is essential to establish a priori bounds where some Liouville results are necessary, see for example [1, 3, 4, 8, 14, 23, 25, 32, 33, 36, 39, 40, 41] and references therein. However, these arguments need strongly that the nonlinearity f is asymptotic to a power near ∞, which is not a necessary condition in our case.

Therefore, our goal here is to prove the existence of positive solutions of (1.2), imposing a new hypothesis, which implies a property of pseudo homogeneity on the nonlinearity $f(x, u)$.

To achieve our results, we will study radial solutions of the problem (1.2). Then, by a change of variables, the problem (1.2) can be rewritten as:

$$\begin{cases} -z''(t) = (1 + (N-2)t)^{\frac{1}{2-N}(2(N-1)+\alpha)} f(z(t)), \quad t \in (0, +\infty) \\ z(0) = z'(+\infty) = 0. \end{cases} \tag{1.3}$$

Therefore, our problem here reduces to study the existence of positive solutions of the problem (1.3).

Also, in this paper, we are interested to investigate a discrete formulation of our problem, since it allows us to present numerical simulations, investigating better the behavior of the solutions. It is a known fact that depending on our differential problem, it may not be possible to calculate its solutions explicitly, however, its discrete formulation allows us to obtain them explicitly, employing computational methods. This fact allows us to study the behavior of the solutions of our differential problem by approximmation. A very useful strategy for this is to consider the formulation of the problem for the discrete scale $\mathbb{T} = h\mathbb{Z}$ and then, to calculate the solutions explicitly for this case. So, when $h \to 0$, the solutions of the discrete problem approach to the solutions of the continuous problem, allowing us a carefully study for these last ones ([15]).

Motivated by these facts, we will study this problem on the setting of time scales theory. In this case, a reformulation of problem (1.3) by a general case of dynamic equation on time scales can be given by (see [5, 6, 7]):

$$\begin{cases} -z^{\Delta\Delta}(t) = (1 + (N-2)\sigma(t))^{\frac{1}{2-N}(2(N-1)+\alpha)} f(z(\sigma(t))), \quad t \in \mathbb{T}_0^+ \\ z(0) = z^{\Delta}(+\infty) = 0, \end{cases} \tag{1.4}$$

where $\mathbb{T}_0$ is a time scale satisfying $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$ and the function $f : \mathbb{R} \to \mathbb{R}^+$ satisfies the following conditions:

- (H0) f is a continuous function such that $f(u) > 0$, $u > 0$.
- (H1) $\lim_{u \to 0} \dfrac{f(u)}{u} = 0$;
- (H2) $\lim_{u \to +\infty} \dfrac{f(u)}{u} = +\infty$;
- (H3) There exist an rd-continuous function $\varphi : \mathbb{T}_0^+ \to \mathbb{T}_0^+$, $M > 0$ and $\bar{\tau} \in \mathbb{T}_0^+$ such that

$$\int_1^{+\infty} \varphi(\sigma(\tau))\sigma(\tau)^{-a_N} \Delta\tau < +\infty \quad \text{and} \quad \frac{f(u \cdot \sigma(\tau))}{f(u)} \le M\varphi(\sigma(\tau))$$

  for all $u > 0$ and for all $\sigma(\tau) > \bar{\tau}$, where $a_N = \dfrac{N+\alpha}{N-2}$.

Notice that when $\mathbb{T}_0 = \mathbb{R}$, our problem reduces to the problem (1.3). If $\mathbb{T}_0 = \mathbb{Z}$ or even $\mathbb{T}_0 = h\mathbb{Z}$, we have a discrete formulation of our problem. On the other hand, our equation also encompasses several types of equations, depending on the chosen time scales. For more details, see [6, 7, 9, 10, 17, 18, 28, 29, 42] and the references therein. Therefore, through our equation (1.4), we are able to study the hybrid equations, difference equations, h-difference equations, quantum difference equations, differential equations, among others. We point out that our results are completely new considering any time scales.

Finally, we present some surprising simulations for our problem to illustrate our main result. These simulations allow a better understanding of the behavior of the solutions. We present two different simulations. The first one shows the behavior of the solutions when the values of α are very small, close to zero. Further, it is a known fact that we can investigate the behavior of the solution of the equation for the case $\mathbb{T}_0 = \mathbb{R}$ only studying the solutions for the case $\mathbb{T}_0 = h\mathbb{Z}$. In this case, when h approaches to 0, the solution of h-difference equations approaches to the solution of the differential equation (case $\mathbb{T}_0 = \mathbb{R}$). Therefore, we present here the solutions of the h-difference equations when h approaches to 0 and also, the smooth solution (case $\mathbb{T}_0 = \mathbb{R}$) to understand how this approximation occurs.

The outline of this paper is as follows: The second section is devoted to present the basic results and concepts to prove the main results of the paper. In the third section, we prove the existence of positive solutions for the dynamic equation on time scales given by (1.4). Finally, the last section is dedicated to present the simulations of the solutions of our problem.

2. Preliminaries

In this section, we review some basic concepts and results concerning time scales which will be essential to prove our main results. For more details, the reader may consult [5, 6, 7].

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T},\ s < t\}.$$

In this definition, we consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then t is called *right-dense*. Analogously, if $\rho(t) < t$, then t is called *left-scattered*, whereas if $\rho(t) = t$ and $t > \inf \mathbb{T}$, then t is *left-dense*. Define the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ by $\mu(t) = \sigma(t) - t$.

Definition 2.1 ([6])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$. We denote the class of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.If the function $f : \mathbb{T} \to \mathbb{R}$ is continuous at each right-dense point and each left-dense point, then f is said to be *continuous* on $\mathbb{T}$.

Given a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T};\ a \leq t \leq b\}$.

We define the set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \backslash \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.2 ([6])**.** For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define $y^\Delta(t)$ to be the number (provided it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|y(\sigma(t)) - y(s) - y^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. We call $y^{\Delta}(t)$ the delta derivative of y at t.

Theorem 2.3 (See [5, Theorem 2.98]). *Let f be twice delta differentiable on $(a,b)_{\mathbb{T}}$ and $f^{\Delta\Delta}(t) \leq 0$ for all $t \in (a,b)_{\mathbb{T}}$, then f is concave.*

Theorem 2.4 (See [5, Corollary 2.47]). *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that has a delta derivative at each point of $[a,b)_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a,b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$ and $f^{\Delta}(t) \leq 0$ for all $t \in [a,b)_{\mathbb{T}}$, respectively.*

Below, we present some important properties of delta-integrals.

Theorem 2.5 (See [6, Theorem 1.75]). *If $f \in \mathcal{C}_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = f(t)\mu(t).$$

Theorem 2.6 (See [6, Theorem 1.76]). *If $f^{\Delta} \geq 0$, then f is nondecreasing.*

The next result is the integration by parts for delta-integrals.

Theorem 2.7 (See [6, Theorem 1.77(vi)]). *If $a, b \in \mathbb{T}$ and $f, g \in \mathcal{C}_{rd}$, then*

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t.$$

3. Second-Order Dynamic Equations on Time Scales

In this section, our goal is to ensure the existence of a positive solution for the following problem:

$$\begin{cases} -\Delta u = |x|^{\alpha} f(u), & x \in B, \\ u(x) = 0, & x \in \partial B \end{cases} \tag{3.1}$$

where $B = \left\{x \in \mathbb{R}^N : |x| < 1\right\}$, $\alpha > -2$ and $f : \mathbb{R} \to \mathbb{R}^+$ satisfies the conditions:

- (B0) f is a continuous function such that $f(u) > 0$, $u > 0$.
- (B1) $\lim\limits_{u \to 0} \dfrac{f(u)}{u} = 0$;
- (B2) $\lim\limits_{u \to +\infty} \dfrac{f(u)}{u} = +\infty$;
- (B3) There exist a continuous function $\varphi : [0,+\infty) \to [0,+\infty)$, $M > 0$ and $\bar{\tau} \in \mathbb{R}_+$ such that

$$\int_1^{+\infty} \varphi(\tau)\tau^{-a_N}\Delta\tau < +\infty$$

and

$$\frac{f(u \cdot \tau)}{f(u)} \leq M\varphi(\tau)$$

for all $u > 0$ and for all $\tau > \bar{\tau}$, where $a_N = \dfrac{N+\alpha}{N-2}$.

Notice that radial solutions of problem (3.1) satisfy

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}r^{\alpha}f(u), & u \in (0,1) \\ u(1) = u'(0) = 0. \end{cases} \tag{3.2}$$

Consider $a : (0,1] \to [0,+\infty)$ defined by

$$a(r) = \frac{r^{2-N} - 1}{N-2}.$$

Performing the change of variables

$$t = a(r) \quad \text{and} \quad z(t) = u(r(t)),$$

the equation (3.2) can be rewritten as:

$$\begin{cases} -z'' = r^{2(N-1)}(t)h(t)f(z(t)), & t \in (0,+\infty) \\ z(0) = z'(+\infty) = 0, \end{cases}$$

where $h(t) = (a^{-1}(t))^{\alpha}$. On the other hand, we have:

$$\begin{aligned} r^{2(N-1)}(t)h(t) &= (1+(N-2)t)^{\frac{2(N-1)}{2-N}}(1+(N-2)t)^{\frac{\alpha}{2-N}} \\ &= (1+(N-2)t)^{\frac{1}{2-N}(2(N-1)+\alpha)}. \end{aligned}$$

More precisely, we obtain the system (1.3). Our goal is to investigate this system in the setting of dynamic equations on time scales. Therefore, we consider, throughout this paper, that $\mathbb{T}_0$ is a time scale satisfying $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$. Also, denote by $\mathbb{T}_0^+ = \mathbb{T}_0 \cup [0,\infty)$. Here, we will consider the following system:

$$\begin{cases} -z^{\Delta\Delta}(t) = (1+(N-2)\sigma(t))^{\frac{1}{2-N}(2(N-1)+\alpha)} f(z(\sigma(t))), & t \in \mathbb{T}_0^+ \\ z(0) = z^{\Delta}(+\infty) = 0, \end{cases} \tag{3.3}$$

where the function $f : \mathbb{R} \to \mathbb{R}^+$ satisfies the following conditions:

(H0) f is a continuous function such that $f(u) > 0$, $u > 0$.

(H1) $\lim\limits_{u\to 0} \dfrac{f(u)}{u} = 0$;

(H2) $\lim\limits_{u\to +\infty} \dfrac{f(u)}{u} = +\infty$;

(H3) There exist an rd-continuous $\varphi : \mathbb{T}_0^+ \to \mathbb{T}_0^+$ and $M > 0$ and $\bar{\tau} \in \mathbb{T}_0^+$ such that

$$\int_1^{+\infty} \varphi(\sigma(\tau))\sigma(\tau)^{-a_N}\Delta\tau < +\infty$$

and

$$\frac{f(u\cdot\sigma(\tau))}{f(u)} \le M\varphi(\sigma(\tau))$$

for all $u > 0$ and for all $\sigma(\tau) > \bar{\tau}$, where $a_N = \dfrac{N+\alpha}{N-2}$.

Notice that when the time scale $\mathbb{T} = \mathbb{R}$, our system reduces to the problem (1.3). However, we can have more general systems, depending on the chosen time scale.

Also, by integration of problem (3.3), we obtain the following delta integral equation which is equivalent to equation (3.3):

$$z(t) = \int_0^t \int_s^{+\infty} G_\alpha(\sigma(\tau)) f(z(\sigma(\tau)))\Delta\tau\Delta s, \tag{3.4}$$

for every $t, s \in \mathbb{T}_0^+$ and $G_\alpha(\sigma(\tau)) = (1 + (N-2)\sigma(\tau))^{\frac{1}{2-N}(2(N-1)+\alpha)}$. Therefore, the solutions of the delta integral equation (3.4) are the fixed points of the operator

$$(Fz)(t) = \int_0^t \int_s^{+\infty} G_\alpha(\sigma(\tau)) f(z(\sigma(\tau)))\Delta\tau\Delta s.$$

To our purposes, we will use the following well-known lemma. See, for instance, [12, 26].

Lemma 3.1. *Let X be a Banach space with norm $|\cdot|$ and let $\mathcal{C} \subset X$ be a cone in X. For $R > 0$, define $\mathcal{C}_R = \mathcal{C} \cap B[0, R]$ where $B[0, R] = \{x \in X : |x| \le R\}$ denotes the closed ball of radius R centered at the origin of X, which is a completely continuous map for which there exists $0 < r < R$ such that*

$$|Fx| < |x|, \quad x \in \partial\mathcal{C}_r \quad \text{and} \quad |Fx| > |x|, \ x \in \partial\mathcal{C}_R$$

or

$$|Fx| > |x|, \quad x \in \partial\mathcal{C}_r \quad \text{and} \quad |Fx| < |x|, \quad x \in \partial\mathcal{C}_R,$$

where $\partial\mathcal{C}_R = \{x \in \mathcal{C} : |x| = R\}$. Then F has a fixed point $u \in \mathcal{C}$ with $r < |u| < R$.

Let $X = \mathcal{C}_{rd}(\mathbb{T}_0^+, \mathbb{R})$ with the norm $\|z\|_\infty = \sup\limits_{t\in\mathbb{T}_0^+} |z(t)|$ and define $\mathcal{C}_1$ as the cone of the nonnegative and concave functions of X such that $z(0) = 0$. Note that $z \in \mathcal{C}_1$ implies that z is an increasing function and z is nonnegative.

Let us assume throughout the paper that f is an increasing function, $N \ge 3$, $\lim\limits_{t\to\infty} z(t)$ exists and is finite, where z is given by (3.4) and also, that there exists $\beta \in [1, \frac{2(N-1)+\alpha}{N-2})$ such that $\sigma(t) = O(t^\beta)$ as $t \to \infty$, when $\alpha > -2$.

Lemma 3.2. *If $\alpha > -2$ and the function f satisfies the conditions (H0)-(H3), then F is well-defined, $F(\mathcal{C}_1) \subset \mathcal{C}_1$ and F is completely continuous operator.*

Proof. At first, since $N \ge 3$, it follows by the definition that G_α is a decreasing function. Therefore, if

$$\frac{2(N-1)+\alpha}{2-N} < -2,$$

then clearly by [7, Corollary 5.71], the integral

$$\int_s^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau$$

converges. Notice also that

$$\frac{2(N-1)+\alpha}{2-N} < -2$$

$$\begin{aligned}
&\Leftrightarrow \quad \frac{2(N-1)+\alpha}{2-N}+1<-1\\
&\Leftrightarrow \quad \frac{N+\alpha}{2-N}<-1\\
&\Leftrightarrow \quad \frac{N+\alpha}{N-2}>1\\
&\Leftrightarrow \quad N+\alpha>N-2\\
&\Leftrightarrow \quad \alpha>-2.
\end{aligned}$$

Therefore, F is well defined for all time scales $\mathbb{T}_0^+$. Moreover, since $\alpha > -2$ and by the hypothesis, it is not difficult to see that

$$\int_0^{+\infty}\left(\int_s^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau\right)\Delta s \;\;<\;\; +\infty.$$

Hence, F is well-defined. Also, notice that the function $F(z)(t)$ belongs to the class $\mathcal{C}^2_{rd}$ and its delta-derivative is given by:

$$F^\Delta(z(t)) = \int_t^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau.$$

Indeed, if t is right-dense, the equality above follows immediately. On the other hand, if t is right-scattered, then:

$$\begin{aligned}
F^\Delta(z(t)) &= \frac{F(z(\sigma(t))-F(z(t))}{\mu(t)}\\
&= \frac{\displaystyle\int_0^{\sigma(t)}\int_s^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau\Delta s-\int_0^{t}\int_s^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau\Delta s}{\mu(t)}\\
&= \frac{\displaystyle\int_t^{\sigma(t)}\int_s^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau\Delta s}{\mu(t)}\\
&= \frac{\displaystyle\mu(t)\int_t^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau}{\mu(t)}\\
&= \int_t^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau>0,
\end{aligned}$$

since $z \in \mathcal{C}^1$, where the fourth equality follows by Theorem 2.5. Moreover, notice that

$$F^{\Delta\Delta}(z)(t) = -G_\alpha(\sigma(t))f(z(\sigma(t))).$$

In fact, if t is right-dense, then it follows immediately. Otherwise, by Theorem 2.5, we have:

$$F^{\Delta\Delta}(z(t)) \;=\; \frac{\displaystyle\int_{\sigma(t)}^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau-\int_t^{+\infty} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau}{\mu(t)}$$

$$\begin{aligned} &= -\frac{\displaystyle\int_t^{\sigma(t)} G_\alpha(\sigma(\tau))f(z(\sigma(\tau)))\Delta\tau}{\mu(t)} \\ &= -G_\alpha(\sigma(t))f(z(\sigma(t))) < 0. \end{aligned}$$

since $z \in \mathcal{C}^1$.Then, $F(z)(\tau)$ is increasing and concave by Theorems 2.3 and 2.4. Therefore, $F(\mathcal{C}_1) \subset \mathcal{C}_1$.

It remains to prove that F is a completely continuous operator. Let $(z_n) \in \mathcal{C}_1$ such that $\|z_n\|_\infty \leq c_0$ and $M_1 = \max\{f(u) : u \in [0, c_0]\}$. Thus, it follows that:

$$|F(z_n)(t)| \leq M_1 \int_0^{+\infty}\int_s^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau\Delta s \quad \text{and} \quad \left|F^{\Delta\Delta}(z_n)(t)\right| \leq M_1 \int_0^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau.$$

By the Arzelá-Ascoli Compactness Criterion for uniform convergence on time scales (see [43, Lemma 4]), up to a subsequence, we can assume that $(F(z_n))$ is uniformly convergent on compact subsets of $[0, +\infty)_{\mathbb{T}}$. To prove that there exists a uniformly convergent subsequence of $F(z_n)$, it suffices to recall that given $\varepsilon > 0$, there is $T = T(\varepsilon) > 0$ such that:

$$\int_T^{+\infty}\int_s^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau\Delta s < \varepsilon.$$

We now verify that F is continuous. Let (z_n) be a sequence in $\mathcal{C}_1$ such that $\|z_n - z_0\|_\infty \to 0$ as $n \to \infty$. Thus

$$|F(z_n)(t) - F(z_0)(t)| \leq \int_0^{+\infty} |\Gamma_n(s) - \Gamma_0(s)|\Delta s$$

where

$$\Gamma_n(s) = \int_s^{+\infty} G_\alpha(\sigma(\tau))f(z_n(\sigma(\tau)))\Delta\tau \quad \text{and} \quad \Gamma_0(s) = \int_s^{+\infty} G_\alpha(\sigma(\tau))f(z_0(\sigma(\tau)))\Delta\tau.$$

Since $\|z_n - z_0\|_\infty \to 0$, then $\Gamma_n(s) \to \Gamma_0(s)$ and for each $n \in \mathbb{N}$, $\Gamma_n(s) \leq M_1 \int_s^{+\infty} G_\alpha(\sigma(\tau))\Delta\tau < +\infty$ for all $s \in [0, +\infty)_{\mathbb{T}}$. By the Lebesgue Dominated Convergence Theorem for delta integrals (see [7]), $\|F(z_n) - F(z_0)\|_\infty \to 0$, which implies that F is continuous and we have the desired result. $\square$

Notice that given $z \in \mathcal{C}_1 \setminus \{0\}$, there exists a unique $\tau_1 = \tau_1(z) \in \mathbb{T}_0^+$ such that

$$2z(\tau_1) = \|z\|_\infty.$$

Define

$$\tau^* := \sup\{\tau_1(F(z)) \in \mathbb{T}_0^+ : z \in \mathcal{C}_1\}$$

and

$$\mathcal{C} := \{z \in \mathcal{C}_1 : 2z(t) \geq \|z\|_\infty,\ \forall t \in [\tau^*, \infty)_{\mathbb{T}}\}.$$

Lemma 3.3. *Suppose the hypotheses (H0)-(H3). Then $\tau^* \in \mathbb{T}_0^+$ is finite and $\mathcal{C}$ is a cone invariant under F.*

Proof. At first, we will show that $\tau^* < \infty$. Suppose the contrary, that is, $\tau^* = +\infty$. Then there must exist a sequence $z_n \subset \mathcal{C}_1 \setminus \{0\}$ such that $\tau_n = \tau_1(F(z_n))$ is a strictly increasing sequence in $\mathbb{T}_0^+$ converging to $+\infty$ as $n \to \infty$. Notice that

$$(Fz_n)(\tau_n) = \frac{\|F(z_n)\|_\infty}{2},$$

which implies that

$$\int_0^{\tau_n} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s = \frac{1}{2}\int_0^{+\infty} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s$$

where $H_n(\sigma(\tau)) = G_\alpha(\sigma(\tau))f(z_n(\sigma(\tau)))$. Therefore, we have

$$\frac{1}{2}\int_0^{\tau_n} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s = \frac{1}{2}\int_0^{+\infty} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s - \frac{1}{2}\int_0^{\tau_n} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s,$$

which implies

$$\int_0^{\tau_n} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s = \int_{\tau_n}^{+\infty} \int_s^{+\infty} H_n(\sigma(\tau))\Delta\tau\Delta s. \tag{3.5}$$

According to (3.5) and using integration by parts for delta-integrals (Theorem 2.7), we have

$$\left(\int_{\tau_n}^{+\infty} H_n(\sigma(\tau))\Delta\tau\right)\tau_n + \int_0^{\tau_n} H_n(\sigma(s))\sigma(s)\Delta s = -\left(\int_{\tau_n}^{+\infty} H_n(\sigma(\tau))\Delta\tau\right)\tau_n + \int_{\tau_n}^{+\infty} H_n(\sigma(s))\sigma(s)\Delta s$$

which implies

$$2\tau_n\left(\int_{\tau_n}^{+\infty} H_n(\sigma(\tau))\Delta\tau\right) + \int_0^{\tau_n} H_n(\sigma(s))\sigma(s)\Delta s = \int_{\tau_n}^{+\infty} H_n(\sigma(s))\sigma(s)\Delta s. \tag{3.6}$$

From this fact, it follows that

$$\int_0^{\tau_n} H_n(\sigma(s))\sigma(s)\Delta s \leq \int_{\tau_n}^{+\infty} H_n(\sigma(s))\sigma(s)\Delta s,$$

which implies that

$$\int_0^{\tau_n} G_\alpha(\sigma(s))f(z_n(\sigma(s)))\sigma(s)\Delta s \leq \int_{\tau_n}^{+\infty} G_\alpha(\sigma(s))f(z_n(\sigma(s)))\sigma(s)\Delta s.$$

Therefore, since z_n is concave, it follows that

$$\int_0^{\tau_n} \sigma(\tau)U_n(\sigma(\tau))\Delta\tau \leq \int_{\tau_n}^{+\infty} \sigma(\tau)U_n(\sigma(\tau))\Delta\tau \tag{3.7}$$

where $U_n(\sigma(\tau)) = G_\alpha(\sigma(\tau))f(\alpha_n \cdot \sigma(\tau))$ with $\alpha_n = z_n(\sigma(\tau_n))/\sigma(\tau_n)$. On the other hand, notice that $f(\alpha_n \cdot \sigma(\tau)) \geq f(\alpha_n)$ for all $\sigma(\tau) \geq 1$. Therefore, by using (3.7), we have

$$\int_1^{\tau_n} \sigma(\tau)U_n(\sigma(\tau))\Delta\tau \leq \int_0^{\tau_n} \sigma(\tau)U_n(\sigma(\tau))\Delta\tau \leq \int_{\tau_n}^{+\infty} \sigma(\tau)U_n(\sigma(\tau))\Delta\tau.$$

Hence,

$$\int_1^{\tau_n} \sigma(\tau)G_\alpha(\sigma(\tau))f(\alpha_n)\Delta\tau \quad \leq \quad \int_1^{\tau_n} \sigma(\tau)G_\alpha(\sigma(\tau))f(\alpha_n \cdot \sigma(\tau))\Delta\tau$$

$$\leq \int_{\tau_n}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))f(\alpha_n \cdot \sigma(\tau))\Delta\tau.$$

Therefore,

$$\int_1^{\tau_n} \sigma(\tau)G_\alpha(\sigma(\tau))\Delta\tau \leq \int_{\tau_n}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\frac{f(\alpha_n \cdot \sigma(\tau))}{f(\alpha_n)}\Delta\tau. \tag{3.8}$$

Now, let us consider two cases.

Case 1. There exists a subsequence (α_{n_k}) of (α_n) such that $\alpha_{n_k} < \bar{\alpha}$ for all k. In this case, from (3.8) and (H3), we get

$$\begin{aligned}
\int_1^{\tau_{n_k}} \sigma(\tau)G_\alpha(\sigma(\tau))\Delta\tau &\leq \int_{\tau_{n_k}}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\frac{f(\bar{\alpha} \cdot \sigma(\tau))}{f(\alpha_{n_k})}\Delta\tau \\
&= \int_{\tau_{n_k}}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\frac{f(\bar{\alpha} \cdot \sigma(\tau))}{f(\bar{\alpha})}\frac{f(\bar{\alpha})}{f(\alpha_{n_k})}\Delta\tau \\
&\leq M\int_{\tau_{n_k}}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\frac{f(\bar{\alpha})}{f(\alpha_{n_k})}\varphi(\sigma(\tau))\Delta\tau \\
&\leq C\int_{\tau_{n_k}}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\varphi(\sigma(\tau))\Delta\tau. \qquad (3.9)
\end{aligned}$$

Case 2. Now, suppose that $\alpha_n \geq \bar{\alpha}$ for all n. In this case, from (3.8) and (H3), we obtain

$$\int_1^{\tau_n} \sigma(\tau)G_\alpha(\sigma(\tau))\Delta\tau \leq C\int_{\tau_n}^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\varphi(\sigma(\tau))\Delta\tau. \tag{3.10}$$

In both cases (equations (3.9) and (3.10)), since $G_\alpha(\sigma(\tau)) = (1+(N-2)\sigma(\tau))^{\frac{1}{2-N}(2(N-1)+\alpha)}$, we have for $t \in \mathbb{T}_0$

$$\begin{aligned}
\sigma(\tau)G_\alpha(\sigma(\tau)) &= \sigma(\tau)(1+(N-2)\sigma(\tau))^{\frac{1}{2-N}(2(N-1)+\alpha)} \\
&\leq \sigma(\tau)((N-2)\sigma(\tau))^{\frac{1}{2-N}(2(N-1)+\alpha)} \\
&= (N-2)^{\frac{1}{2-N}(2(N-1)+\alpha)}\sigma(\tau)^{\left(1+\frac{1}{2-N}(2(N-1)+\alpha)\right)} \\
&= (N-2)^{\frac{1}{2-N}(2(N-1)+\alpha)}\sigma(\tau)^{-a_N}.
\end{aligned}$$

Therefore, we have

$$\int_1^{\tau_n} \sigma(\tau)G_\alpha(\sigma(\tau))\Delta\tau \leq K\int_{\tau_n}^{+\infty} \sigma(\tau)^{-a_N}\varphi(\sigma(\tau))\Delta\tau,$$

and from (H3), it follows that the integral of the right hand side of the above inequality converges to zero when n goes to infinity. But this is impossible, since $\int_1^{+\infty} \sigma(\tau)G_\alpha(\sigma(\tau))\Delta\tau > 0$. Finally, it is clear that $\mathcal{C}$ is an invariant cone under F. □

Next, we prove our main result of this section.

Theorem 3.4. *Suppose the hypotheses (H0)-(H3) are satisfied. Then problem* (3.3) *has a positive solution.*

Proof. Using (H1), we have that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(u) \leq \varepsilon u, \quad \forall \ 0 < u \leq \delta.$$

Now, suppose that $\|z\|_\infty = \delta$, then

$$\begin{aligned} \|F(z)\|_\infty &= \max_{t \in \mathbb{T}_0^+} \int_0^t \int_s^{+\infty} G_\alpha(\sigma(\tau)) f(z(\sigma(\tau))) \Delta\tau \Delta s \\ &\leq \left(\varepsilon \int_0^{+\infty} \int_s^{+\infty} G_\alpha(\sigma(\tau)) \Delta\tau \Delta s \right) \|z\|_\infty. \end{aligned}$$

Therefore, taking ε small enough, we have

$$\|F(z)\|_\infty < \|z\|_\infty \quad \text{for} \quad \|z\|_\infty = \delta. \tag{3.11}$$

Hypothesis (H2) implies that for all $M > 0$, there exists $s_M > 0$ such that

$$f(u) \geq Mu, \quad \forall u \geq s_M.$$

Now, suppose $z \in \mathcal{C}$ with $\|z\|_\infty = u$ and $u \geq s_M$. Therefore, we have

$$\begin{aligned} \|F(z)\|_\infty &\geq F(z)(\tau^*) \\ &= \int_0^{\tau^*} \left(\int_s^{+\infty} G_\alpha(\sigma(\tau)) f(z(\sigma(\tau))) \Delta\tau \right) \Delta s \\ &\geq \int_0^{\tau^*} \left(\int_{\tau^*}^{+\infty} G_\alpha(\sigma(\tau)) f(z(\sigma(\tau))) \Delta\tau \right) \Delta s \\ &\geq \int_0^{\tau^*} \left(\int_{\tau^*}^{+\infty} G_\alpha(\sigma(\tau)) \Delta\tau \right) \Delta s \ f\left(\frac{\|z\|_\infty}{2} \right). \end{aligned}$$

If $\dfrac{\|z\|_\infty}{2} = s_M$, then

$$\|F(z)\|_\infty \geq \frac{M}{2} \int_0^{\tau^*} \left(\int_{\tau^*}^{+\infty} G_\alpha(\sigma(\tau)) \Delta\tau \right) \|z\|_\infty.$$

Thus, taking M sufficiently large, we have

$$\|F(z)\|_\infty > \|z\|_\infty, \quad \text{for} \quad \|z\|_\infty = 2s_M. \tag{3.12}$$

Then, from (3.11), (3.12) and Lemma 3.1, we have the existence of a fixed point of the operator F and hence, a solution of problem (3.3). □

4. Numerical Simulations

In this section, our goal is to present some numerical simulations of the solutions of the problem given by

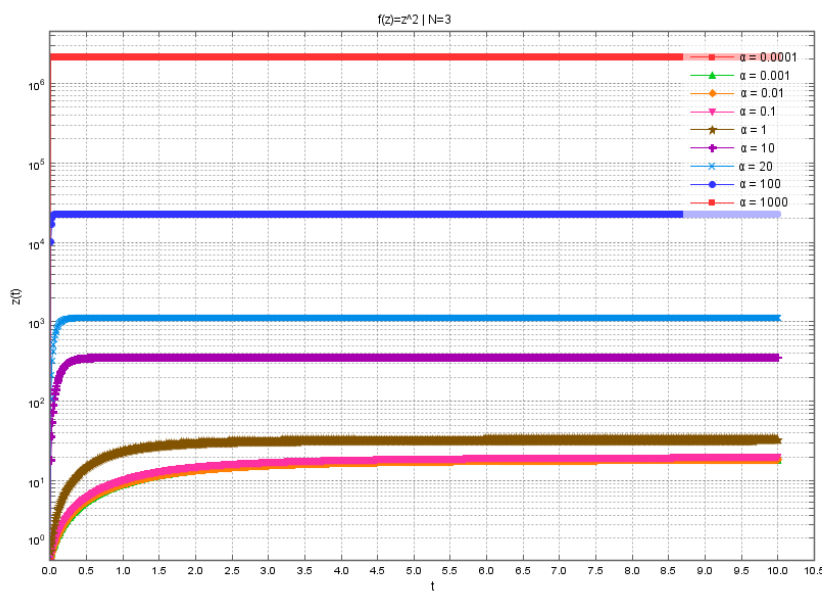
$$\begin{cases} -z^{\Delta\Delta}(t) = (1 + (N-2)\sigma(t))^{\frac{1}{2-N}(2(N-1)+\alpha)} f(z(\sigma(t))), & t \in \mathbb{T}_0^+ \\ z(0) = z^{\Delta}(+\infty) = 0. \end{cases} \tag{4.1}$$

To obtain our simulations of the solutions, we employed the software *Wolfram Mathematica*. This software is powerful and provides a lot of tools for solving differential equations. In our case, the selected function to solve this boundary value problem was *NDSolveValue*, using the *Shooting Method*. The shooting method is a technique for solving boundary value problems by transforming it in a solution of an initial value problem. Different paths in various directions are shoot until a trajectory that has the desired boundary value is found. For more details about the *Shooting Method*, we refer to [35, 37].

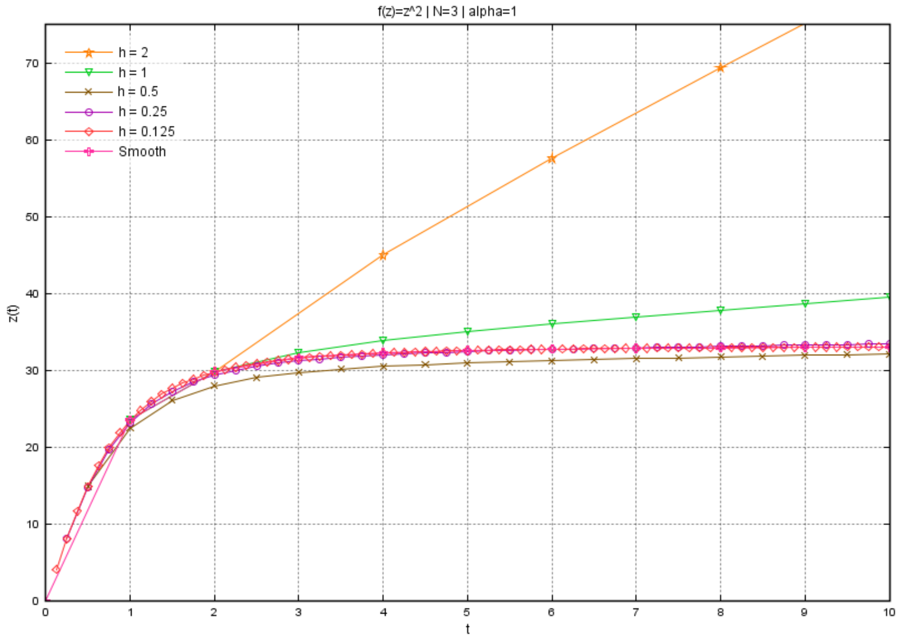
In this section, we will present two different graphs. The first one is devoted to study the behavior of the solution for the case $\mathbb{T}_0 = \mathbb{R}$, when the parameter α is very small. In the second one, we analyze the behavior of the solutions of the problem (4.1) when $\mathbb{T}_0 = h\mathbb{Z}$ and we plot the graphs of the solution of (4.1) for different values of h and for the case $\mathbb{T}_0 = \mathbb{R}$. By these graphs, we notice that the solution of h-difference equation (case $\mathbb{T}_0 = h\mathbb{Z}$) approaches to the solution of differential equation ($\mathbb{T}_0 = \mathbb{R}$) when h approaches to zero.

This fact is very interesting and occurs when we are dealing with first order dynamic equations on time scales, under appropriate conditions. However, the question "if it remains true for second order dynamic equations on time scales" is still open, but our simulations show that this fact seems to be true even when we are dealing with this type of equations. This fact makes the second order h-difference equations very important tool to study second order differential equations.

We start by investigating the behavior of the solutions of (4.1) when α approaches to 0. For it, we consider $\mathbb{T}_0 = \mathbb{R}$, $f(z) = z^2$ and $N = 3$ in the equation (4.1).



Finally, we present the behavior of the solution of (4.1) for the time scale $\mathbb{T}_0 = h\mathbb{Z}$. Therefore, we consider different values of h and by the graphs, it is possible to notice that when h approaches to 0, the solution of h-difference equation approaches to the solution of the differential equation (case $\mathbb{T}_0 = \mathbb{R}$). For this simulation, we consider $f(z) = z^2$, $N = 3$, $\alpha = 1$ in (4.1).



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