# Measure functional differential equations and functional dynamic equations on time scales

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**Abstract**

We study measure functional differential equations and clarify their relation to generalized ordinary differential equations. We show that functional dynamic equations on time scales represent a special case of measure functional differential equations. For both types of equations, we obtain results on the existence and uniqueness of solutions, continuous dependence, and periodic averaging.

**Keywords:** Measure functional differential equations, functional dynamic equations, generalized ordinary differential equations, existence and uniqueness, continuous dependence on a parameter, periodic averaging

**MSC 2010 subject classification:** 34K05, 34K45, 34N05, 34K33

## 1 Introduction

Let $r$, $\sigma > 0$ be given numbers and $t_0 \in \mathbb{R}$. The theory of retarded functional differential equations (see e.g. [10]) is a branch of the theory of functional differential equations concerned with problems of the form

$$x'(t) = f(x_t, t), \quad t \in [t_0, t_0 + \sigma],$$

where $f : \Omega \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, $\Omega \subset C([-r, 0], \mathbb{R}^n)$, and $x_t$ is given by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, for every $t \in [t_0, t_0 + \sigma]$. The equivalent integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}s, \quad t \in [t_0, t_0 + \sigma],$$

where the integral can be considered, for instance, in the sense of Riemann, Lebesgue or Henstock-Kurzweil.

In this paper, we focus our attention on more general problems of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{1.1}$$

where the integral on the right-hand side is the Kurzweil-Stieltjes integral with respect to a nondecreasing function $g$ (see the next section). We call these equations measure functional differential equations. As explained in [3], equation (1.1) is equivalent to

$$\mathrm{D}x = f(x_t, t)\mathrm{D}g,$$

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where D$x$ and D$g$ denote the distributional derivatives of the functions $x$ and $g$ in the sense of L. Schwartz.

While the theories of measure differential equations (including the more general abstract measure differential equations) and measure delay differential equations are well-developed (see e.g. [3], [4], [5], [13], [21]), the literature concerning measure functional differential equations is scarce. We start our exposition by describing the relation between measure functional differential equations and generalized ordinary differential equations, which were introduced by J. Kurzweil in 1957 (see [15]). The idea of converting functional differential equations to generalized ordinary differential equations first appeared in the papers [12], [18] written by C. Imaz, F. Oliva, and Z. Vorel. It was later generalized by M. Federson and Š. Schwabik in the paper [6], which is devoted to impulsive retarded functional differential equations. Using the existing theory of generalized ordinary differential equations, we obtain results on the existence and uniqueness of a solution and on the continuous dependence of a solution on parameters for measure functional differential equations.

Both differential equations and difference equations are important special cases of measure differential equations. Another unification of continuous-time and discrete-time equations is provided by the theory of dynamic equations on time scales, which has its roots in the work of S. Hilger (see [11]), and became increasingly popular during the past two decades (see [1], [2]). In [22], A. Slavík established the relation between dynamic equations on time scales, measure differential equations, and generalized ordinary differential equations. In the present paper, we extend this correspondence to functional dynamic equations on time scales and measure functional differential equations. Using this relation, we obtain theorems on the existence and uniqueness of a solution, continuous dependence of a solution on a parameter, and a periodic averaging theorem for functional dynamic equations on time scales.

Our paper is organized as follows. The second section introduces fundamental concepts and basic results of the Kurzweil integration theory. In the third section, using some ideas from [6], we present the correspondence between measure functional differential equations and generalized ordinary differential equations. We also explain that impulsive functional differential equations represent a special case of measure functional differential equations. The fourth section is devoted to functional dynamic equations on time scales. In this section, we establish the relation between functional dynamic equations on time scales and measure functional differential equations. In the fifth section, we present theorems concerning existence and uniqueness of a solution. In the sixth section, we prove continuous dependence results. Finally, a periodic averaging theorem for functional dynamic equations on time scales is presented in the last section.

## 2 Kurzweil integration

Throughout this paper, we use the following definition of the integral introduced by J. Kurzweil in [15]. Consider a function $\delta : [a,b] \to \mathbb{R}^+$ (called a gauge on $[a,b]$). A tagged partition of the interval $[a,b]$ with division points $a = s_0 \le s_1 \le \cdots \le s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1,\ldots,k$, is called $\delta$-fine if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i = 1,\ldots,k.$$

Let $X$ be a Banach space. A function $U : [a,b] \times [a,b] \to X$ is called Kurzweil integrable on $[a,b]$, if there is an element $I \in X$ such that given $\varepsilon > 0$, there is a gauge $\delta$ on $[a,b]$ such that

$$\left\| \sum_{i=1}^{k} (U(\tau_i, s_i) - U(\tau_i, s_{i-1})) - I \right\| < \varepsilon$$

for every $\delta$-fine tagged partition of $[a,b]$. In this case, $I$ is called the Kurzweil integral of $U$ over $[a,b]$ and will be denoted by $\int_a^b DU(\tau,t)$.

This definition generalizes the well-known Henstock-Kurzweil integral of a function $f : [a,b] \to X$, which is obtained by setting $U(\tau,t) = f(\tau)t$. The Kurzweil-Stieljtes integral of a function $f : [a,b] \to X$

with respect to a function $g:[a,b]\to\mathbb{R}$, which appears in the definition of a measure functional differential equation, corresponds to the choice $U(\tau,t)=f(\tau)g(t)$ and will be denoted by $\int_a^b f(s)\,\mathrm{d}g(s)$ or simply $\int_a^b f\,\mathrm{d}g$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals etc. More information can be found in the book [19].

A function $f:[a,b]\to X$ is called regulated, if the limits

$$\lim_{s\to t-} f(s)=f(t-),\quad t\in(a,b],\quad\text{and}\quad \lim_{s\to t+} f(s)=f(t+),\quad t\in[a,b)$$

exist. The space of all regulated functions $f:[a,b]\to X$ will be denoted by $G([a,b],X)$, and is a Banach space under the usual supremum norm $\|f\|_\infty=\sup_{a\le t\le b}\|f(t)\|$. The subspace of all continuous functions $f:[a,b]\to X$ will be denoted by $C([a,b],X)$.

A proof of the next result can be found in [19], Corollary 1.34; the inequalities follow directly from the definition of the Kurzweil-Stieljtes integral.

**Theorem 2.1.** *If $f:[a,b]\to\mathbb{R}^n$ is a regulated function and $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f\,\mathrm{d}g$ exists and*

$$\left\|\int_a^b f(s)\,\mathrm{d}g(s)\right\|\le\int_a^b\|f(s)\|\,\mathrm{d}g(s)\le\|f\|_\infty(g(b)-g(a)).$$

The following result, which describes the properties of the indefinite Kurzweil-Stieljtes integral, is a special case of Theorem 1.16 in [19].

**Theorem 2.2.** *Let $f:[a,b]\to\mathbb{R}^n$ and $g:[a,b]\to\mathbb{R}$ be a pair of functions such that $g$ is regulated and $\int_a^b f\,\mathrm{d}g$ exists. Then the function*

$$h(t)=\int_a^t f(s)\,\mathrm{d}g(s),\quad t\in[a,b],$$

*is regulated and satisfies*

$$\begin{aligned} h(t+) &= h(t)+f(t)\Delta^+g(t),\quad t\in[a,b),\\ h(t-) &= h(t)-f(t)\Delta^-g(t),\quad t\in(a,b],\end{aligned}$$

*where $\Delta^+g(t)=g(t+)-g(t)$ and $\Delta^-g(t)=g(t)-g(t-)$.*

# 3 Measure functional differential equations and generalized ordinary differential equations

In the present section, our goal is to establish a correspondence between functional differential equations and generalized ordinary differential equations. We start by introducing some basic definitions and notation. More information about generalized ordinary differential equations can be found in [19].

**Definition 3.1.** Let $X$ be a Banach space. Consider a set $O\subset X$, an interval $[a,b]\subset\mathbb{R}$ and a function $F:O\times[a,b]\to X$. A function $x:[a,b]\to O$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t)$$

on the interval $[a,b]$, if

$$x(d) - x(c) = \int_c^d DF(x(\tau), t)$$

for every $c, d \in [a,b]$.

To obtain a reasonable theory of generalized ordinary differential equations, we restrict our attention to equations whose right-hand sides satisfy the following conditions.

**Definition 3.2.** Let $X$ be a Banach space. Consider a set $O \subset X$ and an interval $[a,b] \subset \mathbb{R}$. If $h : [a,b] \to \mathbb{R}$ is a nondecreasing function, we say that a function $F : O \times [a,b] \to X$ belongs to the class $\mathcal{F}(O \times [a,b], h)$, if

$$\|F(x, s_2) - F(x, s_1)\| \le |h(s_2) - h(s_1)|$$

for all $(x, s_2)$, $(x, s_1) \in O \times [a,b]$ and

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \le \|x - y\| \cdot |h(s_2) - h(s_1)|$$

for all $(x, s_2)$, $(x, s_1)$, $(y, s_2)$, $(y, s_1) \in O \times [a,b]$.

When the right-hand side of a generalized ordinary differential equation satisfies the above mentioned conditions, we have the following information about its solutions (see Lemma 3.12 in [19]).

**Lemma 3.3.** *Let $X$ be a Banach space. Consider a set $O \subset X$, an interval $[a,b] \subset \mathbb{R}$ and a function $F : O \times [a,b] \to X$. If $x : [a,b] \to O$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$$

*and $F \in \mathcal{F}(O \times [a,b], h)$, then $x$ is a regulated function.*

Let $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\} \subset G([-r, 0], \mathbb{R}^n)$. Consider a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ and a function $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$. We will show that under certain assumptions, a measure functional differential equation of the form

$$y(t) = y(t_0) + \int_{t_0}^t f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{3.1}$$

where the solution $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is supposed to be a regulated function, can be converted to a generalized ordinary differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \tag{3.2}$$

where $x$ takes values in $O$, i.e. we transform the original measure functional differential equation, whose solution takes values in $\mathbb{R}^n$, into a generalized ordinary equation, whose solution takes values in an infinite-dimensional Banach space. The right-hand side $F$ of this generalized equation will be given by

$$F(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} f(x_s, s)\,\mathrm{d}g(s), & t_0 \le \vartheta \le t \le t_0 + \sigma, \\ \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), & t \le \vartheta \le t_0 + \sigma \end{cases} \tag{3.3}$$

for every $x \in O$ and $t \in [t_0, t_0 + \sigma]$. As we will show, the relation between the solution $x$ of (3.2) and the solution $y$ of (3.1) is described by

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma], \end{cases}$$

where $t \in [t_0, t_0 + \sigma]$.

The following property will be important for us because it ensures that if $y \in O$, then $x(t) \in O$ for every $t \in [t_0, t_0 + \sigma]$.

**Definition 3.4.** Let $O$ be a subset of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. We will say that $O$ has the prolongation property, if for every $y \in O$ and every $\bar{t} \in [t_0 - r, t_0 + \sigma]$, the function $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \le t \le \bar{t}, \\ y(\bar{t}), & \bar{t} < t \le t_0 + \sigma \end{cases}$$

is also an element of $O$.

For example, let $B$ be an arbitrary subset of $\mathbb{R}^n$. Then both the set of all regulated functions $f : [t_0 - r, t_0 + \sigma] \to B$ and the set of all continuous functions $f : [t_0 - r, t_0 + \sigma] \to B$ have the prolongation property.

Recall that $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is nondecreasing.

We introduce the following three conditions, which will be used throughout the rest of this paper:

(A) The integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.

(B) There exists a constant $M > 0$ such that

$$\|f(y, t)\| \le M$$

for every $y \in P$ and every $t \in [t_0, t_0 + \sigma]$.

(C) There exists a constant $L > 0$ such that

$$\|f(y, t) - f(z, t)\| \le L\|y - z\|_\infty$$

for every $y, z \in P$ and every $t \in [t_0, t_0 + \sigma]$.

Before proceeding further, we need the following property of regulated functions.

**Lemma 3.5.** *If $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a regulated function, then $s \mapsto \|y_s\|_\infty$ is regulated on $[t_0, t_0 + \sigma]$.*

*Proof.* We will show that $\lim_{s \to s_0-} \|y_s\|_\infty$ exists for every $s_0 \in (t_0, t_0 + \sigma]$. The function $y$ is regulated, and therefore satisfies the Cauchy condition at $s_0 - r$ and $s_0$: Given an arbitrary $\varepsilon > 0$, there exists a $\delta \in (0, s_0 - t_0)$ such that

$$\|y(u) - y(v)\| < \varepsilon, \quad u, v \in (s_0 - r - \delta, s_0 - r), \tag{3.4}$$

and

$$\|y(u) - y(v)\| < \varepsilon, \quad u, v \in (s_0 - \delta, s_0). \tag{3.5}$$

Now, consider a pair of numbers $s_1$, $s_2$ such that $s_0 - \delta < s_1 < s_2 < s_0$. For every $s \in [s_1 - r, s_2 - r]$, it follows from (3.4) that

$$\|y(s)\| \le \|y(s_2 - r)\| + \varepsilon \le \|y_{s_2}\|_\infty + \varepsilon.$$

It is also clear that $\|y(s)\| \le \|y_{s_2}\|_\infty$ for every $s \in [s_2 - r, s_1]$. Consequently, $\|y_{s_1}\|_\infty \le \|y_{s_2}\|_\infty + \varepsilon$. Using (3.5) in a similar way, we obtain $\|y_{s_2}\|_\infty \le \|y_{s_1}\|_\infty + \varepsilon$. It follows that

$$\big|\|y_{s_1}\|_\infty - \|y_{s_2}\|_\infty\big| \le \varepsilon, \quad s_1, s_2 \in (s_0 - \delta, s_0),$$

i.e. the Cauchy condition for the existence of $\lim_{s \to s_0-} \|y_s\|_\infty$ is satisfied. The existence of $\lim_{s \to s_0+} \|y_s\|_\infty$ for $s_0 \in [t_0, t_0 + \sigma)$ can be proved similarly. □

**Lemma 3.6.** *Let $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function and that $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C)*. Then the function $F : O \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ given by* (3.3) *belongs to the class $\mathcal{F}(O \times [t_0, t_0 + \sigma], h)$, where*

$$h(t) = (L + M)(g(t) - g(t_0)), \quad t \in [t_0, t_0 + \sigma].$$

*Proof.* Condition (A) implies that the integrals in the definition of $F$ exist. Given $y \in O$ and $t_0 \le s_1 < s_2 \le t_0 + \sigma$, we see that

$$F(y, s_2)(\tau) - F(y, s_1)(\tau) = \begin{cases} 0, & t_0 - r \le \tau \le s_1, \\ \int_{s_1}^{\tau} f(y_s, s)\,\mathrm{d}g(s), & s_1 \le \tau \le s_2, \\ \int_{s_1}^{s_2} f(y_s, s)\,\mathrm{d}g(s), & s_2 \le \tau \le t_0 + \sigma. \end{cases}$$

Hence for an arbitrary $y \in O$ and for $t_0 \le s_1 < s_2 \le t_0 + \sigma$, we have by (B)

$$\|F(y, s_2) - F(y, s_1)\|_\infty = \sup_{t_0 - r \le \tau \le t_0 + \sigma} \|F(y, s_2)(\tau) - F(y, s_1)(\tau)\| =$$

$$= \sup_{s_1 \le \tau \le s_2} \|F(y, s_2)(\tau) - F(y, s_1)(\tau)\| = \sup_{s_1 \le \tau \le s_2} \left\| \int_{s_1}^{\tau} f(y_s, s)\,\mathrm{d}g(s) \right\| \le$$

$$\le \int_{s_1}^{s_2} M\,\mathrm{d}g(s) \le h(s_2) - h(s_1).$$

Similarly, using (C), if $y, z \in O$ and $t_0 \le s_1 \le s_2 \le t_0 + \sigma$, then

$$\|F(y, s_2) - F(y, s_1) - F(z, s_2) + F(z, s_1)\|_\infty =$$

$$= \sup_{s_1 \le \tau \le s_2} \left\| \int_{s_1}^{\tau} [f(y_s, s) - f(z_s, s)]\,\mathrm{d}g(s) \right\| \le \sup_{s_1 \le \tau \le s_2} \int_{s_1}^{\tau} L\|y_s - z_s\|_\infty\,\mathrm{d}g(s) \le$$

$$\le \|y - z\|_\infty \int_{s_1}^{s_2} L\,\mathrm{d}g(s) \le \|y - z\|_\infty (h(s_2) - h(s_1))$$

(note that the function $s \mapsto \|y_s - z_s\|_\infty$ is regulated according to Lemma 3.5, and therefore the integral $\int_{s_1}^{\tau} L\|y_s - z_s\|_\infty\,\mathrm{d}g(s)$ exists). □

The following statement is a slightly modified version of Lemma 3.3 from [6] (which is concerned with the special case $g(t) = t$). The proof from [6] can be carried over without any changes and we repeat it here for reader's convenience.

**Lemma 3.7.** *Let $O$ be a subset of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ with the prolongation property and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $\phi \in P$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, and $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is such that the integral $\int_{t_0}^{t_0 + \sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$. Consider $F$ given by* (3.3) *and assume that $x : [t_0, t_0 + \sigma] \to O$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t)$$

*with initial condition $x(t_0)(\vartheta) = \phi(\vartheta)$ for $\vartheta \in [t_0 - r, t_0]$, and $x(t_0)(\vartheta) = x(t_0)(t_0)$ for $\vartheta \in [t_0, t_0 + \sigma]$. If $v \in [t_0, t_0 + \sigma]$ and $\vartheta \in [t_0 - r, t_0 + \sigma]$, then*

$$x(v)(\vartheta) = x(v)(v), \quad \vartheta \ge v, \tag{3.6}$$

*and*

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad v \ge \vartheta. \tag{3.7}$$

*Proof.* Assume that $\vartheta \geq v$. Since $x$ is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t),$$

we have

$$x(v)(v) = x(t_0)(v) + \int_{t_0}^{v} DF(x(\tau),t)(v)$$

and similarly

$$x(v)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{v} DF(x(\tau),t)(\vartheta).$$

Since $x(t_0)(\vartheta) = x(t_0)(v)$ by the properties of the initial condition, we have

$$x(v)(\vartheta) - x(v)(v) = \int_{t_0}^{v} DF(x(\tau),t)(\vartheta) - \int_{t_0}^{v} DF(x(\tau),t)(v).$$

It follows from the existence of the integral $\int_{t_0}^{v} DF(x(\tau),t)$ that for every $\varepsilon > 0$, there is a gauge $\delta$ on $[t_0, t_0+\sigma]$ such that if $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \ldots, k\}$ is a $\delta$-fine tagged partition of $[t_0, v]$, then

$$\left\| \sum_{i=1}^{k} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})) - \int_{t_0}^{v} DF(x(\tau),t) \right\|_{\infty} < \varepsilon.$$

Therefore we have

$$\|x(v)(\vartheta) - x(v)(v)\| < 2\varepsilon + \left\| \sum_{i=1}^{k} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(\vartheta) - \sum_{i=1}^{k} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(v) \right\|.$$

By the definition of $F$ in (3.3), it is a matter of routine to check that, for every $i \in \{1, \ldots, k\}$, we have

$$F(x(\tau_i), s_i)(\vartheta) - F(x(\tau_i), s_{i-1})(\vartheta) = F(x(\tau_i), s_i)(v) - F(x(\tau_i), s_{i-1})(v),$$

and consequently

$$\|x(v)(\vartheta) - x(v)(v)\| < 2\varepsilon.$$

Since this holds for an arbitrary $\varepsilon > 0$, the relation (3.6) is satisfied.

To prove the second statement, assume that $\vartheta \leq v$. Similarly to the first part of the proof, we have

$$x(v)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{v} DF(x(\tau),t)(\vartheta)$$

and

$$x(\vartheta)(\vartheta) = x(t_0)(\vartheta) + \int_{t_0}^{\vartheta} DF(x(\tau),t)(\vartheta).$$

Hence

$$x(v)(\vartheta) - x(\vartheta)(\vartheta) = \int_{\vartheta}^{v} DF(x(\tau),t)(\vartheta).$$

Now, if $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \ldots, k\}$ is an arbitrary tagged partition of $[\vartheta, v]$, it is straightforward to check by (3.3) that, for every $i \in \{1, \ldots, k\}$, we have

$$F(x(\tau_i), s_i)(\vartheta) - F(x(\tau_i), s_{i-1})(\vartheta) = 0.$$

This means that $\int_{\vartheta}^{v} DF(x(\tau),t)(\vartheta) = 0$ and $x(v)(\vartheta) = x(\vartheta)(\vartheta)$. □

The proofs of the following two theorems are inspired by similar proofs from the paper [6], which describes the special case $g(t) = t$, i.e. the usual type of functional differential equations.

**Theorem 3.8.** *Assume that* $X$ *is a closed subspace of* $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $O$ *is a subset of* $X$ *with the prolongation property,* $P = \{x_t;\ x \in O,\ t \in [t_0, t_0 + \sigma]\}$, $\phi \in P$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ *is a nondecreasing function,* $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ *satisfies conditions* (A), (B), (C). *Let* $F : O \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *be given by* (3.3) *and assume that* $F(x, t) \in X$ *for every* $x \in O$, $t \in [t_0, t_0 + \sigma]$. *Let* $y \in O$ *be a solution of the measure functional differential equation*

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} &= \phi. \end{aligned}$$

*For every* $t \in [t_0 - r, t_0 + \sigma]$, *let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

*Then the function* $x : [t_0, t_0 + \sigma] \to O$ *is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t).$$

*Proof.* We will show that, for every $v \in [t_0, t_0 + \sigma]$, the integral $\int_{t_0}^{v} DF(x(\tau), t)$ exists and

$$x(v) - x(t_0) = \int_{t_0}^{v} DF(x(\tau), t).$$

Let an arbitrary $\varepsilon > 0$ be given. Since $g$ is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) \geq \varepsilon/M$; denote these points by $t_1, \ldots, t_m$. Consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{\frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m\right\}, \quad \tau \in [t_0, t_0 + \sigma]$$

and

$$\delta(\tau) < \min\{|\tau - t_k|;\ k = 1, \ldots, m\}, \quad \tau \in [t_0, t_0 + \sigma].$$

These conditions assure that if a point-interval pair $(\tau, [c, d])$ is $\delta$-fine, then $[c, d]$ contains at most one of the points $t_1, \ldots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c, d]$.

Since $y_{t_k} = x(t_k)_{t_k}$, it follows from Theorem 2.2 that

$$\lim_{s \to t_k+} \int_{t_k}^{s} L\|y_s - x(t_k)_s\|_\infty\,\mathrm{d}g(s) = L\|y_{t_k} - x(t_k)_{t_k}\|_\infty \Delta^+ g(t_k) = 0$$

for every $k \in \{1, \ldots, m\}$. Thus the gauge $\delta$ might be chosen in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L\|y_s - x(t_k)_s\|_\infty\,\mathrm{d}g(s) < \frac{\varepsilon}{2m + 1}, \quad k \in \{1, \ldots, m\}.$$

Using Theorem 2.2 again, we see that

$$\|y(\tau+) - y(\tau)\| = \|f(y_\tau, \tau)\Delta^+ g(\tau)\| < M\frac{\varepsilon}{M} = \varepsilon, \quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}.$$

Thus we can assume that the gauge $\delta$ is such that

$$\|y(\rho) - y(\tau)\| \le \varepsilon$$

for every $\tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}$ and $\rho \in [\tau, \tau + \delta(\tau))$.

Assume now that $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \ldots, l\}$ is a $\delta$-fine tagged partition of the interval $[t_0, v]$. Using the definition of $x$, it can be easily shown that

$$\big(x(s_i) - x(s_{i-1})\big)(\vartheta) = \begin{cases} 0, & \vartheta \in [t_0 - r, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} f(y_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i,] \\ \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0 + \sigma]. \end{cases}$$

Similarly, it follows from the definition of $F$ that

$$\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta) = \begin{cases} 0, & \vartheta \in [t_0 - r, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} f(x(\tau_i)_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0 + \sigma]. \end{cases}$$

By combination of the previous equalities, we obtain

$$\big(x(s_i) - x(s_{i-1})\big)(\vartheta) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta) =$$

$$= \begin{cases} 0, & \vartheta \in [t_0 - r, s_{i-1}], \\ \int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0 + \sigma]. \end{cases}$$

Consequently,

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\infty =$$

$$= \sup_{\vartheta \in [t_0 - r, t_0 + \sigma]} \|\big(x(s_i) - x(s_{i-1})\big)(\vartheta) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(\vartheta)\| =$$

$$= \sup_{\vartheta \in [s_{i-1}, s_i]} \left\| \int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) \right\|.$$

By the definition of $x$, we see that $x(\tau_i)_s = y_s$ whenever $s \le \tau_i$. Thus

$$\int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) = \begin{cases} 0, & \vartheta \in [s_{i-1}, \tau_i], \\ \int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s), & \vartheta \in [\tau_i, s_i]. \end{cases}$$

Then condition (C) implies

$$\left\| \int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s) \right\| \le \int_{\tau_i}^{\vartheta} L\|y_s - x(\tau_i)_s\|_\infty\,\mathrm{d}g(s) \le \int_{\tau_i}^{s_i} L\|y_s - x(\tau_i)_s\|_\infty\,\mathrm{d}g(s).$$

Given a particular point-interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point $t_k = \tau_i$.

(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

In case (i), it follows from the definition of the gauge $\delta$ that

$$\int_{\tau_i}^{s_i} L\|y_s - x(\tau_i)_s\|_\infty \,\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

i.e.

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\infty \le \frac{\varepsilon}{2m+1}.$$

In case (ii), we have

$$\|y_s - x(\tau_i)_s\|_\infty = \sup_{\rho\in[\tau_i,s]} \|y(\rho) - y(\tau_i)\| \le \varepsilon, \quad s \in [\tau_i, s_i]$$

by the definition of the gauge $\delta$. Thus

$$\|x(s_i) - x(s_{i-1}) - \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\|_\infty \le \varepsilon \int_{\tau_i}^{s_i} L \,\mathrm{d}g(s).$$

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\left\| x(v) - x(t_0) - \sum_{i=1}^{l} \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big) \right\|_\infty \le \varepsilon \int_{t_0}^{t_0+\sigma} L \,\mathrm{d}g(s) + \frac{2m\varepsilon}{2m+1}.$$

Since $\varepsilon$ is arbitrary, it follows that

$$x(v) - x(t_0) = \int_{t_0}^{v} DF(x(\tau), t). \qquad \square$$

**Theorem 3.9.** *Assume that $X$ is a closed subspace of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $O$ is a subset of $X$ with the prolongation property, $P = \{x_t;\ x \in O,\ t \in [t_0, t_0 + \sigma]\}$, $\phi \in P$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C). *Let $F : O \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be given by* (3.3) *and assume that $F(x,t) \in X$ for every $x \in O$, $t \in [t_0, t_0 + \sigma]$. Let $x : [t_0, t_0 + \sigma] \to O$ be a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t),$$

*with the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

*Then the function $y \in O$ defined by*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0, \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases}$$

*is a solution of the measure functional differential equation*

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s) \,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} &= \phi. \end{aligned}$$

*Proof.* The equality $y_{t_0} = \phi$ follows easily from the definitions of $y$ and $x(t_0)$. It remains to prove that if $v \in [t_0, t_0 + \sigma]$, then

$$y(v) - y(t_0) = \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s).$$

Using Lemma 3.7, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} DF(x(\tau), t)\right)(v).$$

Thus

$$y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) = \left(\int_{t_0}^{v} DF(x(\tau), t)\right)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s). \tag{3.8}$$

Let an arbitrary $\varepsilon > 0$ be given. Since $g$ is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) \geq \varepsilon/(L + M)$; denote these points by $t_1, \dots, t_m$. Consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{\frac{t_k - t_{k-1}}{2},\ k = 2, \dots, m\right\},\quad \tau \in [t_0, t_0 + \sigma]$$

and

$$\delta(\tau) < \min\{|\tau - t_k|;\ k = 1, \dots, m\},\quad \tau \in [t_0, t_0 + \sigma].$$

As in the proof of Theorem 3.8, these conditions assure that if a point-interval pair $(\tau, [c, d])$ is $\delta$-fine, then $[c, d]$ contains at most one of the points $t_1, \dots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c, d]$.

Again, the gauge $\delta$ might be chosen in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L\|y_s - x(t_k)_s\|_\infty\,\mathrm{d}g(s) < \frac{\varepsilon}{2m + 1},\quad k \in \{1, \dots, m\}.$$

According to Lemma 3.6, the function $F$ given by (3.3) belongs to the class $\mathcal{F}(O \times [t_0, t_0 + \sigma], h)$, where

$$h(t) = (L + M)(g(t) - g(t_0)).$$

Since

$$\|h(\tau+) - h(\tau)\| = \|(L + M)\Delta^+ g(\tau)\| < \varepsilon,\quad \tau \in [t_0, t_0 + \sigma]\backslash\{t_1, \dots, t_m\},$$

we can assume that the gauge $\delta$ satisfies

$$\|h(\rho) - h(\tau)\| \leq \varepsilon \text{ for every } \rho \in [\tau, \tau + \delta(\tau)).$$

Finally, the gauge $\delta$ should be such that

$$\left\|\int_{t_0}^{v} DF(x(\tau), t) - \sum_{i=1}^{l}\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)\right\|_\infty < \varepsilon \tag{3.9}$$

for every $\delta$-fine partition $\{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, l\}$ of $[t_0, v]$. The existence of such a gauge follows from the definition of the Kurzweil integral. Choose a particular $\delta$-fine partition $\{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, l\}$ of $[t_0, v]$. By (3.8) and (3.9), we have

$$\left\|y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s)\right\| = \left\|\left(\int_{t_0}^{v} DF(x(\tau), t)\right)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s)\right\| <$$

$$< \varepsilon + \left\| \sum_{i=1}^{l} \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) \right\| \le$$

$$\le \varepsilon + \sum_{i=1}^{l} \left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right\|.$$

The definition of $F$ yields

$$\big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) = \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s),$$

which implies

$$\left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right\| =$$

$$= \left\| \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right\| = \left\| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_s, s) - f(y_s, s)]\,\mathrm{d}g(s) \right\|.$$

By Lemma 3.7, for every $i \in \{1, \ldots, l\}$, we have $x(\tau_i)_s = x(s)_s = y_s$ for $s \in [s_{i-1}, \tau_i]$ and $y_s = x(s)_s = x(s_i)_s$ for $s \in [\tau_i, s_i]$. Therefore

$$\left\| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_s, s) - f(y_s, s)]\,\mathrm{d}g(s) \right\| = \left\| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s, s) - f(y_s, s)]\,\mathrm{d}g(s) \right\| =$$

$$= \left\| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s, s) - f(x(s_i)_s, s)]\,\mathrm{d}g(s) \right\| \le \int_{\tau_i}^{s_i} L\|x(\tau_i)_s - x(s_i)_s\|_\infty\,\mathrm{d}g(s),$$

where the last inequality follows from condition (C). Again, we distinguish two cases:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point $t_k = \tau_i$.

(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

In case (i), it follows from the definition of the gauge $\delta$ that

$$\int_{\tau_i}^{s_i} L\|y_s - x(\tau_i)_s\|_\infty\,\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

i.e.

$$\left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right\| \le \frac{\varepsilon}{2m+1}.$$

In case (ii), we use Lemma 3.3 to obtain the estimate

$$\|x(s_i)_s - x(\tau_i)_s\|_\infty \le \|x(s_i) - x(\tau_i)\|_\infty \le h(s_i) - h(\tau_i) \le \varepsilon$$

for every $s \in [\tau_i, s_i]$, and thus

$$\left\| \big(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big)(v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right\| \le \varepsilon \int_{\tau_i}^{s_i} L\,\mathrm{d}g(s).$$

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\sum_{i=1}^{l}\left\|\big(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1})\big)(v)-\int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s)\right\| \le \varepsilon\int_{t_0}^{t_0+\sigma} L\,\mathrm{d}g(s)+\frac{2m\varepsilon}{2m+1} <$$

$$< \varepsilon\left(1+\int_{t_0}^{t_0+\sigma} L\,\mathrm{d}g(s)\right).$$

Consequently,

$$\left\|y(v)-y(t_0)-\int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s)\right\| < \varepsilon\left(2+\int_{t_0}^{t_0+\sigma} L\,\mathrm{d}g(s)\right),$$

which completes the proof. □

**Remark 3.10.** It follows from Lemma 3.7 that the relation

$$y(\vartheta)=\begin{cases} x(t_0)(\vartheta), & t_0-r\le\vartheta\le t_0,\\ x(\vartheta)(\vartheta), & t_0\le\vartheta\le t_0+\sigma.\end{cases}$$

from the previous theorem can be replaced by a single equality

$$y(\vartheta)=x(t_0+\sigma)(\vartheta),\quad t_0-r\le\vartheta\le t_0+\sigma.$$

**Remark 3.11.** Before proceeding further, we stop for a moment to discuss conditions (A), (B), (C), which appear in the statements of Theorem 3.8 and Theorem 3.9.

Condition (A) requires the existence of the Kurzweil-Stieltjes integral $\int_{t_0}^{t_0+\sigma} f(y_t,t)\,\mathrm{d}g(t)$ for every $y\in O$. The class of integrable functions is quite large; for example, if $g(t)=t$, the Kurzweil-Stieltjes integral reduces to the well-known Henstock-Kurzweil integral, which generalizes both Lebesgue and Newton integrals (see e.g. [9]). For a general nondecreasing function $g$, Theorem 2.1 provides a useful sufficient condition: the Kurzweil-Stieltjes integral exists whenever $t\mapsto f(y_t,t)$ is a regulated function.

Conditions (B) and (C) can be replaced by slightly weaker statements: An inspection of the proofs of Theorem 3.8 and Theorem 3.9 shows that it is enough to assume the existence of constants $M$, $L>0$ such that

$$\left\|\int_a^b f(y_t,t)\,\mathrm{d}g(t)\right\|\le M(g(b)-g(a))$$

and

$$\left\|\int_a^b (f(y_t,t)-f(z_t,t))\,\mathrm{d}g(t)\right\|\le L\int_a^b \|y_t-z_t\|\,\mathrm{d}g(t)$$

for every $a$, $b\in[t_0,t_0+\sigma]$, $y,z\in O$.

**Remark 3.12.** The paper [6] deals with impulsive functional differential equations of the form

$$\begin{aligned} y'(t) &= f(y_t,t),\quad t\in[t_0,t_0+\sigma]\backslash\{t_1,\dots,t_m\},\\ \Delta^+y(t_k) &= I_k(y(t_k)),\quad k\in\{1,\dots,m\},\\ y_{t_0} &= \phi,\end{aligned}$$

where $f: P\times[t_0,t_0+\sigma]\to\mathbb{R}^n$, $P\subset G([-r,0],\mathbb{R}^n)$, $t_0\le t_1<\dots<t_m<t_0+\sigma$ and $I_k:\mathbb{R}^n\to\mathbb{R}^n$ for every $k\in\{1,\dots,m\}$. The solution $y$ is assumed to be differentiable on $[t_0,t_0+\sigma]\backslash\{t_1,\dots,t_m\}$ and left-continuous at the points $t_1,\dots,t_m$. The integral form of this impulsive problem is

$$\begin{aligned} y(t) &= y(t_0)+\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}s+\sum_{k;\,t_0\le t_k<t} I_k(y(t_k)) = y(t_0)+\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}s+\sum_{k=1}^{m} I_k(y(t_k))H_{t_k}(t),\\ y_{t_0} &= \phi,\end{aligned}$$

where $H_v$ denotes the characteristic function of $(v,\infty)$, i.e. $H_v(t)=0$ for $t\le v$ and $H_v(t)=1$ for $t>v$. We claim that this problem is equivalent to

$$\begin{aligned} y(t) &= y(t_0)+\int_{t_0}^{t}\tilde f(y_s,s)\,\mathrm{d}g(s), \quad t\in[t_0,t_0+\sigma], \\ y_{t_0} &= \phi, \end{aligned} \tag{3.10}$$

where $g(t)=t+\sum_{k=1}^{m}H_{t_k}(t)$ and

$$\tilde f(y,t)=\begin{cases} f(y,t) & \text{if } t\in[t_0,t_0+\sigma]\backslash\{t_1,\dots,t_m\}, \\ I_k(y(0)) & \text{if } t=t_k \text{ for some } k\in\{1,\dots,m\} \end{cases}$$

for every $t\in[t_0,t_0+\sigma]$ and $y\in P$. Indeed, assume that $y$ satisfies (3.10). Then we have

$$y(t)=y(t_0)+\int_{t_0}^{t}\tilde f(y_s,s)\,\mathrm{d}g(s)=y(t_0)+\int_{t_0}^{t}\tilde f(y_s,s)\,\mathrm{d}s+\sum_{k=1}^{m}\int_{t_0}^{t}\tilde f(y_s,s)\,\mathrm{d}H_{t_k}(s)=$$

$$=y(t_0)+\int_{t_0}^{t}f(y_s,s)\,\mathrm{d}s+\sum_{k;\,t_0\le t_k<t}\tilde f(y_{t_k},t_k)\Delta^{+}H_{t_k}(t_k)=y(t_0)+\int_{t_0}^{t}f(y_s,s)\,\mathrm{d}s+\sum_{k;\,t_0\le t_k<t}I_k(y(t_k)).$$

Note that if $f$ is bounded and Lipschitz-continuous and if the impulse operators $I_k$ are bounded and Lipschitz-continuous, then $\tilde f$ has the same properties.

Thus we see that our measure functional differential equations are general enough to encompass impulsive behavior and there is no need to consider impulses separately like in [6].

# 4 Functional dynamic equations on time scales

This section starts with a short overview of some basic concepts in the theory of time scales. Then we suggest a new approach to functional dynamic equations on time scales and explain their relation to measure functional differential equations.

Let $\mathbb{T}$ be a time scale, i.e. a closed nonempty subset of $\mathbb{R}$. For every $t\in\mathbb{T}$, we define the forward jump operator by $\sigma(t)=\inf\{s\in\mathbb{T},\ s>t\}$ (where we make the convention that $\inf\emptyset=\sup\mathbb{T}$) and the graininess function by $\mu(t)=\sigma(t)-t$. If $\sigma(t)>t$, we say that $t$ is right-scattered; otherwise, $t$ is right-dense.

A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$.

Given a pair of numbers $a,\ b\in\mathbb{T}$, the symbol $[a,b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, i.e. $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T};\ a\le t\le b\}$. On the other hand, $[a,b]$ is the usual closed interval on the real line, i.e. $[a,b]=\{t\in\mathbb{R};\ a\le t\le b\}$. This notational convention should help the reader to distinguish between ordinary and time scale intervals.

In the time scale calculus, the usual derivative $f'(t)$ is replaced by the $\Delta$-derivative $f^{\Delta}(t)$. Similarly, the usual integral $\int_a^b f(t)\,\mathrm{d}t$ is replaced by the $\Delta$-integral $\int_a^b f(t)\,\Delta t$, where $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$. The definitions and properties of the $\Delta$-derivative and $\Delta$-integral can be found in [1] and [2].

In the rest of this paper, we use the same notation as in [22]: Given a real number $t\le\sup\mathbb{T}$, let

$$t^*=\inf\{s\in\mathbb{T};s\ge t\}.$$

(Note that $t^*$ might be different from $\sigma(t)$.) Since $\mathbb{T}$ is a closed set, we have $t^*\in\mathbb{T}$. Further, let

$$\mathbb{T}^*=\begin{cases} (-\infty,\sup\mathbb{T}] & \text{if } \sup\mathbb{T}<\infty, \\ (-\infty,\infty) & \text{otherwise.} \end{cases}$$

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, we consider its extension $f^* : \mathbb{T}^* \to \mathbb{R}^n$ given by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

According to the following theorem (which was proved in [22]), the $\Delta$-integral of a time scale function $f$ is in fact equivalent to the Kurzweil-Stieltjes integral of the extended function $f^*$.

**Theorem 4.1.** *Let $f : \mathbb{T} \to \mathbb{R}^n$ be a rd-continuous function. Choose an arbitrary $a \in \mathbb{T}$ and define*

$$\begin{aligned} F_1(t) &= \int_a^t f(s)\,\Delta s, \quad t \in \mathbb{T}, \\ F_2(t) &= \int_a^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*, \end{aligned}$$

*where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$; in particular, $F_2(t) = F_1(t)$ for every $t \in \mathbb{T}$.*

However, it is useful to note that the Kurzweil-Stieltjes integral $\int_a^b f^*\,\mathrm{d}g$ does not change if we replace $f^*$ by a different function which coincides with $f$ on $[a,b] \cap \mathbb{T}$. This is the content of the next theorem.

**Theorem 4.2.** *Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$, $[a,b] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2 : [a,b] \to \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [a,b] \cap \mathbb{T}$. If $\int_a^b f_1\,\mathrm{d}g$ exists, then $\int_a^b f_2\,\mathrm{d}g$ exists as well and both integrals have the same value.*

*Proof.* Denote $I = \int_a^b f_1\,\mathrm{d}g$. Given an arbitrary $\varepsilon > 0$, there is a gauge $\delta_1 : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^k f_1(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| < \varepsilon$$

for every $\delta_1$-fine partition with division points $a = s_0 \le s_1 \le \cdots \le s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \dots, k$. Now, let

$$\delta_2(t) = \begin{cases} \delta_1(t) & \text{if } t \in [a,b] \cap \mathbb{T}, \\ \min\left(\delta_1(t), \frac{1}{2} \inf\{|t-s|, s \in \mathbb{T}\}\right) & \text{if } t \in [a,b] \backslash \mathbb{T}. \end{cases}$$

Note that each $\delta_2$-fine partition is also $\delta_1$-fine. Consider an arbitrary $\delta_2$-fine partition with division points $a = s_0 \le s_1 \le \cdots \le s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \dots, k\}$. For every $i \in \{1, \dots, k\}$, there are two possibilities: Either $[s_{i-1}, s_i] \cap \mathbb{T} = \emptyset$, or $\tau_i \in \mathbb{T}$. In the first case, $g(s_{i-1}) = g(s_i)$, and therefore

$$f_2(\tau_i)(g(s_i) - g(s_{i-1})) = 0 = f_1(\tau_i)(g(s_i) - g(s_{i-1})).$$

In the second case, $f_1(\tau_i) = f_2(\tau_i)$ and

$$f_2(\tau_i)(g(s_i) - g(s_{i-1})) = f_1(\tau_i)(g(s_i) - g(s_{i-1})).$$

Thus we have

$$\left\| \sum_{i=1}^k f_2(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| = \left\| \sum_{i=1}^k f_1(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| < \varepsilon.$$

Since $\varepsilon$ can be arbitrarily small, we conclude that $\int_a^b f_2\,\mathrm{d}g = I$. $\square$

We would like to study dynamic equations on time scales such that the $\Delta$-derivative of the unknown function $x:\mathbb{T}\to\mathbb{R}^n$ at $t\in\mathbb{T}$ depends on the values of $x(s)$, where $s\in[t-r,t]\cap\mathbb{T}$. But, unlike the classical case, there is a difficulty: The function $x_t$ is now defined on a subset of $[-r,0]$, and this subset can depend on $t$. We overcome this problem by considering the function $x_t^*$ instead; throughout this and the following sections, $x_t^*$ stands for $(x^*)_t$. Clearly, $x_t^*$ contains the same information as $x_t$, but it is defined on the whole interval $[-r,0]$. Thus, it seems reasonable to consider functional dynamic equations of the form

$$x^\Delta(t)=f(x_t^*,t).$$

We now show that this equation is equivalent to a certain measure functional differential equation. The symbol $C([a,b]_{\mathbb{T}},\mathbb{R}^n)$ will be used to denote the set of all continuous functions $f:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$.

**Theorem 4.3.** *Let $[t_0-r,t_0+\sigma]_{\mathbb{T}}$ be a time scale interval, $t_0\in\mathbb{T}$, $B\subset\mathbb{R}^n$, $C=C([t_0-r,t_0+\sigma]_{\mathbb{T}},B)$, $P=\{x_t^*;\,x\in C,t\in[t_0,t_0+\sigma]\}$, $f:P\times[t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$, $\phi\in C([t_0-r,t_0]_{\mathbb{T}},B)$. Assume that for every $x\in C$, the function $t\mapsto f(x_t^*,t)$ is rd-continuous on $[t_0,t_0+\sigma]_{\mathbb{T}}$. Define $g(s)=s^*$ for every $s\in[t_0,t_0+\sigma]$. If $x:[t_0-r,t_0+\sigma]_{\mathbb{T}}\to B$ is a solution of the functional dynamic equation*

$$\begin{aligned} x^\Delta(t) &= f(x_t^*,t),\ \ t\in[t_0,t_0+\sigma]_{\mathbb{T}}, &(4.1)\\ x(t) &= \phi(t),\ \ t\in[t_0-r,t_0]_{\mathbb{T}}, &(4.2)\end{aligned}$$

*then $x^*:[t_0-r,t_0+\sigma]\to B$ satisfies*

$$\begin{aligned} x^*(t) &= x^*(t_0)+\int_{t_0}^t f(x_s^*,s^*)\,\mathrm{d}g(s),\ \ t\in[t_0,t_0+\sigma],\\ x_{t_0}^* &= \phi^*.\end{aligned}$$

*Conversely, if $y:[t_0-r,t_0+\sigma]\to B$ is a solution of the measure functional differential equation*

$$\begin{aligned} y(t) &= y(t_0)+\int_{t_0}^t f(y_s,s^*)\,\mathrm{d}g(s),\ \ t\in[t_0,t_0+\sigma],\\ y_{t_0} &= \phi^*,\end{aligned}$$

*then $y=x^*$, where $x:[t_0-r,t_0+\sigma]_{\mathbb{T}}\to B$ satisfies* (4.1) *and* (4.2).

*Proof.* Assume that

$$x^\Delta(t)=f(x_t^*,t),\ \ t\in[t_0,t_0+\sigma]_{\mathbb{T}}.$$

Then

$$x(t)=x(t_0)+\int_{t_0}^t f(x_s^*,s)\,\Delta s,\ \ t\in[t_0,t_0+\sigma]_{\mathbb{T}},$$

and, by Theorem 4.1,

$$x^*(t)=x^*(t_0)+\int_{t_0}^t f(x_{s^*}^*,s^*)\,\mathrm{d}g(s),\ \ t\in[t_0,t_0+\sigma].$$

Since $f(x_{s^*}^*,s^*)=f(x_s^*,s^*)$ for every $s\in\mathbb{T}$, we can use Theorem 4.2 to conclude that

$$x^*(t)=x^*(t_0)+\int_{t_0}^t f(x_s^*,s^*)\,\mathrm{d}g(s),\ \ t\in[t_0,t_0+\sigma].$$

Conversely, assume that $y$ satisfies

$$y(t)=y(t_0)+\int_{t_0}^t f(y_s,s^*)\,\mathrm{d}g(s),\ \ t\in[t_0,t_0+\sigma].$$

Note that $g$ is constant on every interval $(\alpha, \beta]$, where $\beta \in \mathbb{T}$ and $\alpha = \sup\{\tau \in \mathbb{T}; \tau < \beta\}$. Thus $y$ has the same property and it follows that $y = x^*$ for some $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$. Using Theorem 2.2, it is easy to see that $x$ is continuous on $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$. By reversing our previous reasoning, we conclude that $x$ satisfies (4.1) and (4.2). □

**Example 4.4.** There is a fairly large number of papers devoted to delay dynamic equations of the form

$$x^{\Delta}(t) = h(t, x(t), x(\tau_1(t)), \ldots, x(\tau_k(t))), \tag{4.3}$$

where $\tau_i : \mathbb{T} \to \mathbb{T}$ are functions corresponding to the delays, i.e. $\tau_i(t) \le t$ for every $t \in \mathbb{T}$ and every $i = 1, \ldots, k$, and where the function $t \mapsto h(t, x(t), x(\tau_1(t)), \ldots, x(\tau_k(t)))$ is rd-continuous whenever $x$ is a continuous function.

Suppose that the delays are bounded, i.e. there exists a constant $r > 0$ such that $t - r \le \tau_i(t) \le t$, or equivalently $-r \le \tau_i(t) - t \le 0$. Then it is possible to write (4.3) in the form

$$x^{\Delta}(t) = f(x_t^*, t)$$

by taking

$$f(y, t) = h(t, y(0), y(\tau_1(t) - t), \ldots, y(\tau_k(t) - t))$$

for every $y : [-r, 0] \to \mathbb{R}^n$. Also, if $x$ is continuous on $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$, then the function $t \mapsto f(x_t^*, t)$ is rd-continuous on $[t_0, t_0 + \sigma]_{\mathbb{T}}$.

An important special case is represented by linear delay dynamic equations of the form

$$x^{\Delta}(t) = \sum_{i=1}^{k} p_i(t) x(\tau_i(t)) + q(t),$$

where $q, p_1, \ldots, p_k$ and $\tau_1, \ldots, \tau_k$ are rd-continuous functions on $[t_0, t_0 + \sigma]_{\mathbb{T}}$. The corresponding functional dynamic equation is

$$x^{\Delta}(t) = f(x_t^*, t),$$

where

$$f(y, t) = \sum_{i=1}^{k} p_i(t) y(\tau_i(t) - t) + q(t)$$

for every $y : [-r, 0] \to \mathbb{R}^n$. Again, we see that $t \mapsto f(x_t^*, t)$ is rd-continuous on $[t_0, t_0 + \sigma]_{\mathbb{T}}$ whenever $x$ is continuous on $[t_0 - r, t_0 + \sigma]_{\mathbb{T}}$. Moreover, for each pair of functions $y, z : [-r, 0] \to \mathbb{R}^n$, we have

$$\|f(y, t) - f(z, t)\| \le \sum_{i=1}^{k} \|p_i(t)(y(\tau_i(t) - t) - z(\tau_i(t) - t))\| \le \left( \sum_{i=1}^{k} \|p_i(t)\| \right) \|y - z\|_{\infty} \le L \|y - z\|_{\infty}$$

with

$$L = \sup_{t \in [t_0, t_0 + \sigma]_{\mathbb{T}}} \sum_{i=1}^{k} \|p_i(t)\|,$$

i.e. $f$ is Lipschitz-continuous in the first variable.

# 5 Existence-uniqueness theorems

The following existence-uniqueness theorem for generalized ordinary differential equations was proved in [6], Theorem 2.15.

**Theorem 5.1.** *Assume that $X$ is a Banach space, $O \subset X$ an open set, and $F : O \times [t_0, t_0 + \sigma] \to X$ belongs to the class $\mathcal{F}(O \times [t_0, t_0 + \sigma], h)$, where $h : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function. If $x_0 \in O$ is such that $x_0 + F(x_0, t_0+) - F(x_0, t_0) \in O$, then there exists a $\delta > 0$ and a function $x : [t_0, t_0 + \delta] \to X$ which is a unique solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad x(t_0) = x_0.$$

**Remark 5.2.** An estimate for the value $\delta$ which corresponds to the length of interval where the solution exists can be obtained by inspection of the proof of Theorem 2.15 in [6]. Let $\overline{B}(x_0, r)$ denote the closed ball $\{x \in X, \ \|x - x_0\| \le r\}$. Then we find that $\delta \in (0, \sigma]$ can be any number such that $\overline{B}(x_0, h(t_0 + \delta) - h(t_0+)) \subset O$ and $h(t_0 + \delta) - h(t_0+) < 1$. (Note that the proof in [6] assumes that $h(t_0 + \delta) - h(t_0+) < 1/2$, but a careful examination reveals that $h(t_0 + \delta) - h(t_0+) < 1$ is sufficient).

We now use the previous result to obtain an existence-uniqueness theorem for measure functional differential equations.

**Theorem 5.3.** *Assume that $X$ is a closed subspace of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $O$ is an open subset of $X$ with the prolongation property, $P = \{x_t; \ x \in O, \ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (A), (B), (C)*. Let $F : O \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be given by* (3.3) *and assume that $F(x, t) \in X$ for every $x \in O$, $t \in [t_0, t_0 + \sigma]$. If $\phi \in P$ is such that the function*

$$z(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + f(\phi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0 + \sigma] \end{cases}$$

*belongs to $O$, then there exists a $\delta > 0$ and a function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ which is a unique solution of the measure functional differential equation*

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s) \, \mathrm{d}g(s), \\ y_{t_0} &= \phi. \end{aligned}$$

*Proof.* According to Lemma 3.6, the function $F$ belongs to the class $\mathcal{F}(O \times [a, b], h)$, where

$$h(t) = (M + L)(g(t) - g(t_0)).$$

Further, let

$$x_0(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in [t_0 - r, t_0], \\ \phi(0), & \vartheta \in [t_0, t_0 + \sigma]. \end{cases}$$

It is clear that $x_0 \in O$. We also claim that $x_0 + F(x_0, t_0+) - F(x_0, t_0) \in O$. First, note that $F(x_0, t_0) = 0$. The limit $F(x_0, t_0+)$ is taken with respect to the supremum norm and we know it must exist since $F$ is regulated with respect to the second variable (this follows from the fact that $F \in \mathcal{F}(O \times [a, b], h)$). Thus it is sufficient to calculate the pointwise limit $F(x_0, t_0+)(\vartheta)$ for every $\vartheta \in [t_0 - r, t_0 + \sigma]$. Using Theorem 2.2, we obtain

$$F(x_0, t_0+)(\vartheta) = \begin{cases} 0, & t \in [t_0 - r, t_0], \\ f(\phi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0 + \sigma]. \end{cases}$$

It follows that $x_0 + F(x_0, t_0+) - F(x_0, t_0) = z \in O$.

Since all the assumptions of Theorem 5.1 are satisfied, there exists a $\delta > 0$ and a unique solution $x : [t_0, t_0 + \delta] \to X$ of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t), \quad x(t_0) = x_0. \tag{5.1}$$

According to Theorem 3.9, the function $y : [t_0 - r, t_0 + \delta]$ given by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0, \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta \le t_0 + \delta \end{cases}$$

is a solution of the measure functional differential equation

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \\ y_{t_0} &= \phi. \end{aligned}$$

This solution must be unique; otherwise, Theorem 3.8 would imply that $x$ is not the only solution of the generalized ordinary differential equation (5.1). □

**Remark 5.4.** Since the assumptions of Theorem 5.3 might look complicated, we mention two typical choices for the sets $X$, $O$ and $P$:

- $g(t) = t$ for every $t \in [t_0, t_0 + \sigma]$, $X = C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $B \subset \mathbb{R}^n$ is an open set, $O = C([t_0 - r, t_0 + \sigma], B)$, $P = C([-r, 0], B)$. Both conditions $F(x, t) \in X$ and $z \in O$ from Theorem 5.3 are always satisfied (by Theorem 2.2, $F(x, t)$ is a continuous function and therefore $F(x, t) \in X$).
- $X = G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, $B \subset \mathbb{R}^n$ is an open set, $O = G([t_0 - r, t_0 + \sigma], B)$, $P = G([-r, 0], B)$. The condition $F(x, t) \in X$ from Theorem 5.3 is always satisfied (by Theorem 2.2, $F(x, t)$ is a regulated function and therefore $F(x, t) \in X$). The condition $z \in O$ reduces to $\phi(0) + f(\phi, t_0)\Delta^+ g(t_0) \in B$. Note that if
$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s),$$
then $y(t_0+) = \phi(0) + f(\phi, t_0)\Delta^+ g(t_0)$. In other words, the condition ensures that the solution does not leave the set $B$ immediately after time $t_0$.

In both cases, we can use Remark 5.2 to obtain an estimate for the value $\delta$ which corresponds to the length of interval where the solution exists. Assume there exists a $\rho > 0$ such that $\|y - \phi(t)\| < \rho$ implies $y \in B$ for every $t \in [-r, 0]$ (in other words, a $\rho$-neighborhood of $\phi$ is contained in $B$). Since we have

$$h(t) = (M + L)(g(t) - g(t_0)),$$

we see that $\delta \in (0, \sigma]$ can be any number such that

$$g(t_0 + \delta) - g(t_0+) < \frac{\min(1, \rho)}{M + L}.$$

We now prove an existence-uniqueness theorem for functional dynamic equations (cf. [14], [16]).

**Theorem 5.5.** *Let $[t_0-r,t_0+\sigma]_{\mathbb{T}}$ be a time scale interval, $t_0\in\mathbb{T}$, $B\subset\mathbb{R}^n$ open, $C=C([t_0-r,t_0+\sigma]_{\mathbb{T}},B)$, $P=\{y_t^*;\, y\in C, t\in[t_0,t_0+\sigma]\}$, $f:P\times[t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$ a bounded function, which is Lipschitz-continuous in the first argument and such that $t\mapsto f(y_t^*,t)$ is rd-continuous on $[t_0,t_0+\sigma]_{\mathbb{T}}$ for every $y\in C$. If $\phi:[t_0-r,t_0]_{\mathbb{T}}\to B$ is a continuous function such that $\phi(t_0)+f(\phi_{t_0}^*,t_0)\mu(t_0)\in B$, then there exists a $\delta>0$ such that $\delta\geq\mu(t_0)$ and $t_0+\delta\in\mathbb{T}$, and a function $y:[t_0-r,t_0+\delta]_{\mathbb{T}}\to B$ which is a unique solution of the functional dynamic equation*

$$\begin{aligned} y^{\Delta}(t) &= f(y_t^*,t), \quad t\in[t_0,t_0+\delta],\\ y(t) &= \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}. \end{aligned}$$

*Proof.* Let $X=\{y^*;\, y\in C([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)\}$, $O=\{y^*;\, y\in C\}$, and $g(t)=t^*$ for every $t\in[t_0,t_0+\sigma]$. Note that $C([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)$ is a closed (Banach) space and the operator $T:C([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)\to X$ given by $T(y)=y^*$ is an isometric isomorphism; it follows that $X$ is a closed subspace of $G([t_0-r,t_0+\sigma],\mathbb{R}^n)$. The function $F:O\times[t_0,t_0+\sigma]\to G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ given by (3.3) satisfies $F(x,t)\in X$ for every $x\in O$, $t\in[t_0,t_0+\sigma]$. Indeed, by definition (3.3), we see that $F(x,t)$ is constant on $[t_0-r,t_0]$ and on every interval $(\alpha,\beta)\subset[t_0,t_0+r]$ which contains no time scale points (because $g$ is constant on such intervals). The function $g$ is left-continuous, and it follows from Theorem 2.2 that $F(x,t)$ is left-continuous on $[t_0,t_0+\sigma]$ and right-continuous at all points of $[t_0,t_0+\sigma]$ where $g$ is right-continuous, i.e. at all right-dense points of $[t_0,t_0+r]_{\mathbb{T}}$. Thus $F(x,t)$ must have the form $y^*$ for some rd-continuous function $y:[t_0-r,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$ and $F(x,t)\in X$. It is also clear that $O$ is an open subset of $X$ and has the prolongation property.

Let $f^*(y,t)=f(y,t^*)$ for every $y\in P$ and $t\in[t_0,t_0+\sigma]$. Consider an arbitrary $y\in O$. Since $t\mapsto f(y_t,t)$ is rd-continuous on $[t_0,t_0+\sigma]_{\mathbb{T}}$, the integral $\int_{t_0}^{t_0+\sigma} f(y_t,t)\Delta t$ exists. Using Theorem 4.1 and Theorem 4.2, we have

$$\int_{t_0}^{t_0+\sigma} f(y_t,t)\Delta t=\int_{t_0}^{t_0+\sigma} f(y_{t^*},t^*)\,\mathrm{d}g(t)=\int_{t_0}^{t_0+\sigma} f(y_t,t^*)\,\mathrm{d}g(t)=\int_{t_0}^{t_0+\sigma} f^*(y_t,t)\,\mathrm{d}g(t),$$

i.e. the last integral exists. Since $\Delta^+g(t_0)=\mu(t_0)$ and $\phi(t_0)+f(\phi_{t_0}^*,t_0)\mu(t_0)\in B$, it follows that the function

$$z(t)=\begin{cases}\phi_{t_0}^*(t-t_0), & t\in[t_0-r,t_0],\\ \phi_{t_0}^*(0)+f(\phi_{t_0}^*,t_0)\Delta^+g(t_0), & t\in(t_0,t_0+\sigma]\end{cases}$$

belongs to $O$. Therefore the functions $f^*$, $g$ and $\phi_{t_0}^*$ satisfy all assumptions of Theorem 5.3, and there exists a $\delta>0$ and a function $u:[t_0-r,t_0+\delta]\to B$ which is the unique solution of

$$\begin{aligned} u(t) &= u(t_0)+\int_{t_0}^{t} f^*(u_s,s)\,\mathrm{d}g(s), t\in[t_0,t_0+\delta]\\ u_{t_0} &= \phi_{t_0}^*. \end{aligned}$$

By Theorem 4.3, $u=y^*$, where $y:[t_0-r,t_0+\delta]_{\mathbb{T}}\to B$ is a solution of

$$\begin{aligned} y^{\Delta}(t) &= f(y_t^*,t), \quad t\in[t_0,t_0+\delta]\\ y(t) &= \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}. \end{aligned}$$

Without loss of generality, we can assume that $\delta\geq\mu(t_0)$; otherwise, let $y(\sigma(t_0))=\phi(t_0)+f(\phi_{t_0}^*,t_0)\mu(t_0)$ to obtain a solution defined on $[t_0-r,t_0+\mu(t_0)]_{\mathbb{T}}$. Again by Theorem 4.3, it follows that the solution $y$ is unique. □

**Remark 5.6.** Similarly to the previous existence-uniqueness results, we can estimate the value of $\delta$ which corresponds to length of interval where the solution exists. Assume there exists a $\rho>0$ such that

$\|y-\phi(t)\| < \rho$ implies $y \in B$ for every $t \in [t_0-r,t_0]_{\mathbb{T}}$ (in other words, a $\rho$-neighborhood of $\phi$ is contained in $B$). By Remark 5.4, we know that $\delta \in (0,\sigma]$ can be any number such that

$$g(t_0+\delta)-g(t_0+) < \frac{\min(1,\rho)}{M+L},$$

where $M$ is the bound and $L$ is the Lipschitz constant for the function $f$ on $P \times [t_0, t_0+\sigma]_{\mathbb{T}}$. In our particular case, we have $g(t) = t^*$ for every $t \in [t_0, t_0+\sigma]$. Since $g(t_0+\delta) = t_0+\delta$ and $g(t_0+) = \sigma(t_0)$, we obtain

$$\delta < \mu(t_0) + \frac{\min(1,\rho)}{M+L}.$$

# 6 Continuous dependence results

In this section, we use an existing continuous dependence theorem for generalized ordinary differential equations to derive continuous dependence theorems for measure functional differential equations and for functional dynamic equations on time scales.

We need the following proposition from [8], Theorem 2.18.

**Theorem 6.1.** *The following conditions are equivalent:*

1. *A set $\mathcal{A} \subset G([\alpha,\beta],\mathbb{R}^n)$ is relatively compact.*
2. *The set $\{x(\alpha); x \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta : [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and an increasing function $K : [\alpha,\beta] \to \mathbb{R}$ such that*

$$\|x(t_2)-x(t_1)\| \le \eta(K(t_2)-K(t_1))$$

*for every $x \in \mathcal{A}$, $\alpha \le t_1 \le t_2 \le \beta$.*

The following continuous dependence result for generalized ordinary differential equations is a Banach space version of Theorem 2.4 from [7]; the proof for the case $X = \mathbb{R}^n$ from [7] is still valid in this more general setting.

**Theorem 6.2.** *Let $X$ be a Banach space, $O \subset X$ an open set, and $h_k : [a,b] \to \mathbb{R}$, $k \in \mathbb{N}_0$, a sequence of nondecreasing left-continuous functions such that $h_k(b) - h_k(a) \le c$ for some $c > 0$ and every $k \in \mathbb{N}_0$. Assume that for every $k \in \mathbb{N}_0$, $F_k : O \times [a,b] \to X$ belongs to the class $\mathcal{F}(O \times [a,b], h_k)$, and that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t), \quad x \in O,\ t \in [a,b],$$

$$\lim_{k\to\infty} F_k(x,t+) = F_0(x,t+) \quad x \in O,\ t \in [a,b).$$

*For every $k \in \mathbb{N}$, let $x_k : [a,b] \to O$ be a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_k(x,t).$$

*If there exists a function $x_0 : [a,b] \to O$ such that $\lim_{k\to\infty} x_k(t) = x_0(t)$ uniformly for $t \in [a,b]$, then $x_0$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_0(x,t), \quad t \in [a,b].$$

It should be remarked that Theorem 2.4 in [7] assumes that the functions $F_k$ are defined on $O\times(-T,T)$, where $[a,b]\subset(-T,T)$, and similarly the functions $h_k$ are defined in the open interval $(-T,T)$. However, it is easy to extend the functions defined on $[a,b]$ to $(-T,T)$ by letting $F_k(x,t)=F_k(x,a)$ for $t\in(-T,a)$, $F_k(x,t)=F_k(x,b)$ for $t\in(b,T)$, and similarly for $h_k$. Note that the extended functions $F_k$ now belong to the class $\mathcal{F}(O\times(-T,T),h_k)$, as assumed in [7].

We are now ready to prove a continuous dependence theorem for measure functional differential equations.

**Theorem 6.3.** *Assume that $X$ is a closed subspace of $G([t_0-r,t_0+\sigma],\mathbb{R}^n)$, $O$ is an open subset of $X$ with the prolongation property, $P=\{y_t;\ y\in O,\ t\in[t_0,t_0+\sigma]\}$, $g:[t_0,t_0+\sigma]\to\mathbb{R}$ is a nondecreasing left-continuous function, and $f_k:P\times[t_0,t_0+\sigma]\to\mathbb{R}^n$, $k\in\mathbb{N}_0$, is a sequence of functions which satisfy the following conditions:*

1. *The integral $\int_{t_0}^{t_0+\sigma} f_k(y_t,t)\,\mathrm{d}g(t)$ exists for every $k\in\mathbb{N}_0$, $y\in O$.*
2. *There exists a constant $M>0$ such that*
$$\|f_k(y,t)\|\le M$$
*for every $k\in\mathbb{N}$, $y\in P$ and $t\in[t_0,t_0+\sigma]$.*
3. *There exists a constant $L>0$ such that*
$$\|f_k(y,t)-f_k(z,t)\|\le L\|y-z\|_\infty$$
*for every $k\in\mathbb{N}$, $y,z\in P$ and $t\in[t_0,t_0+\sigma]$.*
4. *For every $y\in O$,*
$$\lim_{k\to\infty}\int_{t_0}^{t} f_k(y_s,s)\,\mathrm{d}g(s)=\int_{t_0}^{t} f_0(y_s,s)\,\mathrm{d}g(s)$$
*uniformly with respect to $t\in[t_0,t_0+\sigma]$.*
5. *For every $k\in\mathbb{N}$, $x\in O$, $t\in[t_0,t_0+\sigma]$, the function $F_k(x,t):[t_0-r,t_0+\sigma]\to\mathbb{R}^n$ given by*
$$F_k(x,t)(\vartheta)=\begin{cases}0, & t_0-r\le\vartheta\le t_0,\\ \int_{t_0}^{\vartheta} f_k(x_s,s)\,\mathrm{d}g(s), & t_0\le\vartheta\le t\le t_0+\sigma,\\ \int_{t_0}^{t} f_k(x_s,s)\,\mathrm{d}g(s), & t\le\vartheta\le t_0+\sigma.\end{cases}$$
*is an element of $X$.*

*Consider a sequence of functions $\phi_k\in P$, $k\in\mathbb{N}_0$, such that $\lim_{k\to\infty}\phi_k=\phi_0$ uniformly on $[-r,0]$. Let $y_k\in O$, $k\in\mathbb{N}$, be solutions of*

$$\begin{aligned} y_k(t) &= y_k(t_0)+\int_{t_0}^{t} f_k((y_k)_s,s)\,\mathrm{d}g(s), \quad t\in[t_0,t_0+\sigma],\\ (y_k)_{t_0} &= \phi_k.\end{aligned}$$

*If there exists a function $y_0\in O$ such that $\lim_{k\to\infty}y_k=y_0$ on $[t_0,t_0+\sigma]$, then $y_0$ is a solution of*

$$\begin{aligned} y_0(t) &= y_0(t_0)+\int_{t_0}^{t} f_0((y_0)_s,s)\,\mathrm{d}g(s), \quad t\in[t_0,t_0+\sigma],\\ (y_0)_{t_0} &= \phi_0.\end{aligned}$$

*Proof.* The assumptions imply that for every $x \in O$, $\lim_{k\to\infty} F_k(x,t) = F_0(x,t)$ uniformly with respect to $t \in [t_0, t_0+\sigma]$. By the Moore-Osgood theorem, we have $\lim_{k\to\infty} F_k(x,t+) = F_0(x,t+)$ for every $x \in O$ and $t \in [t_0, t_0+\sigma)$. Also, since $X$ is a closed subspace, we have $F_0(x,t) \in X$.

It follows from Lemma 3.6 that $F_k \in \mathcal{F}(O \times [t_0, t_0+\sigma], h)$ for every $k \in \mathbb{N}$, where

$$h(t) = (L+M)(g(t) - g(t_0)), \quad t \in [t_0, t_0+\sigma].$$

Since $\lim_{k\to\infty} F_k(x,t) = F_0(x,t)$, we have $F_0 \in \mathcal{F}(O \times [t_0, t_0+\sigma], h)$.

For every $k \in \mathbb{N}_0$, $t \in [t_0, t_0+\sigma]$, let

$$x_k(t)(\vartheta) = \begin{cases} y_k(\vartheta), & \vartheta \in [t_0 - r, t], \\ y_k(t), & \vartheta \in [t, t_0+\sigma]. \end{cases}$$

According to Theorem 3.8, the function $x_k$, where $k \in \mathbb{N}$, is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_k(x,t).$$

When $k \in \mathbb{N}$ and $t_0 \le t_1 \le t_2 \le t_0+\sigma$, we have

$$\|y_k(t_2) - y_k(t_1)\| = \left\| \int_{t_1}^{t_2} f_k((y_k)_s, s)\, \mathrm{d}g(s) \right\| \le M(g(t_2) - g(t_1)) \le \eta(K(t_2) - K(t_1)),$$

where $\eta(t) = Mt$ for every $t \in [0,\infty)$ and $K(t) = g(t) + t$ for every $t \in [t_0, t_0+\sigma]$; note that $K$ is an increasing function. Moreover, the sequence $\{y_k(t_0)\}_{k=1}^\infty$ is bounded. Thus we see that condition 2 from Theorem 6.1 is satisfied and it follows that $\{y_k\}_{k=1}^\infty$ contains a subsequence which is uniformly convergent in $[t_0, t_0+\sigma]$. Without loss of generality, we can denote this subsequence again by $\{y_k\}_{k=1}^\infty$. Since $(y_k)_{t_0} = \phi_k$, we see that $\{y_k\}_{k=1}^\infty$ is in fact uniformly convergent in $[t_0 - r, t_0+\sigma]$.

By the definition of $x_k$, we have

$$\lim_{k\to\infty} x_k(t) = x_0(t)$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$. It follows from Theorem 6.2 that $x_0$ is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF_0(x,t)$$

on $[t_0, t_0+\sigma]$. The proof is finished by applying Theorem 3.9, which guarantees that $y_0$ satisfies

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f_0((y_0)_s, s)\, \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

□

**Remark 6.4.** We remind the reader that although assumption 5 in the previous theorem looks complicated, it is automatically satisfied if either $g(t) = t$ for every $t \in [t_0, t_0+\sigma]$ and $X = C([t_0 - r, t_0+\sigma], \mathbb{R}^n)$, or if $X = G([t_0 - r, t_0+\sigma], \mathbb{R}^n)$; see Remark 5.4.

Using the previous result, we prove a continuous dependence theorem for functional dynamic equations on time scales.

**Theorem 6.5.** *Let $[t_0 - r, t_0+\sigma]_{\mathbb{T}}$ be a time scale interval, $t_0 \in \mathbb{T}$, $B \subset \mathbb{R}^n$ open, $C = C([t_0 - r, t_0+\sigma]_{\mathbb{T}}, B)$, $P = \{y_t^*;\ y \in C, t \in [t_0, t_0+\sigma]\}$. Consider a sequence of functions $f_k : P \times [t_0, t_0+\sigma]_{\mathbb{T}} \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, such that the following conditions are satisfied:*

*1. For every $y \in C$ and $k \in \mathbb{N}_0$, the function $t \mapsto f_k(y_t^*, t)$ is rd-continuous on $[t_0, t_0 + \sigma]_{\mathbb{T}}$.*

*2. There exists a constant $M > 0$ such that*

$$\|f_k(y,t)\| \le M$$

*for every $k \in \mathbb{N}_0$, $y \in P$ and $t \in [t_0 - r, t_0 + \sigma]_{\mathbb{T}}$.*

*3. There exists a constant $L > 0$ such that*

$$\|f_k(y,t) - f_k(z,t)\| \le L\|y - z\|_\infty$$

*for every $k \in \mathbb{N}_0$, $y, z \in P$ and $t \in [t_0 - r, t_0 + \sigma]_{\mathbb{T}}$.*

*4. For every $y \in C$,*

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(y_s^*, s)\Delta s = \int_{t_0}^{t} f_0(y_s^*, s)\Delta s$$

*uniformly with respect to $t \in [t_0, t_0 + \sigma]_{\mathbb{T}}$.*

*Assume that $\phi_k \in C([t_0 - r, t_0]_{\mathbb{T}}, B)$, $k \in \mathbb{N}_0$, is a sequence of functions such that $\lim_{k\to\infty} \phi_k = \phi_0$ uniformly on $[t_0 - r, t_0]_{\mathbb{T}}$. Let $y_k \in C$, $k \in \mathbb{N}$ be solutions of*

$$\begin{aligned} y_k^\Delta(t) &= f_k((y_k^*)_t, t), \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}}, \\ y_k(t) &= \phi_k(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

*If there exists a function $y_0 \in C$ such that $\lim_{k\to\infty} y_k = y_0$ on $[t_0, t_0 + \sigma]_{\mathbb{T}}$, then $y_0$ is a solution of*

$$\begin{aligned} y_0^\Delta(t) &= f_0((y_0^*)_s, s), \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}}, \\ y_0(t) &= \phi_0(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

*Proof.* Let $X = \{y^*;\, y \in C([t_0 - r, t_0 + \sigma]_{\mathbb{T}}, \mathbb{R}^n)\}$, $O = \{y^*;\, y \in C\}$, and $g(t) = t^*$ for every $t \in [t_0, t_0 + \sigma]$. Note that $O$ is an open subset of $X$ and has the prolongation property. Further, let $f_k^*(y,t) = f_k(y, t^*)$ for every $k \in \mathbb{N}_0$, $y \in P$ and $t \in [t_0, t_0 + \sigma]$. Consider an arbitrary $y \in O$ and $k \in \mathbb{N}_0$. Since $t \mapsto f_k(y_t^*, t)$ is rd-continuous on $[t_0, t_0 + \sigma]_{\mathbb{T}}$, the integral $\int_{t_0}^{t_0+\sigma} f_k(y_t, t)\Delta t$ exists. Using Theorem 4.1 and Theorem 4.2, we have

$$\int_{t_0}^{t_0+\sigma} f_k(y_t, t)\Delta t = \int_{t_0}^{t_0+\sigma} f_k(y_{t^*}, t^*)\,\mathrm{d}g(t) = \int_{t_0}^{t_0+\sigma} f_k(y_t, t^*)\,\mathrm{d}g(t) = \int_{t_0}^{t_0+\sigma} f_k^*(y_t, t)\,\mathrm{d}g(t),$$

i.e. the last integral exists. Using Theorem 4.1 again, we obtain

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k^*(y_s, s)\,\mathrm{d}g(s) = \lim_{k\to\infty} \int_{t_0}^{t^*} f_k(y_s, s)\Delta s = \int_{t_0}^{t^*} f_0(y_s, s)\Delta s = \int_{t_0}^{t} f_0^*(y_s, s)\,\mathrm{d}g(s)$$

uniformly with respect to $t \in [t_0, t_0 + \sigma]$. Further, it is clear that $\lim_{k\to\infty} y_k^* = y_0^*$ on $[t_0, t_0 + \sigma]$, and $\lim_{k\to\infty} \phi_k^* = \phi_0^*$ uniformly on $[t_0 - r, t_0]$. By Theorem 4.3, we have

$$\begin{aligned} y_k^*(t) &= y_k^*(t_0) + \int_{t_0}^{t} f_k((y_k^*)_s, s^*)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_k^*)_{t_0} &= \phi_k^*. \end{aligned}$$

for every $k \in \mathbb{N}_0$. The functions $f_k^*$, $y_k^*$ and $\phi_k^*$, $k \in \mathbb{N}_0$, satisfy the assumptions of Theorem 6.3, and we conclude that

$$\begin{aligned} {y_0}^*(t) &= y_0^*(t_0) + \int_{t_0}^t f_0((y_0^*)_s, s^*)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_0^*)_{t_0} &= \phi_0^*. \end{aligned}$$

By Theorem 4.3, it follows that $y_0 : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies

$$\begin{aligned} y_0^{\Delta}(t) &= f_0((y_0^*)_s, s), \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}}, \\ y_0(t) &= \phi_0(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{aligned}$$

□

**Remark 6.6.** An inspection of the proof of Theorem 6.3 reveals that the hypothesis $\lim_{k\to\infty} y_k = y_0$ is not necessary to conclude that $\{y_k\}_{k=1}^{\infty}$ has a uniformly convergent subsequence. However, we need a condition guaranteeing that the limit function belongs to $O$. Thus, instead of requiring that $\lim_{k\to\infty} y_k = y_0 \in O$, it is possible to assume the existence of a closed set $O' \subset O$ such that $y_k \in O'$ for every $k \in \mathbb{N}$. Then it follows that $\{y_k\}_{k=1}^{\infty}$ has a subsequence which is uniformly convergent to a function $y_0 \in O$ such that

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^t f_0((y_0)_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

Similarly, in Theorem 6.5, we may assume the existence of a closed set $B' \subset B$ such that $y_k$ takes values in $B'$ for every $k \in \mathbb{N}$, and omit the condition $\lim_{k\to\infty} y_k = y_0$.

# 7 Periodic averaging theorems

Averaging theorems provide a useful tool for approximating solutions of a non-autonomous equation by solutions of an autonomous equation whose right-hand side is obtained by averaging the original right-hand side with respect to $t$ (see e.g. [20]). The approximation is especially good in the case when the original right-hand side is periodic with respect to $t$.

In this section, we use the following periodic averaging theorem for measure functional differential equations from [17] to obtain a new theorem on periodic averaging for functional dynamic equations.

**Theorem 7.1.** *Let $\varepsilon_0 > 0$, $L > 0$, $T > 0$, $B \subset \mathbb{R}^n$, $X = G([-r, 0], B)$. Consider a pair of bounded functions $f : X \times [0,\infty) \to \mathbb{R}^n$, $g : X \times [0,\infty) \times (0, \varepsilon_0] \to \mathbb{R}^n$ and a nondecreasing left-continuous function $h : [0,\infty) \to \mathbb{R}$ such that the following conditions are satisfied:*

1. *The integral $\int_0^b f(y_t, t)\,\mathrm{d}h(t)$ exists for every $b > 0$ and $y \in G([-r, b], B)$.*
2. *$f$ is Lipschitz-continuous in the first variable.*
3. *$f$ is $T$-periodic in the second variable.*
4. *There is a constant $\alpha > 0$ such that $h(t + T) - h(t) = \alpha$ for every $t \geq 0$.*
5. *The integral*
$$f_0(x) = \frac{1}{T}\int_0^T f(x, s)\,\mathrm{d}h(s)$$
*exists for every $x \in X$.*

*Let $\phi \in X$. Suppose that for every $\varepsilon \in (0, \varepsilon_0]$, the initial-value problems*

$$
\begin{aligned}
x(t) &= x(0) + \varepsilon \int_0^t f(x_s, s)\,\mathrm{d}h(s) + \varepsilon^2 \int_0^t g(x_s, s, \varepsilon)\,\mathrm{d}h(s), \quad x_0 = \phi, \\
y(t) &= y(0) + \varepsilon \int_0^t f_0(y_s)\,\mathrm{d}s, \quad y_0 = \phi
\end{aligned}
$$

*have solutions $x_\varepsilon$, $y_\varepsilon : [-r, L/\varepsilon] \to B$. Then there exists a constant $J > 0$ such that*

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| \le J\varepsilon$$

*for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L/\varepsilon]$.*

To be able to speak about periodic functions on time scales, we need the following concept of a periodic time scale.

**Definition 7.2.** Let $T > 0$ be a real number. A time scale $\mathbb{T}$ is called $T$-periodic if $t \in \mathbb{T}$ implies $t+T \in \mathbb{T}$ and $\mu(t) = \mu(t+T)$.

We now proceed to the periodic averaging theorem for functional dynamic equations on time scales.

**Theorem 7.3.** *Assume that $T > 0$, $\mathbb{T}$ is a $T$-periodic time scale, $t_0 \in \mathbb{T}$, $\varepsilon_0 > 0$, $L > 0$, $B \subset \mathbb{R}^n$. Consider a pair of bounded functions $f : G([-r, 0], B) \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$, $g : G([-r, 0], B) \times [t_0, \infty)_{\mathbb{T}} \times (0, \varepsilon_0] \to \mathbb{R}^n$ such that the following conditions are satisfied:*

1. *For every $b > t_0$ and $y \in G([t_0 - r, b], B)$, the function $t \mapsto f(y_t, t)$ is regulated on $[t_0, b]_{\mathbb{T}}$.*
2. *For every $b > t_0$ and $y \in C([t_0 - r, b]_{\mathbb{T}}, B)$, the function $t \mapsto f(y_t^*, t)$ is rd-continuous on $[t_0, b]_{\mathbb{T}}$.*
3. *$f$ is Lipschitz-continuous in the first variable.*
4. *$f$ is $T$-periodic and rd-continuous in the second variable.*

*Denote*

$$f_0(y) = \frac{1}{T} \int_{t_0}^{t_0+T} f(y, s)\Delta s, \quad y \in G([-r, 0], B).$$

*Let $\phi \in C([t_0 - r, t_0]_{\mathbb{T}}, B)$. Suppose that for every $\varepsilon \in (0, \varepsilon_0]$, the functional dynamic equation*

$$
\begin{aligned}
x^\Delta(t) &= \varepsilon f(x_t^*, t) + \varepsilon^2 g(x_t^*, t, \varepsilon), \\
x(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}
\end{aligned}
$$

*has a solution $x_\varepsilon : [t_0 - r, t_0 + L/\varepsilon]_{\mathbb{T}} \to \mathbb{R}^n$, and that the functional differential equation*

$$
\begin{aligned}
y(t) &= y(t_0) + \varepsilon \int_{t_0}^t f_0(y_s)\,\mathrm{d}s, \\
y_{t_0} &= \phi^*
\end{aligned}
$$

*has a solution $y_\varepsilon : [t_0 - r, t_0 + L/\varepsilon] \to \mathbb{R}^n$. Then there exists a constant $J > 0$ such that*

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| \le J\varepsilon,$$

*for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + L/\varepsilon]_{\mathbb{T}}$.*

*Proof.* Without loss of generality, we can assume that $t_0 = 0$; otherwise, consider a shifted problem with the time scale $\widetilde{\mathbb{T}} = \{t - t_0; t \in \mathbb{T}\}$ and the right-hand side $\widetilde{f}(x,t) = f(x, t + t_0)$.

For every $t \in [0,\infty)$, $y \in G([-r,0],B)$ and $\varepsilon \in (0,\varepsilon_0]$, let

$$f^*(y,t) = f(y,t^*) \quad \text{and} \quad g^*(y,t,\varepsilon) = g(y,t^*,\varepsilon).$$

Also, let $h(t) = t^*$ for every $t \in [0,\infty)$. It follows directly from the definition of $h$ and the fact that $\mathbb{T}$ is $T$-periodic that

$$h(t+T) - h(t) = T, \quad t \geq 0.$$

By Theorem 4.3, $x^*_\varepsilon$ satisfies

$$\begin{aligned} x^*_\varepsilon(t) &= x^*_\varepsilon(0) + \varepsilon \int_0^t f^*((x^*_\varepsilon)_s, s)\,\mathrm{d}h(s) + \varepsilon^2 \int_0^t g^*((x^*_\varepsilon)_s, s, \varepsilon)\,\mathrm{d}h(s), \quad t \in [0, L/\varepsilon] \\ (x^*_\varepsilon)_0 &= \phi^*. \end{aligned}$$

for every $\varepsilon \in (0,\varepsilon_0]$. From Theorem 4.1, we have

$$f_0(y) = \frac{1}{T}\int_0^T f(y,s)\Delta s = \frac{1}{T}\int_0^T f^*(y,s)\,\mathrm{d}h(s), \quad y \in G([-r,0],B).$$

For every $b \in [0,\infty)_{\mathbb{T}}$ and $y \in G([-r,b],B)$, the function $u(t) = f(y_t,t)$ is regulated on $[0,b]_{\mathbb{T}}$. Consequently, there is a sequence of continuous functions $u_n : [0,b]_{\mathbb{T}} \to \mathbb{R}^n$, $n \in \mathbb{N}$, which is uniformly convergent to $u$. It follows that $\{u^*_n\}_{n=1}^\infty$ is uniformly convergent to $u^*$ on $[0,b]$. Using Theorem 4.1 and uniform convergence theorems for the Kurzweil-Stieltjes and $\Delta$-integrals, we obtain

$$\int_0^b u(t)\Delta t = \lim_{n\to\infty}\int_0^b u_n(t)\Delta t = \lim_{n\to\infty}\int_0^b u^*_n(t)\,\mathrm{d}h(t) = \int_0^b u^*(t)\,\mathrm{d}h(t) = \int_0^b f(y_{t^*}, t^*)\,\mathrm{d}h(t).$$

Theorem 4.2 implies the existence of $\int_0^b f^*(y_t,t)\,\mathrm{d}h(t)$.

Since $f^*$ and $g^*$ satisfy all assumptions of Theorem 7.1, there exists a constant $J > 0$ such that

$$\|x^*_\varepsilon(t) - y_\varepsilon(t)\| \leq J\varepsilon,$$

for every $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0, L/\varepsilon]$. The proof is finished by observing that $x^*_\varepsilon(t) = x_\varepsilon(t)$ for $t \in [0, L/\varepsilon]_{\mathbb{T}}$. □

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