Averaging principle for functional differential equations with impulses at variable times via Kurzweil equations

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Abstract

We consider a large class of retarded functional differential equations subject to impulse effects at variable times and we present an averaging result for this class of equations by means of the techniques and tools of the theory of generalized ordinary differential equations introduced by J. Kurzweil.

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1 Introduction

The purpose of the averaging methods is to determine conditions under which the solutions of an autonomous differential system can approximate the solutions of a more complicated time varying system. The averaging method is therefore a powerful tool in studying the perturbation theory of differential equations, since it allows one to replace a time-varying small perturbation, acting on a long time interval, by a time-invariant perturbation and, in this process, only a small error is introduced.

Justifications of averaging methods for nonlinear systems were first presented in the works of N. N. Bogolyubov and A. Mitropolskii (see [23]) and of N. N. Krylov and N. N. Bogolyubov (see [6]). In these papers, the description of nonlinear systems was presented in the form we know nowadays

$$\begin{cases} \dot{x} = \varepsilon X\left(t, x\right) \\ x\left(0\right) = x_0, \end{cases} \tag{1.1}$$

where ε is a small parameter and x and X are n-dimensional vectors.

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For system (1.1), the "averaged system" was presented as

$$\begin{cases} \dot{y} = \varepsilon X_0(y) \\ y(0) = x_0, \end{cases} \tag{1.2}$$

where the right-hand side of equation (1.2) is obtained by taking the average or mean of the right-hand side of system (1.1), that is,

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)dt.$$

The first result presented and known as the averaging principle for nonlinear differential systems says that the solutions of (1.1) and (1.2) are close enough to one another, in asymptotically large time, provided system (1.1) admits a solution and the right-hand side of (1.1) is Lipschitzian on the second variable.

While the literature about averaging principles for ordinary differential equations (we write ODEs) is well developed, the theory involving the method of averaging for functional differential equations (we write FDEs) is improving. In the 60's, authors as V. I. Fodčuk [10], A. Halanay [11], J. K. Hale [12], G. N. Medvedev [22] and V. M. Volosov [27] developed methods of averaging for certain FDEs, with small parameter, approximating them by autonomous ODEs. For instance, let $r > 0$ and consider the delay differential equation

$$\dot{x} = \varepsilon f(t, x(t-r)),$$

with $\varepsilon > 0$ being a small parameter. Consider the change of variables $t \mapsto t/\varepsilon$ and $y(t) = x(t/\varepsilon)$. Then

$$x\left(\frac{t}{\varepsilon} - r\right) = x\left(\frac{t-\varepsilon r}{\varepsilon}\right) = y(t-\varepsilon r)$$

and hence

$$\dot{y}(t) = \frac{1}{\varepsilon}\,\dot{x}\left(\frac{t}{\varepsilon}\right) = f\left(\frac{t}{\varepsilon}, y(t-\varepsilon r)\right). \tag{1.3}$$

Taking $\varepsilon \to 0$ in equation (1.3), the delay becomes negligible and, therefore, the averaged equation is an autonomous ODE

$$\dot{y} = f_0(y),$$

with

$$f_0(y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s,y)ds.$$

In the 70's, the investigations about averaging methods for FDEs showed that the classic approximation by solutions of an autonomous ODE could be replaced by an approximation by solutions of an autonomous FDE, if one could treat separately the limiting process involving the delay and the averaging process. Then, it became clear that one advantage of treating these two processes separately was the permanence of an infinite-dimensional phase space. Also, the approximation of solutions was better and this fact could be verified by the order of the approximation and by computational simulations. See, for instance, the works of V. Strygin in [26] and of B. Lehman and S. P. Weibel in [21]. See also [18], [19], [20]. In these papers, the authors consider the FDE

$$\begin{cases} \dot{x} = \varepsilon f\left(t, x_t\right) \\ x_0 = \phi, \end{cases} \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter and $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$, with $r \geq 0$ and $t \geq 0$. The initial function ϕ belongs to the Banach space $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$ of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$, equipped with the usual supremum norm, and the function $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and Lipschitzian on the second variable. The averaged system is then given by

$$\begin{cases} \dot{y} = \varepsilon f_0\left(y_t\right) \\ y_0 = \phi, \end{cases}$$

where, for every $\varphi \in \mathcal{C}$, the following limit exists

$$f_0(\varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s, \varphi) ds.$$

In the late 80's, D. D. Bainov and S. D. Milusheva (see [4]) considered an FDE of neutral type given by

$$\begin{cases} \dot{x} = \varepsilon X\left(t, x(t), x(\Delta(t, x(t)), \dot{x}(\Delta(t, x(t))))\right), \quad t > 0, \ t \neq \tau_i(x), \\ x(t) = \phi(t, \varepsilon), \quad t \in [-r, 0], \\ \dot{x}(t) = \dot{\phi}(t, \varepsilon), \quad t \in [-r, 0] \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter, $r > 0$, $t - r \leq \Delta(t, x(t)) \leq t$, $t \geq 0$, $\phi(t, \varepsilon)$ is the initial function, the surfaces $\tau_i(x)$ are such that $\tau_i(x) < \tau_{i+1}(x)$, $i = 1, 2, \ldots$, and all $\tau_i(x)$ are in the half-space $t > 0$ for $x \in D \subset \mathbb{R}^n$ and $i = 1, 2, \ldots$. They also considered the impulses

$$x_i^+ = x_i^- + \varepsilon I_i(x_i^-), \ i = 1, 2, \ldots \tag{1.6}$$

which a solution of (1.5) undergoes when it encounters the surfaces τ_i, $i = 1, 2, \ldots$.

The averaged system for (1.5) was given by

$$\begin{cases} \dot{y} = \varepsilon X_0\left(y\right) + \varepsilon I_0\left(y\right) \\ y\left(0\right) = x_0. \end{cases} \tag{1.7}$$

where the limits

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}X(s,x,x,0)ds=X_0(x) \quad\text{and}\quad \lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}I_i(x)=I_0(x)$$

exist. (Note that (1.7) is an autonomous ODE and not an autonomous FDE.) The authors also assumed that $X(t,x,y,z)$ and $\Delta(t,x)$ are continuous functions, $\phi(t,\varepsilon)$ is continuously differentiable, the impulse operators $I_i(x), i=1,2,\ldots$, are continuous and the functions $\tau_i(x)$, $i=1,2,\ldots$, are twice continuously differentiable. Then they prove that, under certain conditions, for each $\mu>0$ and each $L>0$, there exists $\varepsilon\in(0,\varepsilon_0]$, $\varepsilon_0=\varepsilon_0(\eta,L)$, such that if $0<\varepsilon\le\varepsilon_0$, then $\|x(t)-y(t)\|<\eta$ for $t\in[0,\frac{L}{\varepsilon}]$, where x is the solution of (1.5)-(1.6) and y is the solution of (1.7).

In the present paper, we consider retarded functional differential equation with impulses at variable times (we write impulsive RFDEs) and we establish an averaging principle, where the averaged system is an autonomous FDE. We consider the initial value problem

$$\begin{cases} \dot{y}\left(t\right)=f\left(y_t,\frac{t}{\varepsilon}\right), & t\neq\tau_k(y(t)),\quad t\ge 0,\\ \Delta y\left(t\right)=\varepsilon I_k\left(y\left(\frac{t}{\varepsilon}\right)\right), & t=\tau_k(y(t)),\quad k=1,2,\ldots,\\ x_0=\phi, \end{cases} \tag{1.8}$$

where $\varepsilon>0$ is a small parameter and the initial function ϕ is a left continuous regulated function defined on $[-r,0]$, with $r>0$.

We assume that for each solution $y:[-r,+\infty)\to\mathbb{R}^n$ of (1.8), the mapping $t\mapsto f(t,y_t)$ is Lebesgue integrable and its indefinite integral satisfies Carathéodory- and Lipschitz-type conditions. Roughly speaking, these conditions on the indefinite integral of f allow the function f to behave "badly", e.g., f may have many discontinuities, and yet we can obtain good results, provided its indefinite integral is "well-behaved".

We consider the impulse operators $I_k(x)$, $k=0,1,2,\ldots$ as being continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and we assume that

$$\Delta y(t)=y(t+)-y(t-)=y(t+)-y(t)$$

that is, y is left continuous.

We denote by $m(\tau_k)$ the number of times at which the integral curves of system (1.8) meet the hypersurface τ_k, $k=1,2,\ldots$. By t_k^i we mean the i-th moment of time at which the integral curves of system (1.8) meet the hypersurface τ_k, with $i=1,\ldots,m(\tau_k)$, and $k=1,2,\ldots$. We assume $m(\tau_k)<\infty$, $k=1,2,\ldots$.

The averaged system for problem (1.8) is given by

$$\begin{cases} \dot{y} = \varepsilon f_0\left(y_t\right) + \varepsilon I_0(y) \\ y_0 = \phi, \end{cases} \tag{1.9}$$

where we assume that the following limits exist

$$f_0(\varphi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s,\varphi)ds \quad \text{and} \quad I_0(x) = \lim_{T\to\infty} \sum_{\substack{0\leq t_k^i<T, \\ i=1,\ldots,m(\tau_k)}} I_k(x).$$

Our averaging principle for the impulsive RFDE (1.8) says that, under the above conditions, given $\mu > 0$ and $L > 0$, $\|x(t) - y(t)\| < \mu$, for $t \in [0, \frac{L}{\varepsilon}]$, where x is a solution of (1.8) and y is a solution of (1.9).

In order to obtain this result, we adapted the theory of generalized ODEs, developed by Š. Schwabik for functions with values in $\mathbb{R}^n$ (see [25]), to the case where the functions take values in a general Banach space X. Because impulsive RFDEs can be regarded as generalized ODEs whose solutions are functions of locally bounded variation (see [9]), it is natural to consider X as the space of regulated functions (which includes functions of locally bounded variation). We also use an averaging result for non-impulsive RFDEs borrowed from [8] to get the main theorem.

This paper is organized as follows. In the second section, we describe the framework of impulsive RFDEs we are going to deal with. In the third section, we prove results on continuous dependence of solutions of this class of impulsive RFDEs on the initial data. The fourth section is dedicated to basic facts of the theory of generalized ODEs. In Section 5, the correspondence between impulsive RFDEs and generalized ODEs is presented. Continuous dependence of solutions of generalized ODEs on the initial value is investigated in the sixth section. In the seventh section, we generalize an averaging result for generalized ODEs by Š. Schwabik (see [25] and [24]) and present two theorems concerning about averaging for generalized ODEs. In the eighth section, we generalize an averaging result for impulsive ODEs (see [25]). The last section is devoted to averaging result for impulsive RFDEs at variable times via generalized ODEs.

2 The frame of impulsive RFDEs

Let X be a Banach space. A function $f : [a,b] \to X$ is called *regulated*, if the following limits exist

$$\lim_{s\to t-} f(s) = f(t-) \in X, \quad t \in (a,b], \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+) \in X, \quad t \in [a,b).$$

In this case, we write $f \in G([a,b],X)$ and we endow $G([a,b],X)$ with the usual supremum norm $\|f\|_\infty = \sup_{a\leq t\leq b} \|f(t)\|$. Then $(G([a,b],X), \|\cdot\|_\infty)$ is a Banach space. Also, any function in $G([a,b],X)$ is the uniform limit of step functions (see [14]).

Define

$$G^{-}([a,b],X)=\{u\in G([a,b],X):u \text{ is left continuous at every } t\in(a,b]\}.$$

In $G^{-}([a,b],X)$, we consider the norm induced by $G([a,b],X)$.

Given a function $y:[t_0-r,t_0+\sigma]\to\mathbb{R}^n$, with $t_0\in\mathbb{R}$, $r>0$ and $\sigma>0$, we consider $y_t:[-r,0]\to\mathbb{R}^n$ given by

$$y_t(\theta)=y(t+\theta), \qquad \theta\in[-r,0],\ t\in[t_0,t_0+\sigma].$$

Then it is clear that for a function $y\in G^{-}([t_0-r,t_0+\sigma],\mathbb{R}^n)$, we have $y_t\in G^{-}([-r,0],\mathbb{R}^n)$ for all $t\in[t_0,t_0+\sigma]$.

Consider the retarded functional differential equation with impulse action:

$$\begin{cases} \dot{y}(t)=f(y_t,t), & t\neq\tau_k(y(t)), \quad t\geq t_0,\\ \Delta y(t)=I_k(y(t)), & t=\tau_k(y(t)), \quad k=1,2,\ldots,\\ y_{t_0}=\phi, \end{cases} \tag{2.1}$$

where $\phi\in G^{-}([-r,0],\mathbb{R}^n)$, $y\mapsto I_k(y)$ maps $\mathbb{R}^n$ into itself, for each $k=1,2,\ldots$, τ_k maps $\mathbb{R}^n$ to $[t_0-r,t_0+\sigma]$, and $\Delta y(t):=y(t+)-y(t-)=y(t+)-y(t)$ for any $t\geq t_0$.

Let $\tau_0(x)\equiv t_0$ for all $x\in\mathbb{R}^n$, and for $k=1,2,\ldots$, consider the set

$$S_k=\{(t,x)\in[t_0-r,t_0+\sigma]\times\mathbb{R}^n : t=\tau_k(x)\}.$$

By $m(S_k)$ we denote the number of times at which the integral curves of system (2.1) meet the hypersurface S_k, $k=1,2,\ldots$. By t_k^i we mean the i^{th} moment of time at which the integral curves of system (2.1) meet the hypersurface S_k, with $i=1,\ldots,m(S_k)$, and $k=1,2,\ldots$. We also assume

(C1) $\tau_k\in C(\mathbb{R}^n,[t_0-r,t_0+\sigma])$, $k=1,2,\ldots$;

(C2) $t_0<\tau_1(x)<\tau_2(x)<\ldots$, for each $x\in\mathbb{R}^n$;

(C3) $\tau_k(x)\to+\infty$ as $k\to+\infty$ uniformly on $x\in\mathbb{R}^n$;

(C4) The integral curves of system (2.1) meet successively each one of the hypersurfaces, $S_1,S_2,\ldots$, only a finite number of times (i.e., $m(S_k)<\infty$, $k=1,2,\ldots$);

(C5) $t_k^i<t_k^{i+1}$, for $i=1,\ldots,m(\tau_k)$, and $k=1,2,\ldots$.

System (2.1) is known to be equivalent to the "integral" equation

$$\begin{cases} y\,(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f\,(y_s, s)\, ds + \sum_{\substack{0< t_k^i < t,\\ i=1,\dots,m(\tau_k)}} I_k(y(t_k^i)) \\ \\ y_{t_0} = \phi, \end{cases} \tag{2.2}$$

when the integral exists in some sense. We will consider Lebesgue integration in (2.2).

Let $PC_1 \subset G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be an open set, in the topology of uniform convergence in $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, with the following property: if y is an element of PC_1 and $\bar{t} \in [t_0 - r, t_0 + \sigma]$, then $\bar{y}$ given by

$$\bar{y}\,(t) = \begin{cases} y\,(t)\,, \ t_0 - r \le t \le \bar{t}, \\ y\,(\bar{t})\,, \ \bar{t} < t \le \infty, \end{cases}$$

is also an element of PC_1. In particular, any open ball in $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ has this property.

We assume that $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ and, for every $y \in PC_1$, the mapping $t \mapsto f\,(y_t, t)$, $t \in [t_0, t_0 + \sigma]$, is Lebesgue integrable, and moreover:

(A) There is a Lebesgue integrable function $M : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} f\,(x_s, s)\, ds \right| \le \int_{u_1}^{u_2} M\,(s)\, ds;$$

(B) There is a Lebesgue integrable function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$,

$$\left| \int_{u_1}^{u_2} [f\,(x_s, s) - f\,(y_s, s)]\, ds \right| \le \int_{u_1}^{u_2} L\,(s)\, \|x_s - y_s\|\, ds.$$

For the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$, we assume:

(A′) There is a constant $K_1 > 0$ such that for all $k = 0, 1, 2, \dots, m$ and all $x \in \mathbb{R}^n$,

$$|I_k(x)| \le K_1;$$

(B′) There is a constant $K_2 > 0$ such that for all $k = 0, 1, 2, \dots, m$ and all $x, y \in \mathbb{R}^n$,

$$|I_k(x) - I_k(y)| \le K_2 |x - y|.$$

Remark 2.1. Note that conditions (A) and (B) are Carathéodory- and Lipschitz-type conditions on the indefinite integral of f and not on "f" itself. Thus the standard assumption that $f(\psi, t)$ is continuous in ψ does not need to be fulfilled. Also, the mapping $t \mapsto f\,(y_t, t)$ does not need to be piecewise continuous, as usually required.

3 Existence and continuous dependence of solutions

In this section, we mention some results borrowed from [7] and [9] about existence and continuous dependence of solutions on the initial data.

Note that in [7] and [9], the authors consider pre-assigned moments of impulses, but here, we consider impulses at variable times. However, the proofs of the below results are very similar to the proofs of the similar results found in [7] and [9]. Therefore, we omit their proofs here.

Theorem 3.1 ([7], Theorem 2.1). *Consider problem (2.1) and suppose conditions* (A)*,* (B)*,* (A′) *and* (B′) *are fulfilled. Then there is a* $\Delta > 0$ *such that on the interval* $[t_0, t_0+\Delta]$ *there exists a unique solution* $y : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ *of problem (2.1) for which* $y_{t_0} = \phi$*.*

For the next theorem, we consider the following sequence of initial value problems

$$\begin{cases} \dot{y}\,(t) = f_p\,(y_t, t)\,,\ t \neq \tau_k(y(t)), \quad t \geq t_0 \\ \Delta y\,(t) = I_k^p\,(y\,(t))\,,\ t = \tau_k(y(t)) \quad k = 0, 1, \ldots, m, \\ y_{t_0} = \phi_p, \end{cases} \tag{3.1}$$

where $\Delta y\,(t) := y\,(t+) - y\,(t-) = y\,(t+) - y\,(t)$ and for each $p = 1, 2, \ldots$ and each $k = 0, 1, \ldots, m$, $x \mapsto I_k^p\,(x)$ maps $\mathbb{R}^n$ into itself. We also consider that conditions (C1) to (C5) are fulfilled.

The next theorem concerns continuous dependence of solutions of problem (3.1) on the initial data.

Theorem 3.2 ([9], Theorem 4.1). *Assume that for* $p = 0, 1, \ldots,$ $\phi_p \in G^-([-r, 0], \mathbb{R}^n)$ *and moreover* $f_p : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ *and* $I_k^p : \mathbb{R}^n \to \mathbb{R}^n$*,* $k = 0, 1, 2, \ldots, m$*, satisfy conditions* (A)*,* (B)*,* (A′) *and* (B′) *for the same functions* M, L *and the same constants* K_1, K_2*. Let the relations*

$$\lim_{p\to\infty} \sup_{\vartheta\in[t_0, t_0+\sigma]} \left| \int_{t_0}^{\vartheta} [f_p\,(y_s, s) - f_0\,(y_s, s)] ds \right| = 0 \tag{3.2}$$

for every $y \in PC_1$ *and*

$$\lim_{p\to\infty} I_k^p(x) = I_k^0(x) \tag{3.3}$$

for every $x \in \mathbb{R}^n$*,* $k = 0, 1, \ldots, m$ *be satisfied. Assume that* $y_p : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$*,* $p = 1, 2, \ldots,$ *is a solution on* $[t_0 - r, t_0 + \sigma]$ *of problem* (3.1) *such that*

$$\lim_{p\to\infty} y_p(s) = y(s) \quad \textit{uniformly on} \quad [t_0 - r, t_0 + \sigma].$$

Then $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *is a solution on* $[t_0 - r, t_0 + \sigma]$ *of the following problem*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), \ t \neq \tau_k(y(t)), \quad t \geq t_0 \\ \Delta y(t) = I_k^0(y(t)), \ t = \tau_k(y(t)) \quad k = 1, \ldots, m \\ y_{t_0} = \phi_0. \end{cases} \tag{3.4}$$

The next result says that, for sufficient large $p \in \mathbb{N}$, $y_p : [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ is a solution of (3.1), provided the sequence of initial data $\{\phi_p\}_{p\geq 1}$ converges uniformly on $[-r, 0]$.

Theorem 3.3 ([7], Theorem 3.3)**.** *Assume that* $f_p : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, $p = 0, 1, 2, \ldots$, *satisfies conditions* (A) *and* (B) *for the same functions* M *and* L*. Let* $I_k^p : \mathbb{R}^n \to \mathbb{R}^n$ $k = 1, 2, ..., p = 0, 1, 2, \ldots$, *be impulse operators which satisfy conditions* (A′) *and* (B′) *for the same constants* K_1 *and* K_2*. Assume that*

$$\lim_{p\to\infty} \int_{t_0}^{t} [f_p(y_s, s) - f_0(y_s, s)]ds = 0, \quad t \in [t_0, t_0 + \sigma] \tag{3.5}$$

for every $y \in O$*, and*

$$\lim_{p\to\infty} I_k^p(x) = I_k^0(x) \tag{3.6}$$

for every $x \in \mathbb{R}^n$ *and* $k = 1, \ldots, m$*. Let* $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *be a unique solution of*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^0(y(t_k)), \ k = 1, \ldots, m \\ y_{t_0} = \phi_0, \end{cases} \tag{3.7}$$

on $[t_0 - r, t_0 + \sigma]$*, where* $\phi_0 \in G^-([-r, 0], \mathbb{R}^n)$*. Suppose there exists* $\rho > 0$ *such that* $\sup\limits_{\theta\in[-r,0]} |u(\theta) - \phi_0(\theta)| < \rho$*, then* $u \in G^-([-r, 0], \mathbb{R}^n)$*. Assume further that* $\phi_p \to \phi_0$ *uniformly on* $[-r, 0]$ *as* $p \to \infty$*. Then, for sufficiently large* $p \in \mathbb{N}$*, there exists a solution* y_p *of*

$$\begin{cases} \dot{y}(t) = f_p(y_t, t), \ t \neq t_k \\ \Delta y(t_k) = I_k^p(y(t_k)), \ k = 1, \ldots, m \\ y_{t_0} = \phi_p, \end{cases} \tag{3.8}$$

on $[t_0 - r, t_0 + \sigma]$ *and*

$$\lim_{p\to\infty} y_p(s) = y(s), \quad \text{for } \ s \in [t_0 - r, t_0 + \sigma]. \tag{3.9}$$

4 Generalized ODEs

In this section, we present the basic notation and terminology of the theory of generalized ODEs and we list the fundamental results we need here.

A *tagged division* of a compact interval $[a, b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a, b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on a set $E \subset [a, b]$ is any function $\delta : E \to (0, +\infty)$. Given a gauge δ on $[a, b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ is δ*-fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a, b] \,;\, |t - \tau_i| < \delta(\tau_i)\}.$$

Let X be a Banach space. In the sequel, we use integration specified by the following definition due to J. Kurzweil [15].

Definition 4.1. A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is *Kurzweil integrable over the interval* $[a, b]$, if there is a unique element $I \in X$ ($I = \int_a^b DU(\tau, t)$) such that given $\varepsilon > 0$, there is a gauge δ of $[a, b]$ such that for every δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$, we have

$$\|S(U, d) - I\| < \varepsilon,$$

where $S(U, d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals, etc.

Let an open set $\Omega \subset X \times \mathbb{R}$ be given. Assume that $G : \Omega \to X$ is a given X-valued function $G(x, t)$ defined for $(x, t) \in \Omega$. In the following two definitions, the integrals have to be understood in the sense of Definition 4.1.

Definition 4.2. A function $x : [\alpha, \beta] \to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x, t) \tag{4.1}$$

on the interval $[\alpha, \beta] \subset \mathbb{R}$, if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_\gamma^v DG(x(\tau), t)$$

holds for every γ, $v \in [\alpha, \beta]$.

Given an initial condition $(z_0, t_0) \in \Omega$, we define the solution of the initial value problem for equation (4.1).

Definition 4.3. A function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (4.1) *with initial condition* $x(t_0) = \widetilde{x}$ *on the interval* $[\alpha, \beta] \subset \mathbb{R}$, if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - \widetilde{x} = \int_{t_0}^{v} DG(x(\tau), t)$$

holds for every $v \in [\alpha, \beta]$.

Let $(a, b) \subset \mathbb{R}$ be an interval with $-\infty < a < b < \infty$ and let $\Omega = O \times [a, b]$, where $O \subset X$ is an open set (e.g. $O = B_c = \{x \in X; \|x\| < c\}$ for some $c > 0$). We introduce a class of functions $G : \Omega \to X$ for which it is possible to get more specific information about the solutions of (4.1).

Definition 4.4. Assume that $h : [a, b] \to \mathbb{R}$ is a nondecreasing function defined on $[a, b]$. A function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, whenever

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{4.2}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\||h(s_2) - h(s_1)|$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

Assume that $G : \Omega \to X$ satisfies condition (4.2), $[\alpha, \beta] \subset [a, b]$ and $x : [\alpha, \beta] \to X$ is a solution of (4.1). Then the inequality

$$\|x(s_1) - x(s_2)\| \leq |h(s_2) - h(s_1)|$$

holds for every $s_1, s_2 \in [\alpha, \beta]$. This implies that every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$ of (4.1). Moreover, x is of bounded variation on $[\alpha, \beta]$ and

$$\operatorname{var}_{\alpha}^{\beta}(x) \leq h(\beta) - h(\alpha) < +\infty.$$

(See [25], Lemma 3.10). Also

$$x(\sigma+) - x(\sigma) = \lim_{s \to \sigma+} x(s) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma), \quad \sigma \in [\alpha, \beta),$$

and

$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s \to \sigma-} x(s) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-), \quad \sigma \in (\alpha, \beta],$$

where $G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s)$, $\sigma \in [\alpha, \beta)$, and $G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s)$, $\sigma \in (\alpha, \beta]$. (See [25], Lemma 3.12).

In the sequel, we mention some results whose proofs can be carried out by straightforward adaptation of the corresponding results from [25] to the cases where the functions taking values in a general Banach space X.

The first theorem we mention concerns the existence and uniqueness of a solution of (4.1).

Theorem 4.1 ([9], Theorem 2.15)**.** *Let $G : \Omega \to X$ belong to the class $\mathcal{F}(\Omega, h)$, where the function h is continuous from the left ($h(t-) = h(t)$ for $t \in (a, b]$). Then for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$, we have $(\widetilde{x}_+, t_0) \in \Omega$ and there exists a $\Delta > 0$ such that on the interval $[t_0, t_0 + \Delta]$ there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ordinary differential equation (4.1) for which $x(t_0) = \widetilde{x}$.*

The next theorem is a continuous dependence result for generalized ODEs.

Theorem 4.2 ([25], Theorem 8.8)**.** *Assume that for $k = 1, \ldots,$ $G_k : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h_k)$, where the functions $h_k : [a, b] \to \mathbb{R}$, $k = 1, 2 \ldots,$ are nondecreasing and left continuous and the function $h_0 : [a, b] \to \mathbb{R}$ is nondecreasing and continuous on $[a, b]$. Assume further that*

$$\limsup_{k\to\infty} [h_k(t_2) - h_k(t_1)] \leq h_0(t_2) - h_0(t_1)$$

for every $a \leq t_1 \leq t_2 \leq b$. Suppose

$$\lim_{k\to\infty} G_k(x, t) = G_0(x, t)$$

for $(x, t) \in \Omega$. Let $x : [\alpha, \beta] \to X$, $[\alpha, \beta] \subset [a, b]$, be a solution of the generalized differential equation

$$\frac{dx}{d\tau} = DG_0(x, t) \tag{4.3}$$

on $[\alpha, \beta]$ which has the following uniqueness property: If $y : [\alpha, \gamma] \to X$, $[\alpha, \gamma] \subset [\alpha, \beta]$ is a solution of (4.3) such that $y(\alpha) = x(\alpha)$, then $y(t) = x(t)$ for every $t \in [\alpha, \gamma]$. Assume further that there is a $\rho > 0$ such that if $s \in [\alpha, \beta]$ and $\|y - x(s)\| < \rho$, then $(y, s) \in \Omega$ and let $y_k \in X$, $k = 1, 2, \ldots,$ satisfy

$$\lim_{k\to\infty} y_k = x(\alpha).$$

Then for every $\mu > 0$, there exists a $k_* \in \mathbb{N}$ such that for $k \in \mathbb{N}$, $k > k_*$ there exists a solution x_k of the generalized differential equation

$$\frac{dx}{d\tau} = DG_k(x, t)$$

on $[\alpha, \beta]$ with $x_k(\alpha) = y_k$ and

$$\|x_k(s) - x(s)\| < \mu, \;\; s \in [\alpha, \beta].$$

Finally, we mention a substitution theorem for Kurzweil integrals.

Theorem 4.3 ([25], Theorem 2.18)**.** *Suppose $-\infty < c < d < +\infty$ and let $\varphi : [c,d] \to \mathbb{R}$ be a continuous function which is strictly monotone on $[c,d]$. Let $U : [\varphi(c), \varphi(d)] \times [\varphi(c), \varphi(d)] \to X$ be a given function. If one of the integrals*

$$\int_{\varphi(c)}^{\varphi(d)} DU(\tau, t), \quad \int_c^d DU(\varphi(\sigma), \varphi(s))$$

exists, then the other integral also exists and we have

$$\int_{\varphi(c)}^{\varphi(d)} DU(\tau, t) = \int_c^d DU(\varphi(\sigma), \varphi(s)).$$

5 Impulsive RFDEs regarded as generalized ODEs

Let t_0 and r be positive real numbers. Consider the framework of impulsive RFDEs as in Section 2, but instead of (C1), consider

(C1*) $\tau_k \in C(\mathbb{R}^n, [t_0, \infty))$, $k = 1, 2, \ldots$.

We also assume that (C2) – (C5) are satisfied.

Consider functions $y : [t_0 - r, \infty) \to \mathbb{R}^n$ which are left continuous, admit the right limits $y(t+)$ at every point and are such that $y(t+) \neq y(t)$ only for $t = t_l$, $l = 0, 1, 2, \ldots$, and $y|_{[t_0 - r, t_0]} \in G^-([t_0 - r, t_0], \mathbb{R}^n)$. It is clear that, for a function y having these properties, $y_t \in G^-([-r, 0], \mathbb{R}^n)$ for every $t \in [t_0, \infty)$. Furthermore, for $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, \infty) \to \mathbb{R}^n$, the mapping $t \mapsto f(y_t, t)$ is well defined for $t \in [t_0, \infty)$.

Let $PC_1 \subset G^-([t_0 - r, \infty), \mathbb{R}^n)$ be an open set (in the topology of locally uniform convergence in $G^-([t_0 - r, \infty), \mathbb{R}^n)$) with the following property: if y is an element of PC_1 and $\bar{t} \in [t_0, \infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leq t \leq \bar{t} \\ y(\bar{t}), & \bar{t} < t \leq \infty \end{cases}$$

is also an element of PC_1.

We assume that for $y \in PC_1$, the mapping $t \mapsto f(y_t, t)$, $t \in [t_0, \infty)$, is locally Lebesgue integrable and, moreover,

(A*) There is a locally Lebesgue integrable function $M : [t_0, \infty) \to \mathbb{R}$ such that for all $x \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(B*) There is a locally Lebesgue integrable function $L : [t_0, \infty) \to \mathbb{R}$ such that for all $x, y \in PC_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s) \|x_s - y_s\|\, ds.$$

For the impulse operators $I_l : \mathbb{R}^n \to \mathbb{R}^n$, $l = 1, 2, \ldots$, we assume the following conditions

(A′*) There is a constant $K_1 > 0$ such that for all $l = 0, 1, 2, \ldots$ and all $x \in \mathbb{R}^n$,

$$|I_l(x)| \leq K_1;$$

(B′*) There is a constant $K_2 > 0$ such that for all $l = 0, 1, 2, \ldots$ and all $x, y \in \mathbb{R}^n$,

$$|I_l(x) - I_l(y)| \leq K_2 |x - y|.$$

Definition 5.1. Consider system (2.1), where $f(\varphi, t) : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, and $t \mapsto f(y_t, t)$ is locally Lebesgue integrable for every $y \in PC_1$, where $t \in [t_0, +\infty)$. If there is a function $y \in PC_1$ satisfying

(i) $\dot{y}(t) = f(y_t, t)$, for almost every $t \in [t_0, +\infty) \setminus \{t : t = \tau_k(y(t)),\ k = 1, 2, \ldots\}$;

(ii) $y(t+) = y(t) + I_k(y(t))$, $t = \tau_k(y(t))$, $\quad k = 1, 2, \ldots$;

(iii) $y_{t_0} = \phi$,

then y is called a *solution* of (2.1) in $[t_0, +\infty)$.

Given $y \in PC_1$ and $t \in [t_0, +\infty)$, we define

$$F(y, t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)\, ds, \quad t_0 \leq \vartheta \leq t < +\infty, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\, ds, \quad t_0 \leq t \leq \vartheta < +\infty, \end{cases} \tag{5.1}$$

and

$$J(y,t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t) H_k^i(\vartheta) I_k(y(t_k^i)),\ \vartheta \in [t_0, \infty). \end{cases} \tag{5.2}$$

for $\vartheta \in [t_0 - r, +\infty)$, where H_k^i is the left continuous Heavyside function concentrated at t_k^i, that is

$$H_k^i(t) = \begin{cases} 0, \quad \text{for} \quad t_0 \leq t \leq t_k^i \\ 1, \quad \text{for} \quad t > t_k^i. \end{cases}$$

Taking $F(y,t)$ and $J(y,t)$ given by (5.1) and (5.2), we define

$$G(y,t) = F(y,t) + J(y,t) \tag{5.3}$$

for $y \in PC_1$ and $t \in [t_0, +\infty)$. Then clearly the values of the function $G(y,t)$ belong to $G^{-}([t_0 - r, +\infty), \mathbb{R})$, that is,

$$G : PC_1 \times [t_0, +\infty) \to G^{-}([t_0 - r, +\infty), \mathbb{R}^n).$$

Moreover, for $s_1, s_2 \in [t_0 - r, +\infty)$ and $x, y \in PC_1$ we have

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{5.4}$$

and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) - G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)|, \tag{5.5}$$

where

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)] ds + \max\{K_1, K_2\} \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t), \quad t \in [t_0, +\infty).$$

is a nondecreasing real function which is continuous from the left at every point, continuous for all $t \neq t_k^i$ and $h(t_k^i+)$ exists for $k = 1, 2, \ldots$, and $i = 1, 2, \ldots$. Thus, the function G defined by (5.3) belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = PC_1 \times [t_0, +\infty)$. For details, see [1].

Now, consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG\left(x, t\right). \tag{5.6}$$

where G is given by (5.3).

The next result gives a one-to-one relation between the solution of the impulsive RFDE (2.1) and the generalized ODE (5.6). For more details, see [1].

Theorem 5.1 (Correspondence of equations)**.**

(i) *Consider system* (2.1)*, where* $f : H_1 \times [t_0 - r, +\infty) \to \mathbb{R}^n$*, for each* $t \in [t_0, +\infty)$*,* $t \mapsto f(y_t, t)$ *is locally Lebesgue integrable over* $[t_0 - r, +\infty)$ *and* (A*), (B*), (A′*), (B′*) *are fulfilled. Assume* (C1*) *and (*C2) *to* (C5) *hold. Let* $y(t)$ *be the solution of problem* (2.1) *in the interval* $[t_0 - r, +\infty)$*. Given* $t \in [t_0, +\infty)$*, let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t] \\ y(t), & \vartheta \in [t, +\infty). \end{cases}$$

Then $x(t) \in G^{-}([t_0 - r, +\infty), \mathbb{R}^n)$ *and* x *is a solution of* (5.6) *in* $[t_0, +\infty)$*, with* G *given by* (5.3)*.*

(ii) *Reciprocally, let* $x(t)$ *be a solution of* (5.6)*, with* G *given by* (5.3)*, in the interval* $[t_0, +\infty)$ *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \leq \vartheta \leq t_0, \\ x(t_0)(t_0), \ t_0 \leq \vartheta \leq +\infty. \end{cases}$$

For every $\vartheta \in [t_0 - r, +\infty)$*, define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), \quad t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), \quad t_0 \leq \vartheta \leq +\infty. \end{cases} \tag{5.7}$$

Then y *is a solution of* (2.1) *in* $[t_0 - r, +\infty)$.

6 Continuous dependence of solutions of generalized ODEs

In this section, we prove a result about continuous dependence on the initial data of solutions of a class of generalized ODEs.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x,t),$$

where G is given by (5.3). Assume that $\phi \in G^-([-r,0],\mathbb{R}^n)$ is given and define a function $\widetilde{x} \in G^-([t_0 - r,\infty),\mathbb{R}^n)$ by

$$\widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ \text{if } \vartheta \in [t_0 - r, t_0], \\ \phi(0), \ \text{if } \vartheta \in [t_0,\infty) \end{cases}$$

For each $k = 0, 1, 2, \ldots$, let f_k satisfy the conditions (A*) and (B*) for the same functions M, L and I_j^k satisfy the conditions (A′*) and (B′*) for the same constants K_1, K_2, for $j = 1, 2, \ldots$. For each $(y,t) \in PC_1 \times [t_0,\infty)$, define

$$F_k(y,t)(\vartheta) = \begin{cases} 0, \ \text{if } t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f_k(y_s,s)ds, \ \text{if } t_0 \leq \vartheta \leq t < \infty \\ \displaystyle\int_{t_0}^{t} f_k(y_s,s)ds, \ \text{if } t_0 \leq t \leq \vartheta < \infty, \end{cases} \tag{6.1}$$

and

$$J_k(y,t)(\vartheta) = \begin{cases} 0, \ t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\sum_{j=1}^{+\infty} \sum_{i=1}^{m(\tau_j)} H_j^i(t) H_k^i(\vartheta) I_j^k(y(t_j^i)), \ \vartheta \in [t_0,\infty). \end{cases} \tag{6.2}$$

Then $F_k \in \mathcal{F}(\Omega, h_1)$ and $J_k \in \mathcal{F}(\Omega, h_2)$, for every $k = 0, 1, 2, \ldots$, where

$$h_1 = \int_{t_0}^{t} [M(s) + L(s)]ds \quad \text{and} \quad h_2 = \max\{K_1, K_2\} \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t).$$

If for each $k = 0, 1, 2, \ldots$ and each $(y, t) \in PC_1 \times [t_0, \infty)$, we define

$$G_k(y, t) = F_k(y, t) + J_k(y, t). \tag{6.3}$$

Then $G_k \in \mathcal{F}(\Omega, h)$, where $h = h_2 + h_1$.

In [25], Theorem 8.6, Š. Schwabik presented a result on continuous dependence of solutions on the initial data for a general class of generalized ODEs taking values in $\mathbb{R}^n$. Here, we consider a class of generalized ODEs taking values in the Frechét space, $G^-([t_0 - r, \infty), \mathbb{R}^n)$, of regulated functions from $[t_0 - r, \infty)$ to $\mathbb{R}^n$ which are left continuous and we prove a similar result. The correspondence of equations (2.1) and (5.6) (see Theorem 5.1) is essential in our proof.

Theorem 6.1. *Suppose for each $k = 0, 1, ...$, $G_k : PC_1 \times [t_0, \infty) \to G^-([t_0 - r, \infty), \mathbb{R}^n)$ is given as in* (6.3) *and the following limits exist*

$$\lim_{k\to\infty} F_k(y, t) = F_0(y, t) \quad \textit{and} \quad \lim_{k\to\infty} J_k(y, t) = J_0(y, t), \tag{6.4}$$

for $(y,\, t) \in PC_1 \times [t_0, \infty)$. Let $x : [t_0, \infty) \to PC_1$ be the unique solution of the generalized differential equation

$$\frac{dx}{d\tau} = DG_0(x, t) = D[F_0(x, t) + J_0(x, t)] \tag{6.5}$$

with initial condition $x\,(t_0) \in PC_1$ given by

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \leq \vartheta \leq t_0, \\ y(t_0), & t_0 \leq \vartheta < \infty. \end{cases} \tag{6.6}$$

Assume further that there is a sequence $\{\phi_k\}_{k\geq 1} \in G^-([-r, 0], \mathbb{R}^n)$ satisfying

$$\lim_{k\to\infty} \phi_k(\vartheta - t_0) = x(t_0)(\vartheta), \quad \textit{uniformly on } [t_0 - r, t_0].$$

Then there exists a positive integer m such that, for all $k > m$, there exists a solution $x_k : [t_0, \infty) \to PC_1$ of the generalized differential equation

$$\frac{dx}{d\tau} = DG_k(x, s) \tag{6.7}$$

on $[t_0,\infty)$, *such that*

$$x_k(t_0)(\vartheta)=\begin{cases}\phi_k(\vartheta-t_0), & t_0-r\le\vartheta\le t_0,\\ y_k(t_0), & t_0\le\vartheta<\infty\end{cases}$$

and $\lim\limits_{k\to+\infty} x_k(s)=x(s)$, $s\in[t_0,\infty)$.

Proof. By (6.4), it is clear that for $\vartheta\in[t_0-r,\infty)$, we have

$$\lim_{k\to\infty}F_k(y,t)(\vartheta)=F_0(y,t)(\vartheta)\quad\text{and}\quad\lim_{k\to\infty}J_k(y,t)(\vartheta)=J_0(y,t)(\vartheta).$$

Therefore, by (6.1) and (6.2), we also have

$$\lim_{k\to\infty}\int_{t_0}^{\vartheta}f_k(y_s,s)ds=\int_{t_0}^{\vartheta}f_0(y_s,s)ds,\quad \vartheta\in[t_0,\infty)$$

and

$$\lim_{k\to\infty}I_j^k(y(t_j^i))=I_j^0(y(t_j^i)),\quad j=0,1,2,\ldots,\quad i=1,2,\ldots m(\tau_k).$$

Let $x:[t_0,\infty)\to PC_1$ be the unique solution of the generalized differential equation (6.5), with initial condition (6.6) that is, $x(t_0)=\tilde{x}$, where $\tilde{x}$ is given by (6.6). Define $y:[t_0-r,\infty)\to\mathbb{R}^n$ as in (5.7). Then, by Proposition 5.1, y is a solution of

$$\begin{cases}\dot{y}\left(t\right)=f_0\left(y_t,t\right), & t\neq\tau_k(y(t)),\quad t\ge t_0,\\ \Delta y\left(t\right)=I_k^0\left(y\left(t\right)\right), & t=\tau_k(y(t)),\quad k=1,2,\ldots,\\ y_{t_0}=\phi,\end{cases}\tag{6.8}$$

on $[t_0-r,\infty)$.

Note that, since $\phi_k(\vartheta-t_0)\to x(t_0)(\vartheta)=\phi(\vartheta-t_0)$ uniformly on $[t_0-r,t_0]$, as $k\to\infty$, then $\phi_k\to\phi$ uniformly on $[-r,0]$. Thus Theorem 3.3 implies that, for sufficiently large $k\in\mathbb{N}$, say for $k>k_1$, there exists a solution y_k of equation

$$\begin{cases}\dot{y}\left(t\right)=f_k\left(y_t,t\right), & t\neq\tau_i(y(t))\\ \Delta y\left(t\right)=I_i^k\left(y\left(t\right)\right), & t=\tau_i(y(t))\quad i=1,2,\ldots\\ y_{t_0}=\phi_k,\end{cases}$$

on $[t_0-r,\infty)$ and $y_k(s)\to y(s)$, as $k\to\infty$, for each $s\in[t_0-r,\infty)$ (in particular, $y_k(\theta)\to y(\theta)$, uniformly on $[-r,0]$, as $k\to\infty$ by hypothesis).

Thus if for each $k=1,2,\ldots$, we define

$$x_k(t)(\vartheta)=\begin{cases}y_k(\vartheta), & t_0-r\le\vartheta\le t,\\ y_k(t), & t\le\vartheta<\infty,\end{cases}$$

where $t \in [t_0, \infty)$, then Proposition 5.1 implies that for $k > k_1$, $x_k(t) \in PC_1$ is a solution of (6.7), with initial condition

$$x_k(t_0)(\vartheta) = \begin{cases} \phi_k(\vartheta - t_0), & t_0 - r \leq \vartheta \leq t_0, \\ \phi_k(0), & t_0 \leq \vartheta < \infty. \end{cases}$$

Also, for every fixed $t \in [t_0, \infty)$, we have by definition

$$\|x_k(t) - x(t)\| = \sup_{\vartheta \in [t_0 - r, \infty)} |x_k(t)(\vartheta) - x(t)(\vartheta)| = \sup_{\vartheta \in [t_0 - r, t]} |y_k(\vartheta) - y(\vartheta)|$$

Then, since $\lim_{k \to \infty} y_k(s) = y(s)$, $s \in [t_0 - r, t]$, it follows that for every $\varepsilon > 0$, there exists $k_2 = k_2(t) \in \mathbb{N}$ such that for $k > k_2$,

$$\sup_{\vartheta \in [t_0 - r, t]} |y_k(\vartheta) - y(\vartheta)| < \varepsilon$$

Hence,

$$\|x_k(t) - x(t)\| \leq \varepsilon, \quad k > k_2.$$

This implies that $\lim_{k \to \infty} x_k(t) = x(t)$. Then, taking $m > \max\{k_1, k_2\}$, we obtain, by the Proposition 5.1, that for $k > m$, there exists a solution x_k of equation

$$\frac{dx}{d\tau} = DG_k(x, t)$$

and x is a solution of

$$\frac{dx}{d\tau} = DG_0(x, t)$$

and we have the desired result. □

7 Averaging principles for generalized ODEs

In this section, we present two theorems concerning about averaging method for generalized ODEs. We inspire our proof in Theorem 8.12 from [25].

In the sequel, we consider $t_0 = 0$. Thus $PC_1 \subset G^-([-r, \infty), \mathbb{R}^n)$, $\Omega = PC_1 \times [0, \infty)$ and $h : [0, \infty) \to \mathbb{R}$.

Let us consider the following generalized differential equation

$$\frac{dx}{d\tau} = DG\left(x, t\right), \tag{7.1}$$

where $G \in \mathcal{F}(\Omega, h)$ is given by (5.3).

Note that if a function $H_0 \in \mathcal{F}(\Omega, h)$ is such that $(x, t) \mapsto H_0(x, t) = G_0(x)t$, for $(x, t) \in \Omega =$

$PC_1 \times [0, \infty)$, then the generalized differential equation

$$\frac{dx}{d\tau} = DH_0\,(x, t) = D[G_0(x)t]$$

can be rewritten in the form

$$\dot{x} = G_0\,(x)\,,$$

which is an abstract autonomous ODE. Indeed, since for every sufficiently fine tagged division, $(\tau_i, [s_{i-1}, s_i])$, of a subinterval $[\alpha, \beta]$ of $[0, \infty)$, we have

$$\begin{aligned} \int_\alpha^\beta DH_0(x(\tau), t) &\approx \sum_i [H_0(x(\tau_i), s_i) - H_0(x(\tau_i), s_{i-1})] \\ &= \sum_i G_0(x(\tau_i))(s_i - s_{i-1}) \approx \int_\alpha^\beta G_0(x(t))dt. \end{aligned}$$

by the properties of the Kurzweil integral (Definition 4.1)

The next theorem is a generalization of Theorem 8.12 from [25].

Theorem 7.1. *Assume that* $\Omega = B \times [0, \infty)$, $B = \{x \in \mathbb{R}^n; \|x\| < c\}$, $c > 0$ *and that* $F \in \mathcal{F}(\Omega, h)$ *where* $h : [0, \infty) \to \mathbb{R}$ *is continuous from the left and nondecreasing. Assume that*

$$\limsup_{r \to \infty} \frac{h(r + \alpha) - h(\alpha)}{r} \leq C, \quad C = constant$$

for every $\alpha \geq 0$ *and*

$$\lim_{r \to \infty} \frac{F(x, r)}{r} = F_0(x), \qquad x \in B.$$

Let $y : [0, \infty) \to \mathbb{R}^n$ *be a uniquely determined solution of the autonomous ordinary differential equation*

$$\dot{y} = F_0(y) \tag{7.2}$$

which belongs to B *together with its* ρ*-neighborhood with* $\rho > 0$*, i.e., there is a* $\rho > 0$ *such that* $\{x \in \mathbb{R}^n; \|x - y(t)\| < \rho\} \subset B$ *for every* $t \in [0, \infty)$*. Then for every* $\mu > 0$ *and* $L > 0$ *there is an* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *the inequality*

$$\|x_\varepsilon(t) - y(t)\| < \mu$$

holds for $t \in [0, \frac{L}{\varepsilon}]$*, where* x_ε *is a solution of the generalized ODE*

$$\frac{dx}{d\tau} = D\left[\varepsilon F\left(x, \frac{t}{\varepsilon}\right)\right]$$

on $[0, \frac{L}{\varepsilon}]$ *such that* $x_\varepsilon(0) = y(0)$*.*

Proof. For $y \in B$, $t \in [0, \infty)$ and $\varepsilon > 0$, define

$$G_\varepsilon(y,t) = \varepsilon F\left(y, \frac{t}{\varepsilon}\right)$$

and take $h_\varepsilon(t) = \varepsilon h\left(\frac{t}{\varepsilon}\right)$ for $t \geq 0$. The function h_ε is evidently nondecreasing and continuous from the left on $[0, \infty)$.

Since $F \in \mathcal{F}(\Omega, h)$, we obtain by definition

$$\begin{aligned} \|G_\varepsilon(y,t_2) - G_\varepsilon(y,t_1)\| &= \varepsilon \left\| F\left(y, \frac{t_2}{\varepsilon}\right) - F\left(y, \frac{t_1}{\varepsilon}\right) \right\| \\ &\leq \varepsilon \left| h\left(\frac{t_2}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right) \right| = h_\varepsilon(t_2) - h_\varepsilon(t_1), \end{aligned}$$

and similarly also

$$\|G_\varepsilon(y,t_2) - G_\varepsilon(y,t_1) - G_\varepsilon(x,t_2) + G_\varepsilon(x,t_1)\| \leq \|y - x\|(h_\varepsilon(t_2) - h_\varepsilon(t_1))$$

whenever $x, y \in B$, $t_1, t_2 \in [0, \infty)$. These inequalities mean that we have $G_\varepsilon \in \mathcal{F}(\Omega, h_\varepsilon)$ for $\varepsilon > 0$. If $y \in B$, then by hypothesis

$$\lim_{r \to \infty} \frac{F(y,r) - F(y,0)}{r} = \lim_{r \to \infty} \frac{F(y,r)}{r} = F_0(y)$$

and therefore, for every $\eta > 0$ there is an $R > 0$ such that for $r > R$ we have

$$\begin{aligned} \|F_0(y)\| &\leq \left\| F_0(y) - \frac{F(y,r) - F(y,0)}{r} \right\| + \frac{\|F(y,r) - F(y,0)\|}{r} \\ &\leq \eta + \frac{h(r) - h(0)}{r} < 2\eta + C \end{aligned}$$

because $F \in \mathcal{F}(\Omega, h)$ implies $\|F(y,r) - F(y,0)\| \leq h(r) - h(0)$. Since $\eta > 0$ can be chosen arbitrarily small, we have

$$\|F_0(y)\| \leq C, \qquad y \in B.$$

Analogously, if $x, y \in B$ then for every $\eta > 0$ there is an $R > 0$ such that for $r > R$ we have

$$\begin{aligned} \|F_0(x) - F_0(y)\| &< \eta + \frac{\|F(y,r) - F(y,0) - F(x,r) + F(x,0)\|}{r} \\ &\leq \eta + \|y - x\| \frac{h(r) - h(0)}{r} \leq \eta(1 + \|y - x\|) + C\|y - x\|, \end{aligned}$$

and again since $\eta > 0$ can be arbitrarily small, we obtain

$$\|F_0(x) - F_0(y)\| \leq C\|y - x\|$$

provided $x, y \in B$.

For $y \in B$, $t > 0$ we obtain

$$\lim_{\varepsilon \to 0^+} G_\varepsilon(y,t) = \lim_{\varepsilon \to 0^+} \varepsilon F\left(y, \frac{t}{\varepsilon}\right) = \lim_{\varepsilon \to 0^+} t\frac{\varepsilon}{t} F\left(y, \frac{t}{\varepsilon}\right) = tF_0(y)$$

and also

$$\lim_{\varepsilon \to 0^+} G_\varepsilon(y,0) = \lim_{\varepsilon \to 0^+} \varepsilon F(y,0) = 0.$$

Denote $G_0(y,t) = tF_0(y)$ for $y \in B$, $t \geq 0$. Then the relations given above imply

$$\lim_{\varepsilon \to 0^+} G_\varepsilon(y,t) = G_0(y,t).$$

By the equations above, we have $G_0 \in \mathcal{F}(\Omega, h_0)$ where $h_0(t) = Ct$, $t \geq 0$. Further, for $0 \leq t_1 < t_2 < \infty$, we obtain by the definition

$$h_\varepsilon(t_2) - h_\varepsilon(t_1) = \varepsilon\left(h\left(\frac{t_2}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right) = (t_2 - t_1)\frac{\varepsilon}{(t_2 - t_1)}\left(h\left(\frac{t_2 - t_1}{\varepsilon} + \frac{t_1}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right)$$

and by hypothesis, we have

$$\limsup_{\varepsilon \to 0^+}(h_\varepsilon(t_2) - h_\varepsilon(t_1)) \leq C(t_2 - t_1) = h_0(t_2) - h_0(t_1)$$

because we have

$$\lim_{\varepsilon \to 0^+} \frac{t_2 - t_1}{\varepsilon} = +\infty$$

It is easy to see that the equation above is also satisfied in the case $t_1 = t_2$.

Using the fact that $y : [0, \infty) \to B$ is a solution of (7.2), we obtain by the properties of the generalized Perron integral the equality

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} F_0(y(\tau))d\tau = \int_{s_1}^{s_2} D[F_0(y(\tau))t] = \int_{s_1}^{s_2} DG_0(y(\tau), t)$$

for $s_1, s_2 \in [0, \infty)$, i.e., y is a solution of the generalized ODE

$$\frac{dy}{d\tau} = DG_0(y,t)$$

on $[0, \infty)$ and by the assumption this solution is uniquely determined. In this way, we have shown that all assumptions of Theorem 4.2 are satisfied for the case of the continuous parameter $\varepsilon \to 0^+$. Using the result of Theorem 4.2, we obtain that for every $\mu > 0$ and $L > 0$ there exists a value

$\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, there is a solution x_ε of the generalized ODE

$$\frac{dy}{d\tau} = DG_\varepsilon(y, t)$$

on the interval $[0, \frac{L}{\varepsilon}]$, such that $x_\varepsilon(0) = y(0)$ and

$$\|x_\varepsilon(s) - y(s)\| \leq \mu,$$

for all $s \in [0, \frac{L}{\varepsilon}]$. $\square$

Theorem 7.2. *Let $\Omega = PC_1 \times [0, \infty)$ and suppose $G : \Omega \to G^-([-r, \infty), \mathbb{R}^n)$ is given by (5.3). Suppose*

$$\limsup_{\varepsilon \to 0^+} \frac{h\left(\frac{t}{\varepsilon} + \alpha\right) - h(\alpha)}{\frac{t}{\varepsilon}} \leq C, \qquad \text{where } \ C = \text{constant}, \tag{7.3}$$

for every $\alpha \geq 0$, and

$$\lim_{\varepsilon \to 0^+} \frac{G\left(x, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)}{\frac{t}{\varepsilon}} = G_0(x)(\vartheta), \tag{7.4}$$

for every $x \in PC_1$. Suppose, also, that for every $\vartheta \geq 0$ and $y \in PC_1$,

$$G(y, 0)(\vartheta) = 0.$$

Let $y : [-r, \infty) \to PC_1$ be the unique solution of the autonomous ordinary differential equation

$$\dot{y} = G_0(y), \tag{7.5}$$

and assume there exists $\rho > 0$ such that $\{x \in PC_1; \|z - y(t)\| < \rho\} \subset PC_1$, for every $t \in [0, \infty)$. Then, for every $\mu > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the inequality

$$\|x_\varepsilon(t) - y(t)\| < \mu$$

holds for $t \in \left[0, \frac{L}{\varepsilon}\right]$, where x_ε is a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = D\left[\varepsilon G\left(x, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)\right] \tag{7.6}$$

on $\left[0, \frac{L}{\varepsilon}\right]$ such that $x_\varepsilon(0) = y(0)$.

Proof. For $y \in PC_1$, $t \in [0, \infty)$ and $\varepsilon > 0$, we define

$$H_\varepsilon(y,t)(\vartheta) = \begin{cases} 0, \quad \text{if } \vartheta \in [-r, 0], \\ \varepsilon G\left(y, \dfrac{t}{\varepsilon}\right)\left(\dfrac{\vartheta}{\varepsilon}\right), \quad \text{if } \vartheta \in [0, \infty), \end{cases}$$

and

$$h_\varepsilon(t) = \varepsilon h\left(\frac{t}{\varepsilon}\right).$$

The function h_ε is evidently nondecreasing and continuous from the left on $[0, \infty)$. Since $G \in \mathcal{F}(\Omega, h)$, we have

$$\begin{aligned} |H_\varepsilon(y,t_2)(\vartheta) - H_\varepsilon(y,t_1)(\vartheta)| &= \left|\varepsilon G\left(y, \frac{t_2}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - \varepsilon G\left(y, \frac{t_1}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)\right| \\ &\leq \varepsilon\left|h\left(\frac{t_2}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right| = |h_\varepsilon(t_2) - h_\varepsilon(t_1)| \end{aligned}$$

and, similarly,

$$|H_\varepsilon(x,t_2)(\vartheta) - H_\varepsilon(x,t_1)(\vartheta) - H_\varepsilon(y,t_2)(\vartheta) + H_\varepsilon(y,t_1)(\vartheta)| =$$

$$= \left|\varepsilon G\left(x, \frac{t_2}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - \varepsilon G\left(x, \frac{t_1}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - \varepsilon G\left(y, \frac{t_2}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) + \varepsilon G\left(y, \frac{t_1}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)\right| \leq$$

$$\leq \|x - y\|\varepsilon\left|h\left(\frac{t_2}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right| = \|x - y\||h_\varepsilon(t_2) - h_\varepsilon(t_1)|,$$

for every $x, y \in PC_1$, $t_1, t_2 \in [0, \infty)$ and $\vartheta \in [0, \infty)$. Therefore,

$$\|H_\varepsilon(y,t_2) - H_\varepsilon(y,t_1)\| \leq |h_\varepsilon(t_2) - h_\varepsilon(t_1)|$$

and

$$\|H_\varepsilon(x,t_2) - H_\varepsilon(x,t_1) - H_\varepsilon(y,t_2) + H_\varepsilon(y,t_1)\| \leq \|x - y\||h_\varepsilon(t_2) - h_\varepsilon(t_1)|$$

and hence $H_\varepsilon \in \mathcal{F}(\Omega, h_\varepsilon)$ for $\varepsilon > 0$.

Consider $y \in PC_1$ and $t \in [0, \infty)$. Then, for $\vartheta \in [0, \infty)$, we have

$$\lim_{\varepsilon \to 0^+} \frac{G\left(y, \dfrac{t}{\varepsilon}\right)\left(\dfrac{\vartheta}{\varepsilon}\right) - G(y,0)\left(\dfrac{\vartheta}{\varepsilon}\right)}{\dfrac{t}{\varepsilon}} = \lim_{\varepsilon \to 0^+} \frac{G\left(y, \dfrac{t}{\varepsilon}\right)\left(\dfrac{\vartheta}{\varepsilon}\right)}{\dfrac{t}{\varepsilon}} = G_0(y)(\vartheta).$$

Hence (7.3) and (7.4) imply that, for every $\eta > 0$, there exists $\varepsilon > 0$ sufficiently small such that for

$\vartheta \in [0,\infty)$,

$$\begin{aligned}|G_0(y)(\vartheta)| &\leq \left|G_0(y)(\vartheta) - \frac{\varepsilon}{t}\left[G\left(y,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - G(y,0)\left(\frac{\vartheta}{\varepsilon}\right)\right]\right| \\ &+ \frac{\varepsilon}{t}\left|G\left(y,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - G(y,0)\left(\frac{\vartheta}{\varepsilon}\right)\right| \leq \eta + \frac{\varepsilon}{t}\left[h\left(\frac{t}{\varepsilon}\right) - h(0)\right] < 2\eta + C,\end{aligned}$$

because $G \in \mathcal{F}(\Omega,h)$ implies that $\|G(y,\frac{t}{\varepsilon}) - G(y,0)\| \leq h(\frac{t}{\varepsilon}) - h(0)$. Then, since $\eta > 0$ can be chosen arbitrarily small, we obtain

$$\|G_0(y)\| \leq C, \quad y \in PC_1.$$

Analogously, if $x, y \in PC_1$ and $t \in [0,\infty)$, then for every $\eta > 0$, there exists $\varepsilon > 0$ sufficiently small such that, for $\vartheta \in [0,\infty)$, we have

$$|G_0(x)(\vartheta) - G_0(y)(\vartheta)| <$$

$$< \eta + \frac{t}{\varepsilon}\left|G\left(y,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) - G(y,0)\left(\frac{\vartheta}{\varepsilon}\right) - G\left(x,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) + G(x,0)\left(\frac{\vartheta}{\varepsilon}\right)\right| \leq$$

$$\leq \eta + \|x-y\|\frac{t}{\varepsilon}\left[h\left(\frac{t}{\varepsilon}\right) - h(0)\right] \leq \eta + (\eta + C)\|y-x\| \leq \eta(1 + \|y-x\|) + C\|y-x\|,$$

and, again, since $\eta > 0$ can be chosen sufficiently small, we obtain

$$\|G_0(x) - G_0(y)\| \leq C\|y-x\|, \qquad x, y \in PC_1. \tag{7.7}$$

On the other hand, for $y \in PC_1$, $t \in (0,\infty)$ and $\vartheta \in [0,\infty)$, we have

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(y,t)(\vartheta) = \lim_{\varepsilon\to 0^+} \varepsilon G\left(y,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) = \lim_{\varepsilon\to 0^+} t\frac{\varepsilon}{t}G\left(y,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) = tG_0(y)(\vartheta)$$

and, for $t = 0$ and $\vartheta \in [0,\infty)$, we have

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(y,0)(\vartheta) = \lim_{\varepsilon\to 0^+} \varepsilon G(y,0)\left(\frac{\vartheta}{\varepsilon}\right) = 0.$$

Thus, defining $H_0(y,t) = tG_0(y)$, for $y \in PC_1$ and $t \geq 0$, then we obtain

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(y,t) = H_0(y,t).$$

Also, by (7.7), $H_0 \in \mathcal{F}(\Omega,h_0)$, where $h_0(t) = Ct$, $t \geq 0$. Furthermore, for $0 \leq t_1 < t_2 < +\infty$, by the definition of h_ε, we obtain

$$\begin{aligned} h_\varepsilon(t_2) - h_\varepsilon(t_1) &= \varepsilon\left[h\left(\frac{t_2}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right] \\ &= (t_2 - t_1)\frac{\varepsilon}{t_2 - t_1}\left[h\left(\frac{t_2 - t_1}{\varepsilon} + \frac{t_1}{\varepsilon}\right) - h\left(\frac{t_1}{\varepsilon}\right)\right] \end{aligned}$$

and by (7.3),

$$\limsup_{\varepsilon\to 0^+}[h_\varepsilon(t_2) - h_\varepsilon(t_1)] \le C(t_2 - t_1) = h_0(t_2) - h_0(t_1), \tag{7.8}$$

since

$$\lim_{\varepsilon\to 0^+}\frac{t_2 - t_1}{\varepsilon} = +\infty.$$

Note that (7.8) is also satisfied when $t_1 = t_2$.

Using the fact that $y \in PC_1$ is a solution of (7.5) and using the properties of the Kurzweil integral, we have

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} G_0(y(\tau))d\tau = \int_{s_1}^{s_2} D[G_0(y(\tau))t] = \int_{s_1}^{s_2} DH_0(y(\tau), t)$$

for every $s_1, s_2 \in [0, +\infty)$. Therefore y is a solution of the generalized ordinary differential equation

$$\frac{dy}{d\tau} = DH_0(y, t)$$

on $[0, +\infty)$ and, by hypothesis and Theorem 4.1, this solution is uniquely determined.

In this way, we showed that all hypotheses of Theorem 4.2 are satisfied. Thus, by Theorem 4.2, for every $\mu > 0$ and every $L > 0$, there is a $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and there exists a solution x_ε of the ordinary generalized differential equation

$$\frac{dx}{d\tau} = DH_\varepsilon(x, t) \tag{7.9}$$

on the interval $[0, L]$ satisfying $x_\varepsilon(0) = y(0)$ and

$$\|x_\varepsilon(s) - y(s)\| \le \mu \tag{7.10}$$

for every $s \in [0, \frac{L}{\varepsilon}] \subset [0, \infty)$, where y is solution of (7.5). □

8 An averaging principle for impulsive ODEs

In this section, using the theorem from the previous section, we prove an averaging method for impulsive ODEs.

Theorem 8.1. *Let $\Omega = B \times [0,\infty)$, where $B = \{x \in \mathbb{R}^n : |x| < c\}$, $c > 0$. Suppose $f : \Omega \to \mathbb{R}^n$ satisfies conditions* (A*) *and* (B*). *Suppose the conditions* (C1*) *and* (C2) *to* (C5) *are fulfilled and assume that*

$$\limsup_{r\to\infty} \sum_{k=1}^{+\infty} \sum_{\substack{\alpha \le t_k^i \le \alpha + r,\\ i=1,\dots,m(\tau_k)}} 1 \le d$$

for every $\alpha \ge 0$. Assume further that $I_i : B \to \mathbb{R}^n$, $i = 0,1,2,\dots$, is a sequence of impulse operators which satisfy conditions $(A'*)$ and $(B'*)$. Suppose

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(y,s)ds = f_0(y), \quad y \in B,$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0 \le t_i < T} I_i(x) = I_0(x), \quad x \in B \qquad \textit{and}$$

$$\limsup_{T\to\infty} \frac{1}{T} \int_\alpha^{T+\alpha} [M(s) + L(s)]ds \le c, \quad c = \textit{constant},$$

for every $t \in [0,+\infty)$ and $\alpha \ge 0$. Let $y \in B$ be the uniquely determined solution of the autonomous differential equation

$$\dot{y} = f_0(y) + I_0(y) \tag{8.1}$$

Then, for every $\mu > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the inequality

$$|y^\varepsilon(t) - y(t)| < \mu$$

holds on $\left[0, \frac{L}{\varepsilon}\right]$, where y^ε is a solution of the impulsive differential equation

$$\begin{cases} \dot{y} = f\left(y, \frac{t}{\varepsilon}\right), \ t \ne \tau_i(y(t)) \\ \Delta y(t) = y(t+) - y(t) = \varepsilon I_i(y(t)), \ t = \tau_i(y(t)) \quad i = 1,2,\dots \end{cases} \tag{8.2}$$

on $[0, \frac{L}{\varepsilon}]$ such that $y^\varepsilon(0) = y(0)$.

Proof. Let

$$F(y,t) = \int_0^t f(y,s)ds + \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t) I_k(y(t_k^i))$$

Note that the generalized ODE below

$$\frac{dx}{d\tau} = D\left[\varepsilon F\left(x, \frac{t}{\varepsilon}\right)\right] \tag{8.3}$$

is equivalent to the ordinary system (8.2). In fact, we have

$$\varepsilon F\left(y, \frac{t}{\varepsilon}\right) = \varepsilon \int_0^{t/\varepsilon} f(y,s)ds + \varepsilon \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i\left(\frac{t}{\varepsilon}\right) I_k(y(t_k^i))$$

Using change of variables, we have

$$\varepsilon F\left(y, \frac{t}{\varepsilon}\right) = \int_0^{t} f\left(y, \frac{u}{\varepsilon}\right) du + \varepsilon \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i\left(\frac{t}{\varepsilon}\right) I_k(y(t_k^i)).$$

In this way, we have the equivalence mentioned before. Here $H_k^i\left(\frac{t}{\varepsilon}\right) = 0$ for $\frac{t}{\varepsilon} \in [0, t_k^i]$ and $H_k^i\left(\frac{t}{\varepsilon}\right) = 1$ for $\frac{t}{\varepsilon} > t_k^i$.

By hypotheses, it is easy to verify that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h)$, where

$$h(t) = \int_0^t [M(s) + L(s)]ds + \max(K_1, K_2) \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(t), \quad t \in [0, \infty).$$

Furthermore, the hypotheses also imply that

$$\begin{aligned} \lim_{T\to\infty} \frac{F(y,T)}{T} &= \lim_{T\to\infty} \left[\frac{1}{T} \int_0^T f(y,s)ds + \frac{1}{T} \sum_{k=1}^{+\infty} \sum_{i=1}^{m(\tau_k)} H_k^i(T) I_k(y(t_k^i)) \right] \\ &= f_0(y) + I_0(y) = F_0(y) \end{aligned}$$

and

$$\limsup_{T\to\infty} \frac{h(T+\alpha) - h(\alpha)}{T} \leq c + \max(K_1, K_2)d,$$

for $x \in B$ and $\alpha \geq 0$.

Thus, the hypotheses of Theorem 7.1 are satisfied and the result follows immediately from the correspondence between the impulsive system (8.2) and the generalized system (8.3) and from the correspondence between the averaged ordinary system (8.1) and $\dot{x} = G_0(x)$. $\square$

9 An averaging principle for RFDEs with impulses at variable times

In the next lines, we present an averaging result for RFDEs without impulses borrowed from [8]. Such result will be used to get our main theorem.

Let $\varepsilon > 0$ be a small parameter and consider the non impulsive initial value problem

$$\begin{cases} \dot{y} = \varepsilon f(y_t, t) \\ y_0 = \phi, \end{cases} \tag{9.1}$$

where $\phi \in G^{-}([-r,0],\mathbb{R}^n)$ and $f : G^{-}([-r,0],\mathbb{R}^n) \times [0,\infty) \to \mathbb{R}^n$ satisfies condition (A^*) and the following condition

(K) There is a constant $C > 0$ such that for $x, y \in PC_1$ and $u_1, u_2 \in [0,+\infty)$,

$$\left| \int_{u_1}^{u_2} [f(y_s,s) - f(x_s,s)]\, ds \right| \leq C \int_{u_1}^{u_2} \|y_s - x_s\|\, ds.$$

Clearly condition (K) implies condition (B^*).

We assume that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\psi,s)\, ds = f_0(\psi) \tag{9.2}$$

exists for every $\psi \in G^{-}([-r,\infty),\mathbb{R}^n)$ and that the averaged equation for problem (9.1) is given by

$$\begin{cases} \dot{y} = f_0(y_t) \\ y_0 = \phi. \end{cases} \tag{9.3}$$

The next lemmas imply that, under the above considerations, a solution of (9.1) and a solution of (9.3) can be made close enough in an interval $[0,\frac{L}{\varepsilon}]$, where $L > 0$ is arbitrary and $\varepsilon \in (0,\varepsilon_0)$ for some $\varepsilon_0 > 0$ (see [8], Corollaries 3.2 and 3.3).

Lemma 9.1 ([8], Corollary 3.2)**.** *Consider the RFDEs* (9.1) *and* (9.3)*, where f satisfies the condition (K). Then for every $\rho > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$, we have*

$$\left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s,s)\, ds - \int_0^t f_0(\overline{y}_s) ds \right\| < \rho, \quad t \in [0,L],$$

where y is a solution of (9.1) *on $[0,\frac{L}{\varepsilon}]$ and $\overline{y}$ is a solution of* (9.3) *on $[0,L]$.*

Lemma 9.2 ([8], Corollary 3.3)**.** *Consider the RFDEs* (9.1) *and* (9.3)*, where f satisfies the condition (K). Then for every $\rho > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$, we have*

$$\|y - \overline{y}\|_\infty < \rho,$$

on $[0,\frac{L}{\varepsilon}]$, where y is a solution of (9.1) *on $[0,\frac{L}{\varepsilon}]$ and $\overline{y}$ is a solution of* (9.3) *on $[0,L]$.*

Thus, using these results above, we can prove the next result that will be essential to prove our averaging principle for impulsive RFDEs.

Lemma 9.3. *Consider the RFDE* (9.1)*. Then for every $\rho > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$, we have*

$$\left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s,s)\, ds - \int_0^t f_0(y_s) ds \right\| < \rho, \quad t \in [0,L],$$

where y is a solution of (9.1) *on* $[0, \frac{L}{\varepsilon}]$.

Proof. Let y and $\overline{y}$ be solutions of (9.1) and (9.3), respectively. Thus, by Lemma 9.2, we have

$$\begin{aligned}
\|f_0(\overline{y}_s) - f_0(y_s)\| &= \left\| \lim_{T\to\infty} \frac{1}{T} \int_0^T [f(\overline{y}_s, s) - f(y_s, s)] ds \right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T} C \int_0^T \|\overline{y}_s - y_s\| ds < \lim_{T\to\infty} \frac{1}{T} C\rho T = C\rho.
\end{aligned}$$

Therefore, for $t \in [0, L]$, we obtain

$$\left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s, s)\, ds - \int_0^t f_0(y_s) ds \right\| \leq \left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s, s)\, ds - \int_0^t f_0(\overline{y}_s) ds \right\| +$$

$$+ \left\| \int_0^t f_0(y_s) ds - \int_0^t f_0(\overline{y}_s) ds \right\| < \rho + \int_0^t \|f_0(y_s) - f_0(\overline{y}_s)\| ds <$$

$$< \rho + C\rho t \leq \rho + C\rho L$$

and we have the desired result.

$\square$

Consider the RFDE without impulses

$$\begin{cases} \dot{y} = f(y_t, t) \\ y_0 = \phi, \end{cases} \tag{9.4}$$

where $\phi \in G^-([-r, 0], \mathbb{R}^n)$ and $f : G^-([-r, 0], \mathbb{R}^n) \times [0, \infty) \to \mathbb{R}^n$ satisfies conditions (A^*) and (K). By Theorem 5.1, the corresponding generalized ODE is given by

$$\begin{cases} \dfrac{dx}{d\tau} = DF(x, t) \\ x(0) = \widetilde{x}, \end{cases}$$

with initial condition

$$x(0)(\vartheta) = \widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta), & 0 - r \leq \vartheta \leq 0, \\ \phi(0), & 0 \leq \vartheta < \infty, \end{cases} \tag{9.5}$$

where for $y \in PC_1$ and $t \in [0, \infty)$, $F : \Omega \to PC_1$ is given by

$$F(y, t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)\, ds, \quad t_0 \leq \vartheta \leq t < \infty, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\, ds, \quad t_0 \leq t \leq \vartheta < \infty. \end{cases}$$

Now, we consider $\varepsilon > 0$ as a small parameter and the RFDE (9.1). Then the corresponding generalized ODE is given by

$$\begin{cases} \dfrac{dx}{d\tau} = D[\varepsilon F(x,t)] \\ x(0) = \widetilde{x}, \end{cases}$$

with $\widetilde{x}$ is defined by (9.5).

Note that, for $y \in PC_1$ and $t \in [0,\infty)$, we have

$$\varepsilon F\left(y, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) = \begin{cases} 0, \quad -r \leq \vartheta \leq 0, \\ \varepsilon \displaystyle\int_0^{\vartheta/\varepsilon} f(y_s, s)ds, \quad 0 \leq \dfrac{\vartheta}{\varepsilon} \leq \dfrac{t}{\varepsilon} < \infty, \\ \varepsilon \displaystyle\int_0^{t/\varepsilon} f(y_s, s)ds, \quad 0 \leq \dfrac{t}{\varepsilon} \leq \dfrac{\vartheta}{\varepsilon} < \infty. \end{cases}$$

Now, for $y \in PC_1$ and $t \in [-r,\infty)$, define $H_0 : \Omega \to PC_1$ by

$$H_0(y,t)(\vartheta) = \begin{cases} 0, \quad -r \leq \vartheta \leq 0, \\ \displaystyle\int_0^{\vartheta} f_0(y_s)ds, \quad 0 \leq \vartheta \leq t < \infty \\ \displaystyle\int_0^{t} f_0(y_s)ds, \quad 0 \leq t \leq \vartheta < \infty. \end{cases} \tag{9.6}$$

Then, by Lemma 9.3, given $y \in PC_1$ and $t \in [0,\infty)$, we have

$$H_0(y,t)(\vartheta) = \begin{cases} 0, \quad \vartheta \in [-r, 0], \\ \displaystyle\lim_{\varepsilon \to 0^+} \varepsilon F\left(y, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right) = t \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} F\left(y, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right), \quad \vartheta \in [0,\infty). \end{cases} \tag{9.7}$$

Define, for $y \in PC_1$ and $t > 0$,

$$F_0(y)(\vartheta) = \begin{cases} 0, \quad -r \leq \vartheta \leq 0, \\ \displaystyle\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} F\left(y, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right), \quad \vartheta \in [0,\infty). \end{cases} \tag{9.8}$$

It is not difficult to prove that F_0 is well-defined and it is independent of $t \in [0,\infty)$. Thus,

$$H_0(y,t) = F_0(y)t, \tag{9.9}$$

for $y \in PC_1$ and $t \in [0,\infty)$, and hence (9.6) defines the generalized ODE

$$\frac{dx}{d\tau} = DH_0(y,t) = D[F_0(y)t] \tag{9.10}$$

which corresponds to the averaged RFDE (9.3).

On the other hand, (9.10) is an abstract ODE, taking values in the Frechét space $G^-([-r,\infty),\mathbb{R}^n)$ of left continuous regulated functions from $[-r,\infty)$ to $\mathbb{R}^n$, and from the properties of the Kurzweil integral, (9.10) can be written in the form

$$\dot{x} = F_0(x).$$

Now, we consider the following RFDEs with impulses

$$\begin{cases} \dot{y} = f\left(y_t, t\right), \ t \neq \tau_i(y(t)) \\ \Delta y(t) = I_i(y(t)), t = \tau_i(y(t)) \quad i = 1, 2, \ldots \\ y_0 = \phi \end{cases} \tag{9.11}$$

and

$$\begin{cases} \dot{y} = f\left(y_t, \dfrac{t}{\varepsilon}\right), \ t \neq \tau_i(y(t)) \\ \Delta y(t) = \varepsilon I_i\left(y(t)\right), \ t = \tau_i(y(t)) \quad i = 1, 2, \ldots \\ y_0 = \phi, \end{cases} \tag{9.12}$$

where $\phi \in G^-([-r,0],\mathbb{R}^n)$ and $f : G^-([-r,\infty),\mathbb{R}^n) \times [0,\infty) \to \mathbb{R}^n$ satisfy conditions (A^*) and (K), the impulse operators $I_i, i = 1, 2, \ldots$, satisfy conditions (A'^*) and (B'^*). Moreover, we assume that conditions (C1*) and (C2) to (C5) are fulfilled.

Let $t > 0$ and assume that the following limit exists

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \sum_{k=1}^{\infty} \sum_{\substack{0 \leq t_k^i < \frac{t}{\varepsilon} \\ i=1,\ldots,m(\tau_k)}} I_i(x) = I^0(x), \quad x \in \mathbb{R}^n.$$

It can be shown that the above limit is independent of $t > 0$. Then, for $y \in PC_1$, we have

$$I^0(y(t_k^i)) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \sum_{k=1}^{\infty} \sum_{\substack{0 \leq t_k^i < \frac{t}{\varepsilon} \\ i=1,\ldots,m(\tau_k)}} I_k(y(t_k^i)) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \sum_{k=1}^{\infty} \sum_{i=1}^{m(\tau_k)} I_k(y(t_k^i)) H_k^i\left(\frac{t}{\varepsilon}\right),$$

where H_k^i is the left continuous Heaviside function concentrated at t_k^i.

For $y \in PC_1$ and $t \in [0,\infty)$, define

$$J(y,t)(\vartheta) = \begin{cases} 0, \quad \vartheta \in [-r, 0], \\ \displaystyle\sum_{k=1}^{\infty} \sum_{i=0}^{m(\tau_k)} I_k(y(t_k^i)) H_k^i(t) H_k^i(\vartheta), \quad \vartheta \in [0,\infty), \end{cases} \tag{9.13}$$

and

$$J_0(y)(\vartheta) = \begin{cases} 0, & \vartheta \in [-r, 0], \\ \lim\limits_{\varepsilon \to 0^+} \dfrac{\varepsilon}{t} J\left(y, \dfrac{t}{\varepsilon}\right)\left(\dfrac{\vartheta}{\varepsilon}\right), & \vartheta \in [0, \infty). \end{cases}$$

Then $J : PC_1 \times [0, \infty) \to G^-([-r, \infty), \mathbb{R}^n)$ and $J_0 : PC_1 \to G^-([-r, \infty), \mathbb{R}^n)$. Furthermore, J_0 is well-defined and its definition is independent of $t > 0$.

The next theorem is an averaging method for RFDEs with impulses.

Theorem 9.1. *Suppose that y and y^ε are the solutions of the impulsive RFDEs* (9.11) *and* (9.12) *respectively, where $\phi \in G^-([-r, 0], \mathbb{R}^n)$ and $f : G^-([-r, 0], \mathbb{R}^n) \times [0, \infty) \to \mathbb{R}^n$ satisfies conditions* (A*) *and* (K)*. Assume that f_0 is given by* (9.2)*. Suppose*

$$\limsup_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \int_\alpha^{\frac{t}{\varepsilon}+\alpha} M(s)ds \leq c, \quad c = constant, \tag{9.14}$$

for every $\alpha \geq 0$ and $t > 0$. Also, suppose that the conditions (C1*) *and* (C2) *to* (C5) *are fulfilled. Let*

$$\limsup_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \sum_{k=1}^{\infty} \sum_{\substack{0 \leq t_k^i < \frac{t}{\varepsilon} \\ i=1,\ldots,m(\tau_k)}} 1 \leq d \tag{9.15}$$

for every $\alpha \geq 0$ and every $t > 0$. Assume further that $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m(\tau_k)$, is a sequence of impulse operators satisfying conditions (A') and (B'*). Suppose*

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t} \sum_{k=1}^{\infty} \sum_{\substack{0 \leq t_k^i < \frac{t}{\varepsilon} \\ i=1,\ldots,m(\tau_k)}} I_k(x) = I^0(x), \quad x \in \mathbb{R}^n.$$

Then, for every $\mu > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the inequality

$$\|(y^\varepsilon)_t - \overline{y}_t\| < \mu$$

holds for every $t \in [0, \frac{L}{\varepsilon}]$, where $\overline{y}$ is the solution of the autonomous RFDE

$$\begin{cases} \dot{y} = f_0(y_t) + I^0(y(t)), \\ y_0 = \phi. \end{cases} \tag{9.16}$$

Proof. In this proof, we consider the notation and terminology introduced in the paragraphs before the theorem. Note that system (9.12) is equivalent to the generalized ODE

$$\frac{dx}{d\tau} = D\left[\varepsilon G\left(x, \frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)\right], \tag{9.17}$$

with initial condition (9.5), where G is given by (5.3). By Theorem 5.1, the solution x_ε of (9.17) is given by

$$x_\varepsilon(t)(\vartheta)=\begin{cases} y^\varepsilon(\vartheta), \ \vartheta\in[-r,t] \\ y^\varepsilon(t), \ \vartheta\in[t,\infty). \end{cases}$$

Again, by Theorem 5.1, if ξ is given by

$$\xi(t)(\vartheta)=\begin{cases} \overline{y}(\vartheta), \ \vartheta\in[-r,t] \\ \overline{y}(t), \ \vartheta\in[t,\infty). \end{cases}$$

where $\overline{y}$ is the solution of (9.16), then ξ is a solution of

$$\frac{dx}{d\tau}=D[G_0(x)],$$

where $G_0(x)=F_0(x)+J_0(x)$. Also, by the comments presented before this theorem, it is clear that

$$\lim_{\varepsilon\to 0^+}\frac{G\left(x,\frac{t}{\varepsilon}\right)\left(\frac{\vartheta}{\varepsilon}\right)}{t/\varepsilon}=G_0(x)(\vartheta), \quad t>0.$$

Remember that the function $h:[0,\infty)\to\mathbb{R}$ is given by

$$h(t)=h_1(t)+h_2(t),$$

where

$$h_1(t)=\int_0^t[M(s)+C]ds \quad \text{and} \quad h_2=\max\{K_1,K_2\}\sum_{k=1}^{+\infty}\sum_{i=1}^{m(\tau_k)}H_k^i(t).$$

For $\alpha>0$ and $t>0$, we have

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+}\frac{h\left(\frac{t}{\varepsilon}+\alpha\right)-h(\alpha)}{t/\varepsilon} &= \limsup_{\varepsilon\to 0^+}\frac{h_2\left(\frac{t}{\varepsilon}+\alpha\right)-h_2(\alpha)+h_1\left(\frac{t}{\varepsilon}+\alpha\right)-h_1(\alpha)}{t/\varepsilon}= \\
&= \limsup_{\varepsilon\to 0^+}\left[\frac{\varepsilon}{t}\int_\alpha^{\frac{t}{\varepsilon}+\alpha}M(s)\,ds+\frac{\varepsilon}{t}\int_\alpha^{\frac{t}{\varepsilon}+\alpha}C\,ds\right]+ \\
&+ \limsup_{\varepsilon\to 0^+}\frac{\varepsilon}{t}\left(\max(K_1,K_2)\sum_{k=1}^{\infty}\sum_{i=1}^{m(\tau_k)}\left[H_k^i\left(\frac{t}{\varepsilon}+\alpha\right)-H_k^i(\alpha)\right]\right) \\
&\leq c+C+\max(K_1,K_2)d,
\end{aligned}$$

Also, note, by the definition, that, if $t=0$ and $\vartheta\geq 0=t$, then

$$G(x,0)(\vartheta)=0,$$

for every $x \in PC_1$. Therefore, all the hypotheses of Theorem 7.2 are satisfied and, hence, for every $\mu > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the inequality

$$\|x_\varepsilon(t) - \xi(t)\| < \mu$$

holds, for $t \in \left[0, \frac{L}{\varepsilon}\right]$. Finally, for every $t \in \left[0, \frac{L}{\varepsilon}\right]$, we have

$$\|y_t^\varepsilon - \overline{y}_t\| = \sup_{\theta \in [-r,0]} |y^\varepsilon(t+\theta) - \overline{y}(t+\theta)| = \sup_{\vartheta \in [t-r,t]} |y^\varepsilon(\vartheta) - \overline{y}(\vartheta)| \leq$$

$$\leq \sup_{\vartheta \in [-r,t]} |y^\varepsilon(\vartheta) - \overline{y}(\vartheta)| = \sup_{\vartheta \in [-r,t]} |x_\varepsilon(t)(\vartheta) - \xi(t)(\vartheta)| \leq \|x_\varepsilon(t) - \xi(t)\| < \mu$$

and we obtain the desired result. □

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