Basic results for functional differential and dynamic equations involving impulses

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We study the relation between measure functional differential equations, impulsive measure functional differential equations, and impulsive functional dynamic equations on time scales. For both types of impulsive equations, we obtain results on the existence and uniqueness of solutions, continuous dependence, and periodic averaging. Along the way, we also clarify the relation between time scale integrals and Kurzweil-Henstock-Stieltjes integrals.

1 Introduction

Let r, $\sigma > 0$ be given numbers and $t_0 \in \mathbb{R}$. In our recent paper [6], we have introduced equations of the form

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi, \end{aligned}$$

where x_t denotes the function $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$, for every $t \in [t_0, t_0+\sigma]$. The integral on the right-hand side should be understood as the Kurzweil-Henstock-Stieltjes integral taken with respect to a nondecreasing function $g : [t_0, t_0+\sigma] \to \mathbb{R}$ (see the next section). These equations are called measure functional differential equations; they generalize the usual type of functional differential equation which corresponds to the case $g(t) = t$.

We have shown in [6] that functional dynamic equations on time scales represent a special case of measure functional differential equations, and obtained various results concerning the existence and uniqueness of solutions, continuous dependence, and periodic averaging for both types of equations.

The present paper is a continuation of [6]. Our aim is to demonstrate that measure functional differential equations represent an adequate tool for dealing with differential and dynamic equations involving impulses. Section 2 summarizes some basic results about Kurzweil-Henstock-Stieltjes integrals. In Section 3, we introduce impulsive measure functional differential equations and show how to transform them into measure functional differential equations without impulses. In Section 4, we present certain facts from the time scale calculus and explain the relation between time scale integrals and Kurzweil-Henstock-Stieltjes integrals (this part is independent of the previous sections and might be useful for

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readers interested in integration theory on time scales). Section 5 discusses impulsive functional dynamic equations on time scales and demonstrates how to convert them to impulsive measure functional differential equations. Using the results from the last two sections, we are able to relate impulsive functional dynamic equations and measure functional differential equations. In the final three sections, we employ this correspondence to obtain theorems on the existence and uniqueness of solutions, continuous dependence of solutions on parameters, and periodic averaging for impulsive equations.

It is worth mentioning here that in the classical theory of functional differential equations of the form

$$x'(t) = f(x_t, t), \quad t \in [t_0, t_0 + \sigma], \tag{1.1}$$

it is common to use the following assumptions on the right-hand side of the equation:

- There exists a constant $M > 0$ such that $\|f(x,t)\| \le M$ for each x in a certain subset of the phase space and every $t \in [t_0, t_0 + \sigma]$.
- There exists a constant $L > 0$ such that $\|f(x,t) - f(y,t)\| \le L\|x - y\|$ for each x, y in a certain subset of the phase space and every $t \in [t_0, t_0 + \sigma]$.

However, it became clear (see e.g. [7, 8]) that it is sufficient to impose certain conditions on the indefinite integral of the function on the right-hand side of (1.1) rather than on the right-hand side itself. In the present paper, we consider the following weaker conditions:

- There exists a constant $M > 0$ such that

  $$\left\| \int_{u_1}^{u_2} f(x_t, t)\, \mathrm{d}g(t) \right\| \le M(g(u_2) - g(u_1))$$

  for all $u_1, u_2 \in [t_0, t_0 + \sigma]$ and all x in a certain subset of the phase space.
- There exists a constant $L > 0$ such that

  $$\left\| \int_{u_1}^{u_2} (f(x_t, t) - f(y_t, t))\, \mathrm{d}g(t) \right\| \le L \int_{u_1}^{u_2} \|x_t - y_t\|_\infty \, \mathrm{d}g(t)$$

  for all $u_1, u_2 \in [t_0, t_0 + \sigma]$ and all x, y in a certain subset of the phase space.

2 Kurzweil-Henstock-Stieltjes integral

Consider a function $\delta : [a,b] \to \mathbb{R}^+$ (called a gauge on $[a,b]$). A tagged partition of the interval $[a,b]$ with division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \ldots, m\}$, is called δ-fine if

$$\tau_i - \delta(\tau_i) \le s_{i-1} < s_i \le \tau_i + \delta(\tau_i), \quad i \in \{1, \ldots, m\}.$$

A function $f : [a,b] \to \mathbb{R}^n$ is called Kurzweil-Henstock-Stieltjes integrable on $[a,b]$ with respect to a function $g : [a,b] \to \mathbb{R}$, if there is a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{m} f(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a,b]$. In this case, I is called the Kurzweil-Henstock-Stieltjes integral of f with respect to g over $[a,b]$ and will be denoted by $\int_a^b f(t)\, \mathrm{d}g(t)$, or simply $\int_a^b f\, \mathrm{d}g$. This Stieltjes-type integral is a special case of the integral studied by J. Kurzweil in [11]; on the other hand, the choice $g(t) = t$ leads to the well-known Kurzweil-Henstock integral, which generalizes both Lebesgue and Newton integrals.

A function $f:[a,b]\to\mathbb{R}^n$ is called regulated, if the limits

$$\lim_{s\to t-} f(s)=f(t-)\in\mathbb{R}^n,\quad t\in(a,b]\ \text{ and }\ \lim_{s\to t+} f(s)=f(t+)\in\mathbb{R}^n,\quad t\in[a,b)$$

exist. The set of all regulated functions $f:[a,b]\to B$, where $B\subset\mathbb{R}^n$, will be denoted by $G([a,b],B)$. Note that $G([a,b],\mathbb{R}^n)$ is a Banach space under the usual supremum norm $\|f\|_\infty=\sup_{a\le t\le b}\|f(t)\|$.

Given a regulated function f, the symbols $\Delta^+f(t)$ and $\Delta^-f(t)$ will be used throughout this paper to denote

$$\Delta^+f(t)=f(t+)-f(t) \text{ and } \Delta^-f(t)=f(t)-f(t-).$$

In the following sections, we often assume the existence of certain Kurzweil-Henstock-Stieltjes integrals. The next result from [15, Corollary 1.34] is not really necessary for us, but we mention it here as it provides a useful sufficient condition for the existence of the Kurzweil-Henstock-Stieltjes integral.

Theorem 2.1. *If $f:[a,b]\to\mathbb{R}^n$ is a regulated function and $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f\,\mathrm{d}g$ exists.*

The following Hake-type theorem for the Kurzweil-Henstock-Stieltjes integral is a special case of Theorem 1.14 in [15] (see also Remark 1.15 in the same book).

Theorem 2.2. *Consider a pair of functions $f:[a,b]\to\mathbb{R}^n$ and $g:[a,b]\to\mathbb{R}$.*

1. *Assume that the integral $\int_a^t f\,\mathrm{d}g$ exists for every $t\in[a,b)$ and*

$$\lim_{t\to b-}\left(\int_a^t f\,\mathrm{d}g+f(b)(g(b)-g(t))\right)=I.$$

Then $\int_a^b f\,\mathrm{d}g=I$.

2. *Assume that the integral $\int_t^b f\,\mathrm{d}g$ exists for every $t\in(a,b]$ and*

$$\lim_{t\to a+}\left(\int_t^b f\,\mathrm{d}g+f(a)(g(t)-g(a))\right)=I.$$

Then $\int_a^b f\,\mathrm{d}g=I$.

We also need the following related result, which is a special case of Theorem 1.16 in [15].

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}^n$ and $g:[a,b]\to\mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f\,\mathrm{d}g$ exists. Then the functions*

$$h(t)=\int_a^t f\,\mathrm{d}g \quad\text{and}\quad k(t)=\int_t^b f\,\mathrm{d}g$$

are regulated on $[a,b]$ and satisfy

$$\begin{aligned}
h(t+)&=h(t)+f(t)\Delta^+g(t),\quad t\in[a,b),\\
h(t-)&=h(t)-f(t)\Delta^-g(t),\quad t\in(a,b],\\
k(t+)&=k(t)-f(t)\Delta^+g(t),\quad t\in[a,b),\\
k(t-)&=k(t)+f(t)\Delta^-g(t),\quad t\in(a,b].
\end{aligned}$$

We remark here that, according to the previous theorem, solutions of measure functional differential equations must be regulated functions.

Lemma 2.4. *Let $m \in \mathbb{N}$, $a \le t_1 < t_2 < \cdots < t_m \le b$. Consider a pair of functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, where g is regulated, left-continuous on $[a,b]$, and continuous at $t_1,\dots,t_m$. Let $\tilde{f} : [a,b] \to \mathbb{R}$ and $\tilde{g} : [a,b] \to \mathbb{R}$ be such that $\tilde{f}(t) = f(t)$ for every $t \in [a,b]\backslash\{t_1,\dots,t_m\}$ and $\tilde{g} - g$ is constant on each of the intervals $[a,t_1]$, $(t_1,t_2],\dots,(t_{m-1},t_m]$, $(t_m,b]$. Then the integral $\int_a^b \tilde{f}\,\mathrm{d}\tilde{g}$ exists if and only if the integral $\int_a^b f\,\mathrm{d}g$ exists; in that case, we have*

$$\int_a^b \tilde{f}\,\mathrm{d}\tilde{g} = \int_a^b f\,\mathrm{d}g + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

Proof. Using the definition of the Kurzweil-Henstock-Stieltjes integral, we obtain

$$\int_a^{t_1} \tilde{f}\,\mathrm{d}(\tilde{g}-g) = 0.$$

It follows from Theorem 2.2 and the definition of the Kurzweil-Henstock-Stieltjes integral that

$$\int_{t_k}^{t_{k+1}} \tilde{f}\,\mathrm{d}(\tilde{g}-g) = \lim_{\tau\to t_k+}\int_\tau^{t_{k+1}} \tilde{f}\,\mathrm{d}(\tilde{g}-g) + \tilde{f}(t_k)\Delta^+(\tilde{g}-g)(t_k) = \tilde{f}(t_k)\Delta^+\tilde{g}(t_k)$$

for every $k \in \{1,\dots,m-1\}$. If $t_m = b$, then $\int_{t_m}^b \tilde{f}\,\mathrm{d}(\tilde{g}-g) = 0$; otherwise,

$$\int_{t_m}^{b} \tilde{f}\,\mathrm{d}(\tilde{g}-g) = \lim_{\tau\to t_m+}\int_\tau^{b} \tilde{f}\,\mathrm{d}(\tilde{g}-g) + \tilde{f}(t_m)\Delta^+(\tilde{g}-g)(t_m) = \tilde{f}(t_m)\Delta^+\tilde{g}(t_m).$$

Consequently, $\int_a^b \tilde{f}\,\mathrm{d}(\tilde{g}-g)$ exists and

$$\int_a^b \tilde{f}\,\mathrm{d}(\tilde{g}-g) = \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

By Theorems 2.2 and 2.3, we have

$$\int_a^{t_1} \tilde{f}\,\mathrm{d}g = \lim_{\tau\to t_1-}\int_a^\tau \tilde{f}\,\mathrm{d}g = \lim_{\tau\to t_1-}\int_a^\tau f\,\mathrm{d}g = \int_a^{t_1} f\,\mathrm{d}g,$$

$$\int_{t_k}^{t_{k+1}} \tilde{f}\,\mathrm{d}g = \lim_{\substack{\sigma\to t_k+,\\ \tau\to t_{k+1}-}}\int_\sigma^\tau \tilde{f}\,\mathrm{d}g = \lim_{\substack{\sigma\to t_k+,\\ \tau\to t_{k+1}-}}\int_\sigma^\tau f\,\mathrm{d}g = \int_{t_k}^{t_{k+1}} f\,\mathrm{d}g, \quad k \in \{1,\dots,m-1\},$$

$$\int_{t_m}^{b} \tilde{f}\,\mathrm{d}g = \lim_{\tau\to t_m+}\int_\tau^b \tilde{f}\,\mathrm{d}g = \lim_{\tau\to t_m+}\int_\tau^b f\,\mathrm{d}g = \int_{t_m}^{b} f\,\mathrm{d}g.$$

These three relations might be read not only from left to right, but also from right to left; in other words, the integrals on the left-hand sides exist if and only if the integrals on the right-hand sides exist. Combining the three relations, we see that $\int_a^b \tilde{f}\,\mathrm{d}g$ exists if and only if $\int_a^b f\,\mathrm{d}g$ exists; in this case, their values are equal. To conclude the proof, it is sufficent to observe that

$$\int_a^b \tilde{f}\,\mathrm{d}\tilde{g} = \int_a^b \tilde{f}\,\mathrm{d}g + \int_a^b \tilde{f}\,\mathrm{d}(\tilde{g}-g) = \int_a^b f\,\mathrm{d}g + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k). \qquad \square$$

3 Impulsive measure functional differential equations

A measure functional differential equation has the form

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi, \end{aligned}$$

where the Kurzweil-Henstock-Stieltjes integral on the right-hand side is taken with respect to a nondecreasing function $g : [t_0, t_0+\sigma] \to \mathbb{R}$; these equations have been studied in [6].

We assume that g is a left-continuous function and consider the possibility of adding impulses at preassigned times $t_1, \dots, t_m$, where $t_0 \le t_1 < \cdots < t_m < t_0 + \sigma$. For every $k \in \{1, \dots, m\}$, the impulse at t_k is described by the operator $I_k : \mathbb{R}^n \to \mathbb{R}^n$. In other words, the solution x should satisfy $\Delta^+ x(t_k) = I_k(x(t_k))$. This leads us to the following problem:

$$\begin{aligned} x(v) - x(u) &= \textstyle\int_u^v f(x_s, s)\, \mathrm{d}g(s), \quad \text{whenever } u, v \in J_k \text{ for some } k \in \{0, \dots, m\}, \\ \Delta^+ x(t_k) &= I_k(x(t_k)), \quad k \in \{1, \dots, m\}, \\ x_{t_0} &= \phi, \end{aligned}$$

where $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1, \dots, m-1\}$, and $J_m = (t_m, t_0 + \sigma]$.

The value of the integral $\int_u^v f(x_s, s)\, \mathrm{d}g(s)$, where $u, v \in J_k$, does not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k (this follows easily from the definition of the Kurzweil-Henstock-Stieltjes integral). Thus, without loss of generality, we can assume that g is such that $\Delta^+ g(t_k) = 0$ for every $k \in \{1, \dots, m\}$. Since g is a left-continuous function, it follows that g is continuous at $t_1, \dots, t_m$. Under this assumption, our problem can be rewritten as

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \dots, m\}, \\ t_k < t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma], \qquad (3.1) \\ x_{t_0} &= \phi. \end{aligned}$$

Indeed, the function $t \mapsto \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s)$ is continuous at $t_1, \dots, t_m$ (see Theorem 2.3), and therefore $\Delta^+ x(t_k) = I_k(x(t_k))$ for every $k \in \{1, \dots, m\}$.

Alternatively, the sum on the right-hand side of (3.1) might be written as $\sum_{k=1}^{m} I_k(x(t_k)) H_{t_k}(t)$, where H_v denotes the characteristic function of (v, ∞), i.e. $H_v(t) = 0$ for $t \le v$ and $H_v(t) = 1$ for $t > v$.

The following theorem shows that impulsive measure functional differential equations of the form (3.1) can always be transformed to measure functional differential equations without impulses.

Theorem 3.1. *Let $m \in \mathbb{N}$, $t_0 \le t_1 < \cdots < t_m < t_0 + \sigma$, $B \subset \mathbb{R}^n$, $I_1, \dots, I_m : B \to \mathbb{R}^n$, $P = G([-r, 0], B)$, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$. Assume that $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a regulated left-continuous function which is continuous at $t_1, \dots, t_m$. For every $y \in P$, define*

$$\tilde{f}(y, t) = \begin{cases} f(y, t), & t \in [t_0, t_0+\sigma] \backslash \{t_1, \dots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1, \dots, m\}. \end{cases}$$

Moreover, let the function $\tilde{g} : [t_0, t_0+\sigma] \to \mathbb{R}$ be given by

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1, \dots, m-1\}, \\ g(t) + m, & t \in (t_m, t_0+\sigma]. \end{cases}$$

Then $x \in G([t_0 - r, t_0 + \sigma], B)$ is a solution of

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \dots, m\}, \\ t_k < t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma], \qquad (3.2) \\ x_{t_0} &= \phi \end{aligned}$$

if and only if

$$
\begin{aligned}
x(t) &= x(t_0) + \int_{t_0}^{t} \tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s), \quad t \in [t_0, t_0+\sigma], \qquad (3.3)\\
x_{t_0} &= \phi.
\end{aligned}
$$

Proof. By the definition of $\tilde{g}$, we have $\Delta^+\tilde{g}(t_k) = 1$ for every $k \in \{1,\dots,m\}$. According to Lemma 2.4, we obtain

$$
\int_{t_0}^{t} \tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s) = \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} f(x_{t_k}, t_k)\Delta^+\tilde{g}(t_k)
$$

$$
= \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)),
$$

i.e. the right-hand sides of (3.2) and (3.3) are indeed identical. □

Remark 3.2. When $g(t) = t$ for every $t \in [t_0, t_0+\sigma]$, Eq. (3.2) reduces to the usual type of impulsive functional differential equation

$$
x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma].
$$

Basic results concerning this type of equations were obtained by M. Federson and Š. Schwabik in [7]; the main tool in their investigations was the theory of generalized ordinary differential equations. The previous theorem suggests a different approach: impulsive functional differential equations represent a special case of measure functional differential equations, and therefore the existing theory of measure equations can be used in the study of impulsive equations.

Lemma 3.3. *Let $m \in \mathbb{N}$, $t_0 \le t_1 < \cdots < t_m < t_0+\sigma$, $B \subset \mathbb{R}^n$, $I_1,\dots,I_m : B \to \mathbb{R}^n$, $P = G([-r,0], B)$, $O = G([t_0-r, t_0+\sigma], B)$. Assume that $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function which is continuous at $t_1,\dots,t_m$. Let $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ be a function such that the integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$. For every $y \in P$, define*

$$
\tilde{f}(y,t) = \begin{cases} f(y,t), & t \in [t_0, t_0+\sigma]\backslash\{t_1,\dots,t_m\},\\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1,\dots,m\}. \end{cases}
$$

Moreover, let the function $\tilde{g} : [t_0, t_0+\sigma] \to \mathbb{R}$ be given by

$$
\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1],\\ g(t)+k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1,\dots,m-1\},\\ g(t)+m, & t \in (t_m, t_0+\sigma]. \end{cases}
$$

Then the following statements are true:

1. *The function $\tilde{g}$ is nondecreasing.*
2. *Assume there exist constants $M_1, M_2 \in \mathbb{R}^+$ such that*

$$
\left\| \int_{u_1}^{u_2} f(y_t, t)\,\mathrm{d}g(t) \right\| \le M_1(g(u_2) - g(u_1))
$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$, $y \in O$, *and*

$$\|I_k(x)\| \le M_2$$

for every $k \in \{1, \dots, m\}$ *and* $x \in B$. *Then*

$$\left\| \int_{u_1}^{u_2} \tilde{f}(y_t, t) \, \mathrm{d}\tilde{g}(t) \right\| \le (M_1 + M_2)(\tilde{g}(u_2) - \tilde{g}(u_1))$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ *and* $y \in O$.

3. *Assume there exist constants* $L_1, L_2 \in \mathbb{R}^+$, *such that*

$$\left\| \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t)) \, \mathrm{d}g(t) \right\| \le L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \, \mathrm{d}g(t)$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$, $y, z \in O$, *and*

$$\|I_k(x) - I_k(y)\| \le L_2 \|x - y\|$$

for every $k \in \{1, \dots, m\}$ *and* x, $y \in B$. *Then*

$$\left\| \int_{u_1}^{u_2} \left(\tilde{f}(y_t, t) - \tilde{f}(z_t, t) \right) \mathrm{d}\tilde{g}(t) \right\| \le (L_1 + L_2) \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \, \mathrm{d}\tilde{g}(t)$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ *and* $y, z \in O$.

Proof. It is clear from the definition of $\tilde{g}$ that it is nondecreasing if g is nondecreasing. Moreover,

$$\tilde{g}(v) - \tilde{g}(u) \ge g(v) - g(u) \tag{3.4}$$

whenever $t_0 \le u \le v \le t_0 + \sigma$.

To prove the second statement, let $t_0 \le u_1 \le u_2 \le t_0 + \sigma$, $y \in O$. From Lemma 2.4, we obtain

$$\int_{u_1}^{u_2} \tilde{f}(y_t, t) \, \mathrm{d}\tilde{g}(t) = \int_{u_1}^{u_2} f(y_t, t) \, \mathrm{d}g(t) + \sum_{\substack{k \in \{1, \dots, m\}, \\ u_1 \le t_k < u_2}} I_k(y(t_k)) \Delta^+ \tilde{g}(t_k),$$

and therefore

$$\left\| \int_{u_1}^{u_2} \tilde{f}(y_t, t) \, \mathrm{d}\tilde{g}(t) \right\| \le M_1(g(u_2) - g(u_1)) + M_2 \sum_{\substack{k \in \{1, \dots, m\}, \\ u_1 \le t_k < u_2}} \Delta^+ \tilde{g}(t_k)$$

$$\le M_1(\tilde{g}(u_2) - \tilde{g}(u_1)) + M_2(\tilde{g}(u_2) - \tilde{g}(u_1)) = (M_1 + M_2)(\tilde{g}(u_2) - \tilde{g}(u_1)).$$

To prove the third statement, let $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ and $y, z \in O$. Using Lemma 2.4 again, we obtain

$$\int_{u_1}^{u_2} \left(\tilde{f}(y_t, t) - \tilde{f}(z_t, t) \right) \mathrm{d}\tilde{g}(t)$$

$$= \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t)) \, \mathrm{d}g(t) + \sum_{\substack{k \in \{1, \dots, m\}, \\ u_1 \le t_k < u_2}} (I_k(y(t_k)) - I_k(z(t_k))) \Delta^+ \tilde{g}(t_k).$$

Consequently,

$$\left\| \int_{u_1}^{u_2} \left(\tilde{f}(y_t, t) - \tilde{f}(z_t, t) \right) \mathrm{d}\tilde{g}(t) \right\| \le L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \, \mathrm{d}g(t) + L_2 \sum_{\substack{k \in \{1, \dots, m\}, \\ u_1 \le t_k < u_2}} \|y(t_k) - z(t_k)\| \Delta^+ \tilde{g}(t_k).$$

Using Eq. (3.4) and the definition of the Kurzweil-Henstock-Stieltjes integral, we see that

$$\int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}g(t) \le \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}\tilde{g}(t).$$

Next, we observe that the function

$$h(s) = \int_{t_0}^{s} \|y_t - z_t\|_\infty \,\mathrm{d}\tilde{g}(t), \quad s \in [t_0, t_0 + \sigma],$$

is nondecreasing and $\Delta^+ h(t_k) = \|y_{t_k} - z_{t_k}\|_\infty \Delta^+ \tilde{g}(t_k)$ for $k \in \{1, \dots, m\}$. Therefore

$$L_2 \sum_{\substack{k\in\{1,\dots,m\},\\ u_1 \le t_k < u_2}} \|y(t_k) - z(t_k)\| \Delta^+ \tilde{g}(t_k) \le L_2 \sum_{\substack{k\in\{1,\dots,m\},\\ u_1 \le t_k < u_2}} \|y_{t_k} - z_{t_k}\|_\infty \Delta^+ \tilde{g}(t_k)$$

$$= L_2 \sum_{\substack{k\in\{1,\dots,m\},\\ u_1 \le t_k < u_2}} \Delta^+ h(t_k) \le L_2 (h(u_2) - h(u_1)) = L_2 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}\tilde{g}(t),$$

and it follows that

$$\left\| \int_{u_1}^{u_2} \left(\tilde{f}(y_t, t) - \tilde{f}(z_t, t) \right) \mathrm{d}\tilde{g}(t) \right\| \le (L_1 + L_2) \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}\tilde{g}(t). \qquad \square$$

4 Integration on time scales

This section presents a short overview of time scales, which were introduced in order to unify and extend continuous, discrete and quantum calculus (see e.g. [3], [4]). We also clarify the relation between time scale integrals and Kurzweil-Henstock-Stieltjes integrals. The results obtained here will be used later in our investigation of functional dynamic equations on time scales.

A time scale is a closed nonempty subset $\mathbb{T}$ of the real line. For every $t \in \mathbb{T}$, we define the forward jump operator by $\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}$ and the backward jump operator by $\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}$; we make the convention that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The graininess function is defined as $\mu(t) = \sigma(t) - t$.

If $\sigma(t) > t$, we say that t is a right-scattered point; otherwise, t is right-dense. Similarly, we distinguish between left-scattered and left-dense points, depending on whether $\rho(t) < t$, or $\rho(t) = t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$.

For each pair of numbers a, $b \in \mathbb{T}$, $a \le b$, let $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Given a set $B \subset \mathbb{R}^n$, the symbol $G([a, b]_{\mathbb{T}}, B)$ will be used to denote the set of all regulated functions $f : [a, b]_{\mathbb{T}} \to B$.

In the time scale calculus, the usual derivative $f'(t)$ and integral $\int_a^b f(t)\,\mathrm{d}t$ of a function $f : [a, b] \to \mathbb{R}$ are replaced by the Δ-derivative $f^\Delta(t)$ and Δ-integral $\int_a^b f(t)\,\Delta t$, where $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$. Similarly to the classical case, there exist various definitions of the Δ-integral $\int_a^b f(t)\,\Delta t$, such as the Riemann Δ-integral or Lebesgue Δ-integral; these definitions as well as the definition of the Δ-derivative can be found in [3], [4]. The more general Kurzweil-Henstock Δ-integral was introduced in [14] (see below).

Given a real number $t \le \sup \mathbb{T}$, let

$$t^* = \inf\{s \in \mathbb{T}; s \ge t\}.$$

(Note that t^* might be different from $\sigma(t)$.) Since $\mathbb{T}$ is a closed set, we have $t^* \in \mathbb{T}$. Further, let

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise.} \end{cases}$$

Finally, given a function $f : \mathbb{T} \to \mathbb{R}^n$, we consider its extension $f^* : \mathbb{T}^* \to \mathbb{R}^n$ given by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

The following theorem from [16] describes the relation between the Δ-integral and the Kurzweil-Henstock-Stieltjes integral.

Theorem 4.1. *Let $f : \mathbb{T} \to \mathbb{R}^n$ be an rd-continuous function. Choose an arbitrary $a \in \mathbb{T}$ and define*

$$F_1(t) = \int_a^t f(s)\Delta s, \quad t \in \mathbb{T},$$

$$F_2(t) = \int_a^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$.

In particular, if $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is an rd-continuous function, we obtain

$$\int_a^b f(s)\Delta s = \int_a^b f^*(s)\,\mathrm{d}g(s). \tag{4.1}$$

Since the solutions of impulsive equations are discontinuous, we need to relax the assumption of rd-continuity. It is not difficult to show that Eq. (4.1) remains true in the more general case where f is a regulated function; it is sufficient to use uniform convergence theorems for both types of integrals and the fact that every regulated function is a uniform limit of continuous functions. Although regulated functions are general enough for our purposes, we take this opportunity to prove a much stronger result: All we need to require for Eq. (4.1) to hold is that f is Δ-integrable in Kurzweil-Henstock's sense. At first, we recall the definition of the Kurzweil-Henstock Δ-integral as introduced by A. Peterson and B. Thompson in [14].

Let $\delta = (\delta_L, \delta_R)$ be a pair of nonnegative functions defined on $[a,b]_{\mathbb{T}}$. We say that δ is a Δ-gauge for $[a,b]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $(a,b] \cap \mathbb{T}$, $\delta_R(t) > 0$ on $[a,b) \cap \mathbb{T}$, and $\delta_R(t) \geq \mu(t)$ for all $t \in [a,b) \cap \mathbb{T}$.

A tagged partition of $[a,b]_{\mathbb{T}}$ consists of division points $s_0, \dots, s_m \in [a,b]_{\mathbb{T}}$ such that $a = s_0 < s_1 < \cdots < s_m = b$, and tags $\tau_1, \dots, \tau_m \in [a,b]_{\mathbb{T}}$ such that $\tau_i \in [s_{i-1}, s_i]$ for every $i \in \{1, \dots, m\}$. Such a partition is called δ-fine if

$$\tau_i - \delta_L(\tau_i) \leq s_{i-1} < s_i \leq \tau_i + \delta_R(\tau_i), \quad i \in \{1, \dots, m\}.$$

A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is called Kurzweil-Henstock Δ-integrable, if there exists a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a Δ-gauge δ on $[a,b]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^m f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a,b]_{\mathbb{T}}$. In this case, I is called the Kurzweil-Henstock Δ-integral of f over $[a,b]_{\mathbb{T}}$ and will be denoted by $\int_a^b f(t)\Delta t$.

Here is the promised result which shows that Δ-integrals are in fact special cases of Kurzweil-Henstock-Stieltjes integrals.

Theorem 4.2. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be an arbitrary function. Define $g(t) = t^*$ for every $t \in [a,b]$. Then the Kurzweil-Henstock Δ-integral $\int_a^b f(t)\Delta t$ exists if and only if the Kurzweil-Henstock-Stieltjes integral $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists; in this case, both integrals have the same value.*

Proof. For an arbitrary tagged partition P of $[a,b]$ consisting of division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_1, \ldots, \tau_m$, let

$$S(P) = \sum_{i=1}^{m} f^*(\tau_i)(g(s_i) - g(s_{i-1})) = \sum_{i=1}^{m} f(\tau_i^*)(s_i^* - s_{i-1}^*). \tag{4.2}$$

At first, assume that $\int_a^b f(t)\Delta t$ exists. Then, given an arbitrary $\varepsilon > 0$, there is a Δ-gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{m} f(\tau_i)(s_i - s_{i-1}) - \int_a^b f(t)\Delta t \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a,b]_{\mathbb{T}}$. We construct a gauge $\tilde{\delta} : [a,b] \to \mathbb{R}^+$ in the following way:

$$\tilde{\delta}(t) = \begin{cases} \min(\delta_L(t), \sup\{d; t + d \in [a,b]_{\mathbb{T}}, d \le \delta_R(t)\}) & \text{if } t \in (a,b) \cap \mathbb{T}, \\ \sup\{d; a + d \in [a,b]_{\mathbb{T}}, d \le \delta_R(a)\} & \text{if } t = a, \\ \delta_L(b) & \text{if } t = b, \\ \frac{1}{2} \inf\{|t - s|, s \in \mathbb{T}\} & \text{if } t \in [a,b] \backslash \mathbb{T}. \end{cases}$$

Let P be an arbitrary $\tilde{\delta}$-fine tagged partition of $[a,b]$ with division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \ldots, m\}$. For every $i \in \{1, \ldots, m\}$, there are two possibilities: either $\tau_i \in \mathbb{T}$, or $[s_{i-1}, s_i] \cap \mathbb{T} = \emptyset$.

The division points $s_0, \ldots, s_m$ and tags $\tau_1, \ldots, \tau_m$ need not belong to $\mathbb{T}$, but we can find a partition P' whose division points and tags belong to $\mathbb{T}$, $S(P) = S(P')$, and P' is δ-fine. We proceed by induction: Clearly, $s_0 = a \in \mathbb{T}$. Now, consider an interval $[s_{i-1}, s_i]$ with $s_{i-1} \in \mathbb{T}$. Since $[s_{i-1}, s_i] \cap \mathbb{T} \neq \emptyset$, we must have $\tau_i \in \mathbb{T}$. If $s_i \notin \mathbb{T}$, we replace the division point s_i by s_i^*, delete all division points s_j belonging to (s_i, s_i^*), and also all tags τ_j belonging to (s_i, s_i^*). This operation keeps the value of the integral sum (4.2) unchanged: The contributions of the intervals $[s_{i-1}, s_i]$ and $[s_{i-1}, s_i^*]$ to the value of the sum are the same, and the contributions of intervals $[s_{j-1}, s_j]$ contained in (s_i, s_i^*) are zero because $s_{j-1}^* = s_j^* = s_i^*$. It remains to check that the modified partition is δ-fine. Let $M = \sup([a, \tau_i + \delta_R(\tau_i)] \cap \mathbb{T})$. Obviously, $M \in [a,b]_{\mathbb{T}}$. Since our original partition was $\tilde{\delta}$-fine, it follows that

$$s_i \le \tau_i + \tilde{\delta}(\tau_i) \le \tau_i + \sup\{d; \tau_i + d \in [a,b]_{\mathbb{T}}, d \le \delta_R(\tau_i)\} = M.$$

But $s_i \notin \mathbb{T}$ and $M \in \mathbb{T}$ implies $s_i^* \le M$, because s_i^* is the smallest time scale point larger than s_i. Consequently, $s_i^* \le M \le \tau_i + \delta_R(\tau_i)$.

Now, P' is a δ-fine tagged partition of $[a,b]_{\mathbb{T}}$, and therefore

$$\left\| S(P) - \int_a^b f(t)\Delta t \right\| = \left\| S(P') - \int_a^b f(t)\Delta t \right\| < \varepsilon,$$

which proves that $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists and equals $\int_a^b f(t)\Delta t$.

Conversely, assume that $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists. Then, given an arbitrary $\varepsilon > 0$, there is a gauge $\tilde{\delta} : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^{m} f(\tau_i^*)(s_i^* - s_{i-1}^*) - \int_a^b f^*(t)\,\mathrm{d}g(t) \right\| < \varepsilon$$

for every $\tilde{\delta}$-fine tagged partition of $[a,b]$. We construct a Δ-gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]_{\mathbb{T}}$ by letting $\delta_L(t) = \tilde{\delta}(t)$ and $\delta_R(t) = \max(\tilde{\delta}(t), \mu(t))$ for every $t \in [a,b]_{\mathbb{T}}$.

Consider an arbitrary δ-fine tagged partition P of $[a,b]_{\mathbb{T}}$ with division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \ldots, m\}$; by definition, all these points belong to $\mathbb{T}$.

Our δ-fine partition need not be $\tilde{\delta}$-fine: for certain values of $i \in \{1,\dots,m\}$, it can happen that $\delta_R(\tau_i)+\tau_i \geq s_i > \tilde{\delta}(\tau_i)+\tau_i$. In this case, we have $\delta_R(\tau_i)=\mu(\tau_i)$, the point τ_i is right-scattered, and $s_i=\sigma(\tau_i)$. We claim that it is possible to find a modified tagged partition P' of $[a,b]$ which is $\tilde{\delta}$-fine and $S(P)=S(P')$. To this end, replace the division point s_i by $\tau_i+\tilde{\delta}(\tau_i)$ while keeping τ_i as the tag for the interval $[s_{i-1},\tau_i+\tilde{\delta}(\tau_i)]$, and cover the interval $[\tau_i+\tilde{\delta}(\tau_i),s_i]$ by an arbitrary $\tilde{\delta}$-fine partition. The equality $S(P)=S(P')$ follows from the fact that $t^*=s_i$ for every $t\in(\tau_i,s_i]$.

The proof is concluded by observing that

$$\left\|\sum_{i=1}^m f(\tau_i)(s_i-s_{i-1})-\int_a^b f^*(t)\,\mathrm{d}g(t)\right\|$$

$$=\left\|S(P)-\int_a^b f^*(t)\,\mathrm{d}g(t)\right\|=\left\|S(P')-\int_a^b f^*(t)\,\mathrm{d}g(t)\right\|<\varepsilon,$$

which implies that $\int_a^b f(t)\Delta t$ exists and equals $\int_a^b f^*(t)\,\mathrm{d}g(t)$. □

Remark 4.3. Several authors have been interested in Stieltjes-type integrals on time scales (see e.g. [10], [13]). For example, the definition of the Riemann-Stieltjes Δ-integral $\int_a^b f(t)\Delta g(t)$ of a function $f:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$ with respect to a function $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ can be obtained in a straightforward way by taking the definition of the Riemann Δ-integral and replacing the usual integral sums by

$$\sum_{i=1}^m f(\tau_i)(g(s_i)-g(s_{i-1})).$$

Alternatively, we can start with the definition of the Kurzweil-Henstock Δ-integral and modify the integral sums in the same way. Using exactly the same reasoning as in the proof of Theorem 4.2, one can show that the resulting Stieltjes-type Δ-integral satisfies

$$\int_a^b f(t)\Delta g(t)=\int_a^b f^*(t)\,\mathrm{d}g^*(t).$$

Consequently, many properties of the Δ-integrals can be simply derived from the known properties of the Kurzweil-Henstock-Stieltjes integrals.

Lemma 4.4. *Let $a,b\in\mathbb{T}$, $a<b$, $g(t)=t^*$ for every $t\in[a,b]$. If $f:[a,b]\to\mathbb{R}^n$ is such that the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists, then*

$$\int_c^d f(t)\,\mathrm{d}g(t)=\int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t)$$

for every $c,d\in[a,b]$.

Proof. Using the definition of the Kurzweil-Henstock-Stieltjes integral and the fact that g is constant on $[c,c^*]$ and on $[d,d^*]$, we see that $\int_c^{c^*} f(t)\,\mathrm{d}g(t)=0$ and $\int_d^{d^*} f(t)\,\mathrm{d}g(t)=0$. Therefore

$$\int_c^d f(t)\,\mathrm{d}g(t)=\int_c^{c^*} f(t)\,\mathrm{d}g(t)+\int_{c^*}^d f(t)\,\mathrm{d}g(t)+\int_d^{d^*} f(t)\,\mathrm{d}g(t)=\int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t).$$

□

Theorem 4.5. *Let $f:\mathbb{T}\to\mathbb{R}^n$ be a function such that the Kurzweil-Henstock integral $\int_a^b f(s)\Delta s$ exists for every $a,b\in\mathbb{T}$, $a<b$. Choose an arbitrary $a\in\mathbb{T}$ and define*

$$F_1(t)=\int_a^t f(s)\Delta s,\quad t\in\mathbb{T},$$

$$F_2(t) = \int_a^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$.

Proof. The statement is a simple consequence of Lemma 4.4 and Theorem 4.2:

$$F_2(t) = \int_a^t f^*(s)\,\mathrm{d}g(s) = \int_a^{t^*} f^*(s)\,\mathrm{d}g(s) = \int_a^{t^*} f(s)\Delta s = F_1(t^*) = F_1^*(t) \qquad \square$$

5 Impulsive functional dynamic equations on time scales

In this section, we focus our attention on functional dynamic equations with impulses. In particular, we explain the relation between this type of equations and impulsive measure functional differential equations, which were discussed in Section 3. In [6], we dealt with functional dynamic equations of the form

$$\begin{aligned} x^{\Delta}(t) &= f(x_t^*, t), \ \ t \in [t_0, t_0+\sigma]_{\mathbb{T}}, \\ x(t) &= \phi(t), \ \ t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

The symbol x_t^* should be understood as $(x^*)_t$; as explained in [6], the advantage of using x_t^* rather than x_t stems from the fact that x_t^* is always defined on the whole interval $[-r, 0]$, whereas x_t is defined only on a subset of $[-r, 0]$; moreover, this subset depends on t.

Our aim here is to study functional dynamic equations with impulses. Several authors have already considered impulsive dynamic equations on time scales (see for example [1], [2], [5], [9]); to this end, let $t_1, \ldots, t_m \in \mathbb{T}$, $t_0 \le t_1 < t_2 < \cdots < t_m < t_0 + \sigma$ and $I_1, \ldots, I_m : \mathbb{R}^n \to \mathbb{R}^n$. The usual condition which can be found in the existing literature is that the solution should satisfy

$$x(t_k+) - x(t_k-) = I_k(x(t_k-)), \quad k \in \{1, \ldots, m\}. \tag{5.1}$$

The convention is that $x(t+) = x(t)$ when $t \in \mathbb{T}$ is a right-scattered point and $x(t-) = x(t)$ when $t \in \mathbb{T}$ is left-scattered. Moreover, it is usually assumed that the solution x should be left-continuous. In this case, Eq. (5.1) reduces to

$$x(t_k+) - x(t_k) = I_k(x(t_k)), \quad k \in \{1, \ldots, m\}. \tag{5.2}$$

Note that if t_k is right-scattered, then the left-hand side of Eq. (5.2) is zero. In other words, it makes sense to consider impulses at right-dense points only (the same assumption is made in [2], [5]).

This motivates us to consider impulsive functional dynamic equations of the form

$$\begin{aligned} x^{\Delta}(t) &= f(x_t^*, t), \ \ t \in [t_0, t_0+\sigma]_{\mathbb{T}} \backslash \{t_1, \ldots, t_m\}, \\ \Delta^+ x(t_k) &= I_k(x(t_k)), \ \ k \in \{1, \ldots, m\}, \\ x(t) &= \phi(t), \ \ t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{aligned}$$

where $t_1, \ldots, t_m \in \mathbb{T}$ are right-dense points, $t_0 \le t_1 < t_2 < \cdots < t_m < t_0 + \sigma$, and $I_1, \ldots, I_m : \mathbb{R}^n \to \mathbb{R}^n$. The solution is assumed to be left-continuous. It is not difficult to see that the above problem can be written more compactly in the form

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t f(x_s^*, s)\Delta s + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x(t_k)), \ \ t \in [t_0, t_0+\sigma]_{\mathbb{T}}, \\ x(t) &= \phi(t), \ \ t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

Our immediate goal is to rewrite this equation as an impulsive measure functional differential equation. We need the following proposition from [6] (see Theorem 4.2 there).

Theorem 5.1. *Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$, $[a, b] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2 : [a, b] \to \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [a, b] \cap \mathbb{T}$. If $\int_a^b f_1(s)\,\mathrm{d}g(s)$ exists, then $\int_a^b f_2(s)\,\mathrm{d}g(s)$ exists as well and both integrals have the same value.*

The following theorem describes the relation between impulsive functional dynamic equations and impulsive measure functional differential equations.

Theorem 5.2. *Let $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$ be a time scale interval, $t_0 \in \mathbb{T}$, $B \subset \mathbb{R}^n$, $f : G([-r, 0], B) \times [t_0, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$, $\phi \in G([t_0 - r, t_0]_{\mathbb{T}}, B)$. Define $g(s) = s^*$ for every $s \in [t_0, t_0 + \sigma]$. If $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ is a solution of the impulsive functional dynamic equation*

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s^*, s)\,\Delta s + \sum_{\substack{k \in \{1,\dots,m\},\\ t_k < t}} I_k(x(t_k)), \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}}, \tag{5.3}$$

$$x(t) = \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \tag{5.4}$$

then $x^* : [t_0 - r, t_0 + \sigma] \to B$ is a solution of the impulsive measure functional differential equation

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s^*)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\},\\ t_k < t}} I_k(y(t_k)), \quad t \in [t_0, t_0 + \sigma], \tag{5.5}$$

$$y_{t_0} = \phi_{t_0}^*. \tag{5.6}$$

Conversely, if $y : [t_0 - r, t_0 + \sigma] \to B$ satisfies (5.5) and (5.6), then it must have the form $y = x^*$, where $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ is a solution of (5.3) and (5.4).

Proof. Assume that x satisfies (5.3) and (5.4). Clearly, $x_{t_0}^* = \phi_{t_0}^*$. By Theorem 4.5,

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} f(x_{s^*}^*, s^*)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\},\\ t_k < t^*}} I_k(x(t_k)), \quad t \in [t_0, t_0 + \sigma].$$

We have $t_k \in \mathbb{T}$ for every $k \in \{1, \dots, m\}$. It follows that $x(t_k) = x^*(t_k)$, and $t_k < t^*$ if and only if $t_k < t$. Moreover, since $f(x_{s^*}^*, s^*) = f(x_s^*, s^*)$ for every $s \in \mathbb{T}$, we can use Theorem 5.1 to conclude that

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} f(x_s^*, s^*)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\},\\ t_k < t}} I_k(x^*(t_k)), \quad t \in [t_0, t_0 + \sigma],$$

which proves the first part.

Conversely, assume that y satisfies (5.5) and (5.6). If $t \in [t_0, t_0 + \sigma] \backslash \mathbb{T}$, then g is constant on $[t, t^*]$ and therefore $y(t) = y(t^*)$. It follows that $y = x^*$, where $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ is the restriction of y to $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$. By reversing our previous reasoning, we conclude that x satisfies (5.3) and (5.4). □

Lemma 5.3. *Let $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$ be a time scale interval, $t_0 \in \mathbb{T}$, $O = G([t_0 - r, t_0 + \sigma], B)$, $P = G([-r, 0], B)$, $f : P \times [t_0, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$ an arbitrary function. Define $g(t) = t^*$ and $f^*(y, t) = f(y, t^*)$ for every $y \in P$ and $t \in [t_0, t_0 + \sigma]$.*

1. *If the integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\Delta t$ exists for every $y \in O$, then the integral $\int_{t_0}^{t_0+\sigma} f^*(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.*

2. *Assume there exists a constant $M > 0$ such that*

$$\left\| \int_{u_1}^{u_2} f(y_t, t)\Delta t \right\| \le M(u_2 - u_1)$$

for every $y \in O$ *and* $u_1, u_2 \in [t_0, t_0+\sigma]_{\mathbb{T}}$, $u_1 \le u_2$. *Then*

$$\left\| \int_{u_1}^{u_2} f^*(y_t, t)\,\mathrm{d}g(t) \right\| \le M(g(u_2) - g(u_1))$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ *and* $y \in O$.

3. *Assume there exists a constant* $L > 0$ *such that*

$$\left\| \int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t))\,\Delta t \right\| \le L \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \Delta t$$

for every $y, z \in O$ *and* $u_1, u_2 \in [t_0, t_0+\sigma]_{\mathbb{T}}$, $u_1 \le u_2$. *Then*

$$\left\| \int_{u_1}^{u_2} (f^*(y_t, t) - f^*(z_t, t))\,\mathrm{d}g(t) \right\| \le L \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}g(t)$$

whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ *and* $y, z \in O$.

Proof. Consider an arbitrary $y \in O$. If the integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\Delta t$ exists, then, using Theorems 4.2 and 5.1, we have

$$\int_{t_0}^{t_0+\sigma} f(y_t, t)\Delta t = \int_{t_0}^{t_0+\sigma} f(y_{t^*}, t^*)\,\mathrm{d}g(t) = \int_{t_0}^{t_0+\sigma} f(y_t, t^*)\,\mathrm{d}g(t) = \int_{t_0}^{t_0+\sigma} f^*(y_t, t)\,\mathrm{d}g(t),$$

i.e. the last integral exists as well. This proves the first part.

The remaining two statements follow from Theorem 5.1, Lemma 4.4, and Theorem 4.2. In the first case, we have

$$\left\| \int_{u_1}^{u_2} f^*(y_t, t)\,\mathrm{d}g(t) \right\| = \left\| \int_{u_1}^{u_2} f(y_{t^*}, t^*)\,\mathrm{d}g(t) \right\| = \left\| \int_{u_1^*}^{u_2^*} f(y_{t^*}, t^*)\,\mathrm{d}g(t) \right\| = \left\| \int_{u_1^*}^{u_2^*} f(y_t, t)\Delta t \right\|$$
$$\le M(u_2^* - u_1^*) = M(g(u_2) - g(u_1)).$$

In the second case, we obtain

$$\left\| \int_{u_1}^{u_2} (f^*(y_t, t) - f^*(z_t, t))\,\mathrm{d}g(t) \right\| = \left\| \int_{u_1}^{u_2} (f(y_{t^*}, t^*) - f(z_{t^*}, t^*))\,\mathrm{d}g(t) \right\|$$
$$= \left\| \int_{u_1^*}^{u_2^*} (f(y_{t^*}, t^*) - f(z_{t^*}, t^*))\,\mathrm{d}g(t) \right\| = \left\| \int_{u_1^*}^{u_2^*} (f(y_t, t) - f(z_t, t))\,\Delta t \right\|$$
$$\le L \int_{u_1^*}^{u_2^*} \|y_t - z_t\|_\infty \Delta t = L \int_{u_1^*}^{u_2^*} \|y_t - z_t\|_\infty \,\mathrm{d}g(t) = L \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}g(t). \qquad \square$$

6 Existence-uniqueness theorems

In this section, we present results on local existence and uniqueness of solutions for impulsive measure functional differential equations and impulsive functional dynamic equations on time scales.

Our main tools in the proofs of these results are the correspondence between measure functional differential equations and impulsive measure functional differential equations presented in Section 3 (see Theorem 3.1), and the relation between this last type of equations and impulsive functional dynamic equations on time scales (see Theorem 5.2). We also make use of the existence-uniqueness theorem for measure functional differential equations, which was proved in [6].

Theorem 6.1. *Assume that $X = G([t_0-r, t_0+\sigma], \mathbb{R}^n)$, $B \subset \mathbb{R}^n$ is an open set, $O = G([t_0-r, t_0+\sigma], B)$, $P = G([-r, 0], B)$, $m \in \mathbb{N}$, $t_0 \le t_1 < t_2 < \ldots < t_m < t_0 + \sigma$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function which is continuous at $t_1, \ldots, t_m$. Also, suppose that $I_1, \ldots, I_m : B \to \mathbb{R}^n$ and $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.*
2. *There exists a constant $M_1 > 0$ such that*
$$\left\|\int_{u_1}^{u_2} f(y_t, t)\,\mathrm{d}g(t)\right\| \le M_1(g(u_2) - g(u_1))$$
whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ and $y \in O$.
3. *There exists a constant $L_1 > 0$ such that*
$$\left\|\int_{u_1}^{u_2} (f(y_t, t) - f(z_t, t))\,\mathrm{d}g(t)\right\| \le L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \,\mathrm{d}g(t)$$
whenever $t_0 \le u_1 \le u_2 \le t_0 + \sigma$ and $y, z \in O$.
4. *There exists a constant $M_2 > 0$ such that*
$$\|I_k(x)\| \le M_2$$
for every $k \in \{1, \ldots, m\}$ and $x \in B$.
5. *There exists a constant $L_2 > 0$ such that*
$$\|I_k(x) - I_k(y)\| \le L_2\|x - y\|$$
for every $k \in \{1, \ldots, m\}$ and x, $y \in B$.

Let $\phi \in P$ and assume that either $t_0 < t_1$ and $\phi(0) + f(\phi, t_0)\Delta^+ g(t_0) \in B$, or $t_0 = t_1$ and $\phi(0) + I_1(\phi(0)) \in B$. Then there exists $\delta > 0$ and a function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ which is a unique solution of the impulsive measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0, t_0 + \delta], \\ y_{t_0} &= \phi. \end{aligned} \tag{6.1}$$

Proof. For every $y \in P$, define

$$\tilde{f}(y, t) = \begin{cases} f(y, t), & t \in [t_0, t_0 + \sigma]\backslash\{t_1, \ldots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1, \ldots, m\}. \end{cases}$$

Moreover, let the function $\tilde{g} : [t_0, t_0 + \sigma] \to \mathbb{R}$ be given by

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1, \ldots, m - 1\}, \\ g(t) + m, & t \in (t_m, t_0 + \sigma]. \end{cases}$$

Since g is nondecreasing and left-continuous, $\tilde{g}$ has the same properties.

We have either $t_0 < t_1$ and $\phi(0) + \tilde{f}(\phi, t_0)\Delta^+\tilde{g}(t_0) = \phi(0) + f(\phi, t_0)\Delta^+ g(t_0) \in B$, or $t_0 = t_1$ and $\phi(0) + \tilde{f}(\phi, t_0)\Delta^+\tilde{g}(t_0) = \phi(0) + I_1(\phi(0)) \in B$.

Using these facts and Lemma 3.3, we see that the functions $\tilde{f}$, $\tilde{g}$, and ϕ satisfy all hypotheses of the existence and uniqueness theorem for measure functional differential equations (see Theorem 5.3 and Remark 3.11 in [6]). Consequently, there exist $\delta > 0$ and a function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ which is a unique solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} \tilde{f}(y_s, s)\,\mathrm{d}\tilde{g}(s), \\ y_{t_0} &= \phi. \end{aligned}$$

Finally, by Theorem 3.1, the function y is also a unique solution of (6.1) on $[t_0 - r, t_0 + \delta]$. □

In the sequel, we prove a result on local existence and uniqueness of solutions of impulsive functional dynamic equations on time scales.

Theorem 6.2. *Assume that $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$ is a time scale interval, $t_0 \in \mathbb{T}$, $B \subset \mathbb{R}^n$ is an open set, $O = G([t_0 - r, t_0 + \sigma], B)$, $P = G([-r, 0], B)$, $m \in \mathbb{N}$, $t_1, \dots, t_m \in [t_0, t_0 + \sigma]_{\mathbb{T}}$ are right-dense points such that $t_0 \le t_1 < \dots < t_m < t_0 + \sigma$. Let $f : P \times [t_0, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$ and $I_1, \dots, I_m : B \to \mathbb{R}^n$ be functions which satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\Delta t$ exists for every $y \in O$.*
2. *There exists a constant $M_1 > 0$ such that*
$$\left\| \int_{u_1}^{u_2} f(y_t, t)\Delta t \right\| \le M_1(u_2 - u_1)$$
for every $y \in O$ and $u_1, u_2 \in [t_0, t_0 + \sigma]_{\mathbb{T}}$, $u_1 \le u_2$.
3. *There exists a constant $L_1 > 0$ such that*
$$\left\| \int_{u_1}^{u_2} \left(f(y_t, t) - f(z_t, t)\right) \Delta t \right\| \le L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \Delta t$$
for every $y, z \in O$ and $u_1, u_2 \in [t_0, t_0 + \sigma]_{\mathbb{T}}$, $u_1 \le u_2$.
4. *There exists a constant $M_2 > 0$ such that*
$$\|I_k(y)\| \le M_2$$
for every $k \in \{1, \dots, m\}$ and $y \in B$.
5. *There exists a constant $L_2 > 0$ such that*
$$\|I_k(x) - I_k(y)\| \le L_2\|x - y\|$$
for every $k \in \{1, \dots, m\}$ and $x, y \in B$.

Let $\phi : [t_0 - r, t_0]_{\mathbb{T}} \to B$ be a regulated function such that either $t_0 < t_1$ and $\phi(t_0) + f(\phi^*_{t_0}, t_0)\mu(t) \in B$, or $t_0 = t_1$ and $\phi(t_0) + I_1(\phi(t_0)) \in B$. Then there exist a $\delta > 0$ such that $\delta \ge \mu(t_0)$ and $t_0 + \delta \in \mathbb{T}$, and a function $y : [t_0 - r, t_0 + \delta]_{\mathbb{T}} \to B$ which is a unique solution of the impulsive functional dynamic equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y^*_s, s)\,\Delta s + \sum_{\substack{k \in \{1,\dots,m\},\\ t_k < t}} I_k(y(t_k)), \quad t \in [t_0, t_0 + \delta]_{\mathbb{T}}, \\ y(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

Proof. Let $g(t) = t^*$ and $f^*(y,t) = f(y,t^*)$ for every $t \in [t_0, t_0+\sigma]$ and $y \in P$. Note that $\Delta^+ g(t_0) = \mu(t_0)$. Using the hypotheses and Lemma 5.3, we see that the functions f^*, g and $\phi^*_{t_0}$ satisfy all assumptions of Theorem 6.1. Consequently, there exist $\delta > 0$ and a function $u : [t_0 - r, t_0 + \delta] \to B$ which is a unique solution of

$$\begin{aligned} u(t) &= u(t_0) + \int_{t_0}^{t} f^*(u_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<t}} I_k(u(t_k)), \quad t \in [t_0, t_0+\delta], \\ u_{t_0} &= \phi^*_{t_0}. \end{aligned}$$

Then, by Theorem 5.2, $u = y^*$, where $y : [t_0 - r, t_0 + \delta]_{\mathbb{T}} \to B$ is a solution of

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y^*_s, s)\,\Delta s + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0, t_0+\delta]_{\mathbb{T}}, \\ y(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

Without loss of generality, we can assume that $\delta \geq \mu(t_0)$; otherwise, t_0 is right-scattered, $t_0 < t_1$, and we can let

$$y(\sigma(t_0)) = \phi(t_0) + f(\phi^*_{t_0}, t_0)\mu(t_0)$$

to obtain a solution defined on $[t_0 - r, t_0 + \mu(t_0)]_{\mathbb{T}}$. Again, by Theorem 5.2, the solution y is unique. □

7 Continuous dependence results

In our paper [6], we have obtained a continuous dependence theorem for measure functional differential equations. Since we already know that impulsive functional differential and dynamic equations are in fact special cases of measure functional different equations, we can use the existing result from [6] to derive continuous dependence theorems for both types of impulsive equations; this is the content of the present section.

Theorem 7.1. *Assume that $X = G([t_0-r, t_0+\sigma], \mathbb{R}^n)$, $B \subset \mathbb{R}^n$ is an open set, $O = G([t_0-r, t_0+\sigma], B)$, $P = G([-r, 0], B)$, $m \in \mathbb{N}$, $t_0 \leq t_1 < t_2 < \ldots < t_m < t_0 + \sigma$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing left-continuous function which is continuous at $t_1, \ldots, t_m$. Finally, let $f_p : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $p \in \mathbb{N}_0$, and $I_1^p, \ldots, I_m^p : B \to \mathbb{R}^n$, $p \in \mathbb{N}_0$, be functions which satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f_p(y_t, t)\,\mathrm{d}g(t)$ exists for every $p \in \mathbb{N}_0$, $y \in O$.*
2. *There exists a constant $M_1 > 0$ such that*

   $$\left\| \int_{u_1}^{u_2} f_p(y_t, t)\,\mathrm{d}g(t) \right\| \leq M_1(g(u_2) - g(u_1))$$

   whenever $p \in \mathbb{N}$, $t_0 \leq u_1 \leq u_2 \leq t_0 + \sigma$ and $y \in O$.
3. *There exists a constant $L_1 > 0$ such that*

   $$\left\| \int_{u_1}^{u_2} (f_p(y_t, t) - f_p(z_t, t))\,\mathrm{d}g(t) \right\| \leq L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \mathrm{d}g(t)$$

   whenever $p \in \mathbb{N}$, $t_0 \leq u_1 \leq u_2 \leq t_0 + \sigma$ and $y, z \in O$.
4. *For every $y \in O$,*

   $$\lim_{p\to\infty} \int_{t_0}^{t} f_p(y_s, s)\,\mathrm{d}g(s) = \int_{t_0}^{t} f_0(y_s, s)\,\mathrm{d}g(s)$$

   uniformly with respect to $t \in [t_0, t_0+\sigma]$.

5. *There exists a constant* $M_2 > 0$ *such that*

$$\|I_k^p(x)\| \le M_2$$

for every $k \in \{1,\dots,m\}$, $p \in \mathbb{N}_0$ *and* $x \in B$.

6. *There exists a constant* $L_2 > 0$ *such that*

$$\|I_k^p(x) - I_k^p(y)\| \le L_2\|x-y\|$$

for every $k \in \{1,\dots,m\}$, $p \in \mathbb{N}_0$ *and* $x, y \in B$.

7. *For every* $y \in B$ *and* $k \in \{1,\dots,m\}$, $\lim_{p\to\infty} I_k^p(y) = I_k^0(y)$.

Consider functions $\phi_p \in P$, $p \in \mathbb{N}_0$, *such that* $\lim_{p\to\infty}\phi_p = \phi_0$ *uniformly on* $[-r,0]$. *Let* $y_p \in O$, $p \in \mathbb{N}$, *be solutions of*

$$y_p(t) = y_p(t_0) + \int_{t_0}^t f_p((y_p)_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^p(y_p(t_k)), \quad t \in [t_0, t_0+\sigma], \tag{7.1}$$

$$(y_p)_{t_0} = \phi_p, \tag{7.2}$$

such that $\lim_{p\to\infty} y_p = y_0 \in O$. *Then* y_0 *satisfies*

$$y_0(t) = y_0(t_0) + \int_{t_0}^t f_0((y_0)_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^0(y_0(t_k)), \quad t \in [t_0, t_0+\sigma], \tag{7.3}$$

$$(y_0)_{t_0} = \phi_0. \tag{7.4}$$

Proof. We already know that (7.1) and (7.2) imply

$$\begin{aligned} y_p(t) &= y_p(t_0) + \int_{t_0}^t \tilde f_p((y_p)_s, s)\,\mathrm{d}\tilde g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_p)_{t_0} &= \phi_p, \end{aligned}$$

where the construction of $\tilde f_p$ and $\tilde g$ is described in Theorem 3.1. Since g is nondecreasing and left-continuous, $\tilde g$ posseses the same properties. For every $t \in [t_0, t_0+\sigma]$, we have

$$\lim_{p\to\infty}\int_{t_0}^t \tilde f_p(y_s, s)\,\mathrm{d}\tilde g(s) = \lim_{p\to\infty}\int_{t_0}^t f_p(y_s, s)\,\mathrm{d}g(s) + \lim_{p\to\infty}\sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^p(y(t_k)) =$$

$$= \int_{t_0}^t f_0(y_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^0(y(t_k)) = \int_{t_0}^t \tilde f_0(y_s, s)\,\mathrm{d}\tilde g(s),$$

where the convergence is uniform with respect to $t \in [t_0, t_0+\sigma]$.

By the continuous dependence theorem for measure functional differential equations (see Theorem 6.3 and Remark 3.11 in [6]), it follows that

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^t \tilde f_0((y_0)_s, s)\,\mathrm{d}\tilde g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

The proof is finished by applying Theorem 3.1, which implies that y_0 satisfies (7.3) and (7.4). □

The second result in this section is a continuous dependence theorem for impulsive functional dynamic equations on time scales.

Theorem 7.2. *Assume that $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$ is a time scale interval, $t_0 \in \mathbb{T}$, $B \subset \mathbb{R}^n$ is an open set, $O = G([t_0 - r, t_0 + \sigma], B)$, $P = G([-r, 0], B)$, $m \in \mathbb{N}$, $t_1, \dots, t_m \in [t_0, t_0 + \sigma]_{\mathbb{T}}$ are right-dense points such that $t_0 \le t_1 < \dots < t_m < t_0 + \sigma$. Let $f_p : P \times [t_0, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$, $p \in \mathbb{N}_0$, and $I_1^p, \dots, I_m^p : B \to \mathbb{R}^n$, $p \in \mathbb{N}_0$, be functions which satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f_p(y_t, t)\Delta t$ exists for every $y \in O$ and $p \in \mathbb{N}_0$.*
2. *There exists a constant $M_1 > 0$ such that*
$$\left\| \int_{u_1}^{u_2} f_p(y_t, t)\Delta t \right\| \le M_1(u_2 - u_1)$$
for every $p \in \mathbb{N}$, $y \in O$ and $u_1, u_2 \in [t_0, t_0 + \sigma]_{\mathbb{T}}$, $u_1 \le u_2$.
3. *There exists a constant $L_1 > 0$ such that*
$$\left\| \int_{u_1}^{u_2} \left(f_p(y_t, t) - f_p(z_t, t)\right)\Delta t \right\| \le L_1 \int_{u_1}^{u_2} \|y_t - z_t\|_\infty \Delta t$$
for every $p \in \mathbb{N}$, $y, z \in O$ and $u_1, u_2 \in [t_0, t_0 + \sigma]_{\mathbb{T}}$, $u_1 \le u_2$.
4. *For every $y \in O$,*
$$\lim_{p\to\infty} \int_{t_0}^{t} f_p(y_s, s)\Delta s = \int_{t_0}^{t} f_0(y_s, s)\Delta s$$
uniformly with respect to $t \in [t_0, t_0 + \sigma]_{\mathbb{T}}$.
5. *There exists a constant $M_2 > 0$ such that*
$$\|I_k^p(x)\| \le M_2$$
for every $k \in \{1, \dots, m\}$, $p \in \mathbb{N}_0$ and $x \in B$.
6. *There exists a constant $L_2 > 0$ such that*
$$\|I_k^p(x) - I_k^p(y)\| \le L_2\|x - y\|$$
for every $k \in \{1, \dots, m\}$, $p \in \mathbb{N}_0$ and $x, y \in B$.
7. *For every $x \in B$ and $k \in \{1, \dots, m\}$, $\lim_{p\to\infty} I_k^p(x) = I_k^0(x)$.*

Assume that $\phi_p \in G([t_0 - r, t_0]_{\mathbb{T}}, B)$, $p \in \mathbb{N}_0$, is a sequence of functions such that $\lim_{p\to\infty} \phi_p = \phi_0$ uniformly on $[t_0 - r, t_0]_{\mathbb{T}}$. Let $y_p : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$, $p \in \mathbb{N}$ be solutions of

$$\begin{aligned} y_p(t) &= y_p(t_0) + \int_{t_0}^{t} f_p((y_p^*)_s, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^p(y_p(t_k)), \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}}, \\ y_p(t) &= \phi_p(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

If there exists a function $y_0 : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ such that $\lim_{p\to\infty} y_p = y_0$, then y_0 satisfies

$$y_0(t) = y_0(t_0) + \int_{t_0}^{t} f_0((y_0^*)_s, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^0(y_0(t_k)), \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}}, \tag{7.5}$$

$$y_0(t) = \phi_0(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \tag{7.6}$$

Proof. Let $g(t) = t^*$ for every $t \in [t_0, t_0 + \sigma]$; then g is a left-continuous nondecreasing function which is continuous at $t_1, \dots, t_m$. Further, let $f_p^*(y,t) = f_p(y,t^*)$ for every $p \in \mathbb{N}_0$, $y \in P$ and $t \in [t_0, t_0 + \sigma]$. By Lemma 5.3, the integral $\int_{t_0}^{t_0+\sigma} f_p^*(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$ and $p \in \mathbb{N}_0$. By Theorems 4.5 and 5.1, we obtain

$$\lim_{p\to\infty} \int_{t_0}^{t} f_p^*(y_s, s)\,\mathrm{d}g(s) = \lim_{p\to\infty} \int_{t_0}^{t^*} f_p(y_s, s)\Delta s = \int_{t_0}^{t^*} f_0(y_s, s)\Delta s$$

$$= \int_{t_0}^{t} f_0(y_{s^*}, s^*)\,\mathrm{d}g(s) = \int_{t_0}^{t} f_0^*(y_s, s)\,\mathrm{d}g(s),$$

where the convergence is uniform with respect to $t \in [t_0, t_0 + \sigma]$.

Further, it is clear that $\lim_{p\to\infty} y_p^* = y_0^*$ on $[t_0, t_0 + \sigma]$, and $\lim_{p\to\infty} \phi_p^* = \phi_0^*$ uniformly on $[t_0 - r, t_0]$. By Theorem 5.2, we have

$$\begin{aligned} y_p^*(t) &= y_p^*(t_0) + \int_{t_0}^{t} f_p^*((y_p^*)_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^p(y_p^*(t_k)), \quad t \in [t_0, t_0 + \sigma], \\ (y_p^*)_{t_0} &= (\phi_p^*)_{t_0}. \end{aligned}$$

for every $p \in \mathbb{N}$. Using Lemma 5.3, we see that all hypotheses of Theorem 7.1 are satisfied. Consequently,

$$\begin{aligned} y_0^*(t) &= y_0^*(t_0) + \int_{t_0}^{t} f_0^*((y_0^*)_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k^0(y_0^*(t_k)), \quad t \in [t_0, t_0 + \sigma], \\ (y_0^*)_{t_0} &= (\phi_0^*)_{t_0}. \end{aligned}$$

By Theorem 5.2, it follows that y_0 satisfies (7.5) and (7.6). □

Remark 7.3. According to Remark 6.6 in [6], the assumptions of Theorem 7.1 might be modified in the following way: Instead of requiring the existence of a function $y_0 \in O$ such that $\lim_{k\to\infty} y_k = y_0$, it is enough to assume the existence of a closed set $B' \subset B$ such that the functions y_k, $k \in \mathbb{N}$, take values in B'. Under this hypothesis, the conclusion is that $\{y_k\}_{k=1}^{\infty}$ has a subsequence which is uniformly convergent to a function $y_0 \in O$ such that

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f_0((y_0)_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

Theorem 7.2 can be modified in a similar way.

8 Periodic averaging theorems

The basic idea behind averaging theorems is that one can approximate solutions of a non-autonomous equation by solutions of an autonomous equation whose right-hand side corresponds to the average of the original right-hand side. The method is quite general and can be applied to many types of equations (see e.g. [12]); it is especially powerful in the case when the original right-hand side is periodic in t.

In this section, we use an existing periodic averaging theorem for measure functional differential equations to obtain periodic averaging theorems for functional differential and dynamic equations involving impulses.

Theorem 8.1. *Assume that $\varepsilon_0 > 0$, $L > 0$, $B \subset \mathbb{R}^n$, $X = G([-r, 0], B)$, $m \in \mathbb{N}$ and $0 \le t_1 < t_2 < \dots < t_m < T$. Consider a pair of bounded functions $f : X \times [0, \infty) \to \mathbb{R}^n$, $g : X \times [0, \infty) \times (0, \varepsilon_0] \to \mathbb{R}^n$ and a nondecreasing left-continuous function $h : [0, \infty) \to \mathbb{R}$ which is continuous at $t_1, \dots, t_m$. Let $I_k : B \to \mathbb{R}^n$, $k \in \{1, 2, \dots, m\}$ be bounded and Lipschitz-continuous functions. For every integer $k > m$, define t_k and I_k by the recursive formulas $t_k = t_{k-m} + T$ and $I_k = I_{k-m}$. Suppose that the following conditions are satisfied:*

1. *The integral* $\int_0^b f(y_t,t)\,\mathrm{d}h(t)$ *exists for every* $b>0$ *and* $y \in G([-r,b],B)$.
2. f *is Lipschitz-continuous with respect to the first variable.*
3. f *is* T*-periodic in the second variable.*
4. *There is a constant* $\alpha>0$ *such that* $h(t+T)-h(t)=\alpha$ *for every* $t \geq 0$.
5. *The integral*

$$f_0(x)=\frac{1}{T}\int_0^T f(x,s)\,\mathrm{d}h(s)$$

exists for every $x \in X$.

Denote

$$I_0(y)=\frac{1}{T}\sum_{k=1}^{m} I_k(y), \quad y \in B.$$

Let $\phi \in X$ *and suppose for every* $\varepsilon \in (0,\varepsilon_0]$, *the initial value problems*

$$\begin{aligned} x(t) &= x(0)+\varepsilon\int_0^t f(x_s,s)\,\mathrm{d}h(s)+\varepsilon^2\int_0^t g(x_s,s,\varepsilon)\,\mathrm{d}h(s)+\varepsilon\sum_{\substack{k\in\mathbb{N},\\ t_k<t}} I_k(x(t_k)),\\ x_0 &= \phi,\\ y(t) &= y(0)+\varepsilon\int_0^t (f_0(y_s)+I_0(y(s)))\,\mathrm{d}s,\\ y_0 &= \phi \end{aligned}$$

have solutions x^ε, $y^\varepsilon : [-r,L/\varepsilon] \to B$. *Then there exists a constant* $J>0$ *such that*

$$\|x^\varepsilon(t)-y^\varepsilon(t)\| \leq J\varepsilon$$

for every $\varepsilon \in (0,\varepsilon_0]$ *and* $t \in [0,L/\varepsilon]$.

Proof. Define the function $\tilde{h}(t):[0,\infty)\to\mathbb{R}$ by

$$\tilde{h}(t)=\begin{cases} h(t), & t\in[0,t_1],\\ h(t)+k, & t\in(t_k,t_{k+1}] \text{ for some } k\in\mathbb{N}. \end{cases}$$

Note that $\Delta^+\tilde{h}(t_k)=1$ for every $k\in\mathbb{N}$, $\tilde{h}$ is nondecreasing, left-continuous, and for every $t\geq 0$, we have $\tilde{h}(t+T)-\tilde{h}(t)=\tilde{\alpha}$, where $\tilde{\alpha}=h(t+T)-h(t)+m=\alpha+m$.

By the assumptions, it follows that

$$x^\varepsilon(t)=x^\varepsilon(0)+\int_0^t \left(\varepsilon f(x_s^\varepsilon,s)+\varepsilon^2 g(x_s^\varepsilon,s,\varepsilon)\right)\mathrm{d}h(s)+\sum_{\substack{k\in\mathbb{N},\\ t_k<t}}\varepsilon I_k(x^\varepsilon(t_k))$$

for every $\varepsilon\in(0,\varepsilon_0]$ and $t\in[0,L/\varepsilon]$. Let

$$F^\varepsilon(y,t)=\begin{cases} \varepsilon f(y,t)+\varepsilon^2 g(y,t,\varepsilon), & t\notin\{t_1,t_2,\ldots\},\\ \varepsilon I_k(y(0)), & t=t_k \text{ for some } k\in\mathbb{N} \end{cases}$$

for every $y\in X$ and $t\geq 0$. By Theorem 3.1, we obtain

$$x^\varepsilon(t)=x^\varepsilon(0)+\int_0^t F^\varepsilon(x_s^\varepsilon,s)\,\mathrm{d}\tilde{h}(s) \tag{8.1}$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L/\varepsilon]$. For every $y \in X$ and $t \geq 0$, we have

$$F^\varepsilon(y,t) = \varepsilon \tilde{f}(y,t) + \varepsilon^2 \tilde{g}(y,t,\varepsilon), \tag{8.2}$$

where

$$\tilde{f}(y,t) = \begin{cases} f(y,t), & t \notin \{t_1, t_2, \ldots\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \mathbb{N} \end{cases}$$

and

$$\tilde{g}(y,t,\varepsilon) = \begin{cases} g(y,t,\varepsilon), & t \notin \{t_1, t_2, \ldots\}, \\ 0, & t = t_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

It follows from (8.1) and (8.2) that for every $\varepsilon \in (0, \varepsilon_0]$, the function $x^\varepsilon : [-r, L/\varepsilon] \to B$ is a solution of the initial value problem

$$\begin{aligned} x(t) &= x(0) + \varepsilon \int_0^t \tilde{f}(x_s, s) \, \mathrm{d}\tilde{h}(s) + \varepsilon^2 \int_0^t \tilde{g}(x_s, s, \varepsilon) \, \mathrm{d}\tilde{h}(s), \\ x_0 &= \phi. \end{aligned}$$

The function $\tilde{f}$ is Lipschitz-continuous with respect to the first variable and T-periodic in the second variable. Using Lemma 2.4, we have

$$\int_0^T \tilde{f}(x,s) \, \mathrm{d}\tilde{h}(s) = \int_0^T f(x,s) \, \mathrm{d}h(s) + \sum_{k=1}^m \tilde{f}(x,t_k) \Delta^+ \tilde{h}(t_k) = \int_0^T f(x,s) \, \mathrm{d}h(s) + \sum_{k=1}^m I_k(x(0))$$

for every $x \in X$. Consequently, the function

$$\tilde{f}_0(x) = \frac{1}{T} \int_0^T \tilde{f}(x,s) \, \mathrm{d}\tilde{h}(s), \quad x \in X,$$

satisfies

$$\tilde{f}_0(x) = \frac{1}{T} \int_0^T f(x,s) \, \mathrm{d}h(s) + \frac{1}{T} \sum_{k=1}^m I_k(x(0)) = f_0(x) + I_0(x(0)), \quad x \in X.$$

By the periodic averaging theorem for measure functional differential equations (see Theorem 13 in [12]), there is a constant $J > 0$ such that $\|x^\varepsilon(t) - y^\varepsilon(t)\| \leq J\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L/\varepsilon]$. □

We now proceed to the periodic averaging theorem for impulsive functional dynamic equations on time scales.

Definition 8.2. Let $T > 0$ be a real number. A time scale $\mathbb{T}$ is called T-periodic if $t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$ and $\mu(t) = \mu(t + T)$.

Theorem 8.3. *Assume that $\mathbb{T}$ is a T-periodic time scale, $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$ a time scale interval, $t_0 \in \mathbb{T}$, $\varepsilon_0 > 0$, $L > 0$, $B \subset \mathbb{R}^n$, $X = G([-r,0], B)$, $m \in \mathbb{N}$, $t_1, \ldots, t_m \in \mathbb{T}$ are right-dense points such that $t_0 \leq t_1 < t_2 < \cdots < t_m < t_0 + T$. Let $I_k : B \to \mathbb{R}^n$, $k \in \{1, 2, \ldots, m\}$ be bounded and Lipschitz-continuous functions. For every integer $k > m$, define t_k and I_k by the recursive formulas $t_k = t_{k-m} + T$ and $I_k = I_{k-m}$. Consider a pair of bounded functions $f : X \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$, $g : X \times [t_0, \infty)_{\mathbb{T}} \times (0, \varepsilon_0] \to \mathbb{R}^n$ such that the following conditions are satisfied:*

1. *The integral $\int_0^b f(y_t, t) \Delta t$ exists for every $b > 0$ and $y \in G([-r,b], B)$.*

2. *f is Lipschitz-continuous with respect to the first variable.*

3. *f is T-periodic in the second variable.*
4. *The integral*
$$f_0(x) = \frac{1}{T}\int_{t_0}^{t_0+T} f(x,s)\Delta s$$
exists for every $x \in X$.

Denote
$$I_0(y) = \frac{1}{T}\sum_{k=1}^{m} I_k(y), \quad y \in B.$$
Let $\phi \in G([t_0-r,t_0]_{\mathbb{T}},B)$ and suppose for every $\varepsilon \in (0,\varepsilon_0]$, the initial value problems
$$\begin{aligned} x(t) &= x(t_0) + \varepsilon\int_{t_0}^{t} f(x_s^*,s)\Delta s + \varepsilon^2\int_{t_0}^{t} g(x_s^*,s,\varepsilon)\Delta s + \varepsilon\sum_{\substack{k\in\mathbb{N},\\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0,t_0+L/\varepsilon]_{\mathbb{T}},\\ x(t) &= \phi(t), \quad t \in [t_0-r,t_0]_{\mathbb{T}},\\ y(t) &= y(t_0) + \varepsilon\int_{t_0}^{t}(f_0(y_s)+I_0(y(s)))\,\mathrm{d}s,\\ y_{t_0} &= \phi_{t_0}^* \end{aligned}$$
have solutions $x^\varepsilon : [t_0-r,t_0+L/\varepsilon]_{\mathbb{T}} \to B$ and $y^\varepsilon : [t_0-r,t_0+L/\varepsilon] \to B$, respectively. Then there exists a constant $J > 0$ such that
$$\|x^\varepsilon(t) - y^\varepsilon(t)\| \le J\varepsilon,$$
for every $\varepsilon \in (0,\varepsilon_0]$ and $t \in [t_0,t_0+L/\varepsilon]_{\mathbb{T}}$.

Proof. Without loss of generality, we can assume that $t_0 = 0$; otherwise, consider a shifted problem with the time scale $\widetilde{\mathbb{T}} = \{t-t_0; t \in \mathbb{T}\}$ and functions $\widetilde{f}(x,t) = f(x,t+t_0)$ and $\widetilde{g}(x,t,\varepsilon) = g(x,t+t_0,\varepsilon)$.

For every $t \in [t_0,\infty)$, $x \in X$ and $\varepsilon \in (0,\varepsilon_0]$, let
$$f^*(x,t) = f(x,t^*) \quad \text{and} \quad g^*(x,t,\varepsilon) = g(x,t^*,\varepsilon).$$
Also, let $h(t) = t^*$ for every $t \in [t_0,\infty)$. Since $\mathbb{T}$ is T-periodic, it follows that
$$h(t+T) - h(t) = T, \quad t \ge 0.$$
From Theorem 4.5, we obtain
$$f_0(x) = \frac{1}{T}\int_0^T f(x,s)\Delta s = \frac{1}{T}\int_0^T f^*(x,s)\,\mathrm{d}h(s)$$
for every $x \in X$.

For every $b > 0$ and $y \in G([-r,b],B)$, the integral $\int_0^b f(y_t,t)\Delta t$ exists. Then, Theorems 4.5 and 5.1 imply
$$\int_0^b f(y_t,t)\Delta t = \int_0^b f(y_{t^*},t^*)\,\mathrm{d}h(t) = \int_0^b f(y_t,t^*)\,\mathrm{d}h(t) = \int_0^b f^*(y_t,t)\,\mathrm{d}h(t),$$
i.e. the last integral exists as well.

It follows from Theorem 5.2 that for every $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0,L/\varepsilon]$, we have
$$\begin{aligned} (x^\varepsilon)^*(t) &= (x^\varepsilon)^*(0) + \varepsilon\int_0^t f^*((x^\varepsilon)_s^*,s)\,\mathrm{d}h(s) + \varepsilon^2\int_0^t g^*((x^\varepsilon)_s^*,s,\varepsilon)\,\mathrm{d}h(s) + \varepsilon\sum_{\substack{k\in\mathbb{N},\\ t_k<t}} I_k((x^\varepsilon)^*(t_k)),\\ (x^\varepsilon)_0^* &= \phi_0^*. \end{aligned}$$
Finally, by Theorem 8.1, there exists a constant $J > 0$ such that $\|(x^\varepsilon)^*(t) - y^\varepsilon(t)\| \le J\varepsilon$ for every $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0,L/\varepsilon]$. We conclude the proof by observing that $(x^\varepsilon)^*(t) = x^\varepsilon(t)$ for $t \in [0,L/\varepsilon]_{\mathbb{T}}$. □

9 Conclusion

In the present paper, we demonstrated how the existing theory of measure functional differential equations can be helpful in the study of impulsive functional differential equations. Moreover, we showed that impulsive functional dynamic equations on time scales represent a special type of measure functional differential equations.

Using known results for measure functional differential equations, we were able to prove theorems concerning the existence and uniqueness of solutions, continuous dependence of solutions on the right-hand side, and periodic averaging for impulsive functional differential and dynamic equations.

These facts demonstrate that the study of measure functional differential equations is fully justified, since they allow us to deal with other types of equations in a unified way.

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