Global mild solutions for a nonautonomous 2D Navier-Stokes equations with impulses at variable times

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Abstract

The present paper deals with existence and uniqueness of global mild solutions for a new model of Navier-Stokes equations on $\mathbb{R}^2$ subjected to impulse effects at variable times. By using the framework of impulsive/nonautonomous dynamical systems we are able to consider impulse effects in the system as well relax conditions on the external forcing term which is, in our case, non-linear and explicitly time-dependent, extending previous results on the specialized literature. Moreover, we also introduce sufficient conditions on the structure of the impulse set which ensure dissipativity for the system, i.e., uniform boundedness of global solutions starting in bounded sets, which is an indicative to the existence of objects as attractors.

1 Introduction

The Navier-Stokes equations, NSEs for short, are a system of evolution partial differential equations derived from Newton's laws of motion for a continuous distribution of matter in the fluid state characterized by an inability to support shear stresses [13]. These equations allow us to determine the velocity field as well the inner pressure of fluids confined on regions of the Euclidean space. They are used as a model to describe plenty of different physics phenomena as water flow, ocean currents, sound propagation in viscous medium, circulation of nervous impulses throughout the nervous system, among many others. Currently the NSEs are of fundamental importance in both: theoretical and applied point of view. They were the cornerstone of the development of some relevant aspects of mathematical analysis and

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nonlinear differential equations, and became crucial in fields as petroleum industry, plasma physics, meteorology, thermo-hydraulics, among others, see [1, 7, 13, 16–18, 22, 24, 26, 27] and the references therein for a relevant list of applicability. Due to this fact, these equations have been attracted the attention of several important scientists since the middle of the 19th century.

On the other hand, it is well known that many relevant phenomena, including some from fluid dynamics, have their behavior drastically modified somehow after an instantaneous change on their state, which may introduce in the model several discontinuities. Properties as velocity, density and viscosity are discontinuous at interfaces between different fluids as presented in [21]. Despite of the extensive literature on NSEs and the recents progress on the impulsive dynamical systems, surprisedly models from fluid dynamics incorporating impulse effects on its structure are somewhat scarce. A model of NSEs incorporating impulses makes sense physically and allows to describe more precisely some of the phenomena modeled by these equations, mainly when there is an instantaneous change of conditions caused by intrinsic inner/outer factors of the system. Motivated by this fact, we investigate the global well posedness, in the sense of Hadamard, as well the large time behavior of mild solutions of the nonautonomous 2D Navier-Stokes equations with impulses at variable times:

$$\begin{cases} \dfrac{\partial u}{\partial t} + q(t)(u\cdot\nabla)u - \nu\Delta u + \nabla p = f(t,u), & (t,x)\in(0,+\infty)\times\Omega, \\ \operatorname{div} u = 0, & (t,x)\in(0,+\infty)\times\Omega, \\ u = 0, & (t,x)\in(0,+\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0 \in \mathbf{V}, \\ I : M \subset \mathbf{V} \to \mathbf{V}, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, the vector function $u = (u_1, u_2) = (u_1(t,x), u_2(t,x))$ denotes the velocity field of a newtonian fluid filling the domain Ω, $p = p(t,x)$ is its scalar pressure and $\nu > 0$ its viscosity. We shall assume that $q = q(t)$ is a bounded function, maybe a periodic scalar function, and $f = (f_1, f_2) = (f_1(t,u), f_2(t,u))$ is the external force applied to the fluid which is a time dependent nonlinear vector function and is not necessarily continuous (see Section 3). The space $\mathbf{V}$ is the closure of the set $\{v \in (C_0^\infty(\Omega))^2 : \operatorname{div} v = 0 \text{ in } \Omega\}$ in $\mathbb{H}_0^1(\Omega)$ as we explain in the sequel. The set M, called the impulsive set, is a non-empty closed subset of $\mathbf{V}$ and the function I, called the impulse function, is supposed be continuous.

The system (1.1) is considered as the usual impulsive systems, that is, we have an impulsive differential system, an impulsive set and an impulse function. The impulse function is responsible by the discontinuities of the system which occur when the solution of the differential equation hits the impulsive set. In our case, the impulses represent the abrupt changes in the velocity field. A theory that we may use to study a system of type (1.1) is the theory of impulsive dynamical systems which has been increased considerably due to its applicability on real-world problems in physics, technology and biology. The reader may consult the works [2–5, 11, 12, 14, 29, 30] as recent trends on this subject. Furthermore, in the context of impulsive dynamical systems, we may mention the work [9] that presents sufficient conditions for impulsive sets and impulse functions in order to the impulsive system possesses a well behavior in its evolution.

Besides of the impulse actions, we relax the regularity on the nonlinearity f which does not need to be necessarily continuous since all the fluids in the environment are multiphase, that is, property variables are discontinuous at the interfaces between different phases, see [21]. We also incorporate a nonautonomous weight on the convective acceleration term in the model (1.1), which from a dynamical system point of view allows to wonder about the existence of periodic (almost periodic, quasi periodic, recurrent, pseudo recurrent, etc.) solutions as well averaging principles for this nonautonomous NSE. When the model (1.1) is written in the abstract form, this nonautonomous convective term gives rise to a nonlinear nonautonomous operator, which is of a different nature from the forcing term f on the right hand side, and the deep understanding of this operator is the cern of the existence of the special solutions mentioned above. This will be the subject of further investigation.

The system (1.1) without impulse conditions was studied in the classical monograph [8], where $f = f(t, x)$ is continuous in the t variable.

In order to study the existence of solutions as well the topological dynamics of system (1.1), we study initially the existence of solutions for its associated non-impulsive system

$$\begin{cases} \dfrac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu\Delta u + \nabla p = f(t, u), & (t, x) \in (0, +\infty) \times \Omega, \\ \operatorname{div} u = 0, & (t, x) \in (0, +\infty) \times \Omega, \\ u = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, \cdot) = u_0 \in \mathbf{V}. \end{cases} \tag{1.2}$$

Once obtained the well posedness of system (1.2), we analyse system (1.1).

In what follows, we describe the organisation of the paper.

In the first part, namely Section 2, we deal with the following abstract nonautonomous Navier-Stokes equations

$$\begin{cases} \dfrac{du}{dt} + Au + \mathscr{B}(\sigma(t, \omega))(u, u) = \mathscr{F}(t, \sigma(t, \omega), u), & t \in J, \\ u(0) = u_0 \in V, \end{cases} \tag{1.3}$$

where the operators $\mathscr{B}$ and $\mathscr{F}$ are non-stationary, driven by a dynamical system in a Hilbert space V, and $J \subset \mathbb{R}$ is an interval which contains 0. At first, we establish sufficient conditions to obtain the existence and uniqueness of mild solutions (Definition 2.1) for the system (1.3), see Theorem 2.2. In the sequel, we present Theorem 2.4 that exhibits conditions for the mild solution of (1.3) to be prolonged on $\mathbb{R}_+$. In Theorem 2.5, we show that system (1.3) generates a cocycle and in Theorem 2.7 we prove that system (1.3) is bounded dissipative.

Section 3 deals with the existence of mild solution and pressure to the non-impulsive system (1.2). We show that system (1.2) is a particular case of system (1.3). For that, we use the abstract theory of nonautonomous systems presented in [8, 25] which is very meaningful to the study of nonautonomous ordinary and partial equations due its applications. For instance, to study the dynamics of a nonautonomous ordinary differential equation

$$x' = h(t, x), \tag{1.4}$$

with $h : \mathbb{R} \times Z \to \mathbb{R}^n$ ($Z \subset \mathbb{R}^n$ is an open set), we consider the hull of h defined by $H(h) = \overline{\{h_\tau : \tau \in \mathbb{R}\}}$ under the compact open topology, where $h_\tau(t, x) = h(t + \tau, x)$ for all $t, \tau \in \mathbb{R}$ and $x \in Z$. Now, for each $g \in H(h)$ we consider the system

$$x' = g(t, x) \tag{1.5}$$

called the $\mathcal{H}-$class along with the equation (1.4). Under additional conditions, system (1.5) admits a unique solution defined in $\mathbb{R}$, passing through a point $(0, x_0)$, for each $g \in H(h)$. Thus $H(h) \times \mathbb{R} \ni (g, t) \mapsto g_t \in H(h)$ defines a flow on $H(h)$ and

$$\mathbb{R} \times \mathbb{R}^n \times H(h) \ni (t, x_0, g) \mapsto x(t, x_0, g) \in \mathbb{R}^n$$

defines a cocycle (see [8]), where $x(t, x_0, g)$ is the solution of (1.5) with initial value x_0 at time $t = 0$. With the aid of the flow and of the cocycle, we may obtain topological results for system (1.4), especially when long time behavior of solutions are involved. The reader may consult [25] for more details.

In what follows, we give the idea to rewrite the model (1.2) as an abstract evolution equation and for this we introduce some functional spaces as well standard functional analytic tools which can be found on classical monographs as [6, 23, 26, 28].

Let $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2$ be the Lebesgue and Sobolev spaces, respectively endowed with the inner products

$$(u, v)_{\mathbb{L}^2} = \sum_{j=1}^{2} \int_\Omega u_j\, v_j\; dx, \quad u = (u_1, u_2),\; v = (v_1, v_2) \in \mathbb{L}^2(\Omega),$$

and

$$(u, v)_{\mathbb{H}_0^1} = \sum_{j=1}^{2} \int_\Omega \nabla u_j \cdot \nabla v_j dx, \quad u = (u_1, u_2),\; v = (v_1, v_2) \in \mathbb{H}_0^1(\Omega),$$

and norms $\|\cdot\|_{\mathbb{L}^2} = (\cdot, \cdot)_{\mathbb{L}^2}^{1/2}$ and $\|\cdot\|_{\mathbb{H}_0^1} = (\cdot, \cdot)_{\mathbb{H}_0^1}^{1/2}$.

We define

$$\mathcal{E} = \left\{ v \in (C_0^\infty(\Omega))^2 :\; \operatorname{div} v = 0 \text{ in } \Omega \right\},$$

$$\mathbf{H} = \text{closure of } \mathcal{E} \text{ in } \mathbb{L}^2(\Omega) \quad \text{and} \quad \mathbf{V} = \text{closure of } \mathcal{E} \text{ in } \mathbb{H}_0^1(\Omega).$$

We also consider the Leray's ortogonal projection $\Pi : \mathbb{L}^2(\Omega) \to \mathbf{H}$. Recall that for each $u \in \mathbb{L}^2(\Omega)$ there is a unique $\varphi \in H^1(\Omega)$ (up to an additive constant for φ) such that $\Pi u = u - \nabla \varphi$, see [15].

It is well known that $\big(\mathbf{H},(\cdot,\cdot)_{\mathbb{L}^2}\big)$ and $\big(\mathbf{V},(\cdot,\cdot)_{\mathbb{H}_0^1}\big)$ are Hilbert spaces, and by Poincaré's inequality $\mathbf{V}\overset{d}{\hookrightarrow}\mathbf{H}$, i.e., $\mathbf{V}$ is a dense subspace of $\mathbf{H}$ and the inclusion $i:\mathbf{V}\to\mathbf{H}$ is continuous.

Let $\mathbf{H}'$ and $\mathbf{V}'$ be the topological dual spaces of $\mathbf{H}$ and $\mathbf{V}$, respectively. By duality we have $\mathbf{H}'\overset{d}{\hookrightarrow}\mathbf{V}'$ and the inclusion $i^*:\mathbf{H}'\to\mathbf{V}'$ is the adjoint operator of i. By Riesz Representation Theorem's we can identify $\mathbf{H}$ and $\mathbf{H}'$ in order to write

$$\mathbf{V}\overset{d}{\hookrightarrow}\mathbf{H}\equiv\mathbf{H}'\overset{d}{\hookrightarrow}\mathbf{V}',$$

i.e., each space is dense in the following one and the inclusion maps are continuous. As a consequence of the previous identifications, the scalar product in $\mathbf{H}$, $(f,u)_{\mathbb{L}^2}$, of $f\in\mathbf{H}$ and $u\in\mathbf{V}$, is the same as the duality product between $\mathbf{V}'$ and $\mathbf{V}$, $\langle f,u\rangle_{\mathbf{V}',\mathbf{V}}$, i.e.,

$$(f,u)_{\mathbb{L}^2}=\langle f,u\rangle_{\mathbf{V}',\mathbf{V}},\quad \text{for all } f\in\mathbf{H} \text{ and for all } u\in\mathbf{V}.$$

Moreover, for each $u\in\mathbf{V}$ the functional

$$v\in\mathbf{V}\mapsto \nu(u,v)_{\mathbb{H}_0^1}\in\mathbb{R}$$

is an element of $\mathbf{V}'$ and by Lax-Milgran's Theorem there exists an isomorphism $\mathbb{A}:\mathbf{V}\to\mathbf{V}'$ such that

$$\langle\mathbb{A}u,v\rangle_{\mathbf{V}',\mathbf{V}}=\nu(u,v)_{\mathbb{H}_0^1},\ \text{for all } u,v\in\mathbf{V}.$$

Let $\mathbb{A}|_{\mathbf{H}}:D(\mathbb{A}|_{\mathbf{H}})\subset\mathbf{H}\to\mathbf{H}$ be the $\mathbf{H}$-realisation of $\mathbb{A}$, i.e., the linear operator defined as $D(\mathbb{A}|_{\mathbf{H}})=\{u\in\mathbf{V}:\mathbb{A}u\in\mathbf{H}\}$ and $\mathbb{A}|_{\mathbf{H}}u=\mathbb{A}u$ for $u\in D(\mathbb{A}|_{\mathbf{H}})$.

We define the Stokes operator on Ω as

$$A=\Pi\mathbb{A}|_{\mathbf{H}},$$

that is,

$$Au=-\nu\Pi\Delta u\quad\text{for all}\quad u\in D(\mathbb{A}|_{\mathbf{H}}).$$

Assuming regularity of the boundary $\partial\Omega$, it is well known [19,28] that $D(A)=\{u\in\mathbb{H}^2(\Omega)\cap\mathbf{H}:\ u=0 \text{ in } \partial\Omega\}$.

Now, for each $t\in\mathbb{R}_+$, we define $b(t):\mathbf{V}\times\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ by

$$b(t)(u,v,w)=q(t)\big(\Pi(u\cdot\nabla)v,w\big)_{\mathbb{L}^2}=q(t)\left(\sum_{i,j=1}^{2}\int_\Omega\Pi\Big(u_i\frac{\partial v_j}{\partial x_i}\Big)w_j dx\right).$$

Assuming proper conditions on f (see Section 3), the Nemitskii operator is well defined as a map $F(t):\mathbf{V}\to\mathbf{V}'$, $F(t)(u)=\Pi f(t,u)$, and we can rewrite model (1.2) as:

$$\begin{cases}\dfrac{d}{dt}(u(t),v)_{\mathbb{L}^2}+(Au(t),v)_{\mathbb{L}^2}+b(t)(u(t),u(t),v)=(F(t)(u(t)),v)_{\mathbb{L}^2}, & v\in\mathbf{V},\ t>0,\\ u(0)=u_0\in\mathbf{V}.\end{cases}\tag{1.6}$$

System (1.6) is the "weak formulation" of the model (1.2). We also can rewrite this formulation in a more conveniently fashion:

$$\begin{cases} \dfrac{du}{dt} + Au + B(t)(u,u) = F(t)(u), & \text{in } \mathbf{V}', \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \end{cases} \tag{1.7}$$

where $B(t) : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$ is the bilinear operator defined by

$$\langle B(t)(u,v), w\rangle_{\mathbf{V}',\mathbf{V}} = b(t)(u,v,w), \text{ for all } u,v,w \in \mathbf{V}.$$

In the next step, we introduce the $\mathcal{H}-$class along with the equation (1.7), represented by $\mathcal{M} = \mathcal{H}(B,F) = \overline{\{(B_\tau, F_\tau) : \tau \in \mathbb{R}_+\}}$. Consequently, we are able to define the mappings $\mathbf{B} : \mathcal{M} \to \mathscr{L}^2(\mathbf{V}, \mathbf{E})$ and $\mathbf{F} : \mathbb{R}_+ \times \mathcal{M} \times \mathbf{V} \to \mathbf{E}$, where $\mathbf{E} = X^\gamma$ for some $-\frac{1}{2} < \gamma < 0$, see Section 3. These mappings allow us to rewrite system (1.7) into the sytem

$$\begin{cases} \dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t, \sigma(t,\omega), u), & t > 0, \\ u(0) = u_0 \in \mathbf{V}, \end{cases} \tag{1.8}$$

which is a particular system of type (1.3). Thus, using the results presented in Section 2, we obtain existence and uniqueness of a global mild solution of system (1.8), see Theorem 3.4. Theorem 3.5 shows that system (1.8) is bounded dissipative and generates a cocycle. Using the projection of Leray, we obtain the existence of a mild solution and a pressure to the system (1.2), see Theorem 3.6.

Finally, in Section 4, we deal with the impulsive 2D Navier-Stokes equations (1.1). Using the previous construction to obtain (1.8), we get the abstract form of (1.1) given by

$$\begin{cases} \dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t, \sigma(t,\omega), u), & t > 0, \\ u(0) = u_0 \in \mathbf{V}, \\ I : M \to \mathbf{V}, \end{cases} \tag{1.9}$$

and we construct an impulsive cocycle for this system using the theory of nonautonomous impulsive dynamical systems. We also prove that system (1.9) is bounded dissipative and we present a convergence result for the impulsive cocycle generated from system (1.9), see Theorem 4.2.

2 The abstract nonautonomous Navier-Stokes equations

Let $(H, (\cdot,\cdot)_H)$ be a separable real or complex Hilbert space and $A : D(A) \subset H \to H$ be a self-adjoint operator such that, for some $a > 0$, satisfies

$$\operatorname{Re}(Au, u)_H \geqslant a\|u\|_H^2, \quad u \in D(A). \tag{2.1}$$

It follows by [7, Lemma 6.20] that $-A$ generates an analytic semigroup $\{e^{-At}\}_{t\geqslant 0} \subset \mathscr{L}(H)$ satisfying

$$\|e^{-At}\|_{\mathscr{L}(H)} \leqslant Ke^{-at}, \quad t \geqslant 0, \tag{2.2}$$

where $\mathscr{L}(H)$ is the space of bounded operators in H equipped with the usual norm, and $a > 0$ is the constant occurring in (2.1).

Assuming that $0 \in \rho(A)$, we consider the scale of Hilbert spaces $X^\alpha = D(A^\alpha)$ of fractional powers of the operator A endowed with the norm $\|\cdot\|_{X^\alpha} = \|A^\alpha \cdot\|_H$ $(X^0 = H)$. If $\beta > \alpha$, it is well known that $X^\beta \overset{d}{\hookrightarrow} X^\alpha$ and

$$\|e^{-At}\|_{\mathscr{L}(X^\alpha, X^\beta)} \leqslant c\, t^{\alpha-\beta},$$

for $t > 0$ and some constant $c = c(\alpha, \beta)$.

We will also assume that there exist separable Hilbert spaces $(V, (\cdot,\cdot)_V)$ and $(E, (\cdot,\cdot)_E)$ such that $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} E$ and

i) $e^{-At} \in \mathscr{L}(E, V), \quad t > 0;$

ii) There exist constants $0 < \alpha_1 < 1$ and $K_1, K_2 > 0$ such that

$$\|e^{-At}\|_{\mathscr{L}(E,V)} \leqslant K_1 t^{-\alpha_1} e^{-at} \quad \text{and} \quad \|e^{-At}\|_{\mathscr{L}(V,V)} \leqslant K_2 e^{-at} \quad t > 0. \tag{2.3}$$

In the concrete case $V = X^{\frac{1}{2}}$ and $E = X^\gamma$, for some $-\frac{1}{2} < \gamma < 0$, see Section 3.

We also denote by $\mathscr{L}^2(V, E)$ the space of all continuous bilinear operators $\mathcal{B} : V \times V \to E$ equipped with the norm

$$\|\mathcal{B}\|_{\mathscr{L}^2(V,E)} = \sup\{\|\mathcal{B}(u, v)\|_E : \|u\|_V, \|v\|_V \leqslant 1\}.$$

We consider now a metric space $(\mathcal{M}, d)$ and a dynamical system $(\mathcal{M}, \mathbb{R}, \sigma)$ on $\mathcal{M}$, i.e., a continuous map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ which satisfies:

i) $\sigma(0, \omega) = \omega, \quad \omega \in \mathcal{M};$

ii) $\sigma(t + s, \omega) = \sigma(s, \sigma(t, \omega)), \quad t, s \in \mathbb{R}, \omega \in \mathcal{M}.$

Let $\mathscr{B} : \mathcal{M} \to \mathscr{L}^2(V, E)$ be a continuous map such that

$$\|\mathscr{B}\|_\infty = \sup_{\omega \in \mathcal{M}} \|\mathscr{B}(\omega)\|_{\mathscr{L}^2(V,E)} < \infty.$$

In the particular case when $\mathcal{M}$ is compact, $\|\cdot\|_\infty$ defines a norm on $C(\mathcal{M}, \mathscr{L}^2(V, E))$ (the space of all continuous maps $\mathscr{H} : \mathcal{M} \to \mathscr{L}^2(V, E)$) and $\big(C(\mathcal{M}, \mathscr{L}^2(V, E)), \|\cdot\|_\infty\big)$ is a Banach space.

Also, let us assume that for all $u, v \in V$ and $\omega \in \mathcal{M}$,

$$\|\mathscr{B}(\omega)(u, u) - \mathscr{B}(\omega)(v, v)\|_E \leqslant \|\mathscr{B}\|_\infty(\|u\|_V + \|v\|_V)\|u - v\|_V, \tag{2.4}$$

and

$$\|\mathscr{B}(\omega)(u,v)\|_E \leqslant \|\mathscr{B}\|_\infty \|u\|_V \|v\|_V. \tag{2.5}$$

Additionally we assume that

$$\mathrm{Re}\,(\mathscr{B}(\omega)(u,v),w)_E = -\mathrm{Re}\,(\mathscr{B}(\omega)(u,w),v)_E, \quad u,v,w \in V, \omega \in \mathcal{M}, \tag{2.6}$$

which implies the orthogonality condition

$$\mathrm{Re}\,(\mathscr{B}(\omega)(u,v),v)_E = 0, \quad u,v \in V, \omega \in \mathcal{M}. \tag{2.7}$$

Let X be a Banach space and $J \subset \mathbb{R}$ be an interval. A function $g\colon J \to X$ is called *regulated in* X if g has only discontinuities of the first kind, i.e., if the lateral limits

$$g(t^-) = \lim_{s\to t^-} g(s) \quad \text{and} \quad g(t^+) = \lim_{s\to t^+} g(s)$$

exist for all $t \in J$, where they make sense. The set of all regulated functions $g\colon J \to X$ will be denoted by $G\,(J,X)$. The space $G\,(J,X)$ equipped with the usual supremum norm $\|g\|_G = \sup\limits_{t\in J} \|g(t)\|_X$ is a Banach space when J is a bounded interval. If J is unbounded, we consider in $G\,(J,X)$ the topology of the locally uniform convergence, see [20, Theorem 3.6] for more details. We notice that $g \in G\,(J,X)$ iff g is the uniform limit of step functions.

Now, let $\mathscr{F} : J \times \mathcal{M} \times V \to E$ be a function satisfying the following conditions:

(C1) For each fixed $t \in J$, $\mathscr{F}(t,\cdot,\cdot)$ is continuous on $\mathcal{M} \times V$.

(C2) For each $\omega \in \mathcal{M}$ and $u \in V$, we have $\mathscr{F}(\cdot,\omega,u) \in G(J,E)$.

(C3) There is a bounded function $M : \mathbb{R} \to \mathbb{R}_+$, such that for any interval $[a,b] \subset J$, we have

$$\int_a^b |\phi(s)| \|\mathscr{F}(s,\omega,u)\|_E ds \leqslant \int_a^b M(s)|\phi(s)| ds$$

for all $\phi \in L^1[a,b]$, $\omega \in \mathcal{M}$ and $u \in V$.

(C4) There is a bounded function $L : \mathbb{R} \to \mathbb{R}_+$, such that for any interval $[a,b] \subset J$, we have

$$\int_a^b |\phi(s)| \|\mathscr{F}(s,\omega_1,u_1) - \mathscr{F}(s,\omega_2,u_2)\|_E \, ds \leqslant \int_a^b L(s)|\phi(s)|(d(\omega_1,\omega_2) + \|u_1 - u_2\|_V) ds$$

for all $\phi \in L^1[a,b]$, $\omega_1,\omega_2 \in \mathcal{M}$ and $u_1,u_2 \in V$.

(C5) $\|\mathscr{F}\|_1 = \sup\{\|\mathscr{F}(t,\omega,u)\|_E :\, t \in J, \omega \in \mathcal{M}, u \in V\} < \infty$.

Given $J \subset \mathbb{R}_+$ an interval containing 0, $\omega \in \mathcal{M}$ and assuming all the conditions above, we consider the following abstract nonautonomous Navier-Stokes system (without impulses) in the state space V:

$$\begin{cases} \dfrac{du}{dt} + Au + \mathscr{B}(\sigma(t,\omega))(u,u) = \mathscr{F}(t,\sigma(t,\omega),u), \quad t \in J, \\ u(0) = u_0 \in V. \end{cases} \tag{2.8}$$

Definition 2.1. We say that a continuous function $u : J \to V$ is a *mild solution* of (2.8) if u satisfies the following integral equation:

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}\mathscr{F}(s,\sigma(s,\omega),u(s))ds - \int_0^t e^{-A(t-s)}\mathscr{B}(\sigma(s,\omega))(u(s),u(s))ds,$$

for all $t \in J$.

In the next result, we present sufficient conditions to obtain the existence and uniqueness of a mild solution to the system (2.8).

Theorem 2.2. *Let $u_0 \in V$ and $r > 0$. Then there exist positive numbers $\delta = \delta(u_0, r) > 0$, $T = T(u_0, r) > 0$ and a function $\varphi : [0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M} \to V$ ($\overline{B(u_0,\delta)} = \{u \in V : \|u - u_0\|_V \leqslant \delta\}$) satisfying the following conditions:*

i) $\varphi(0, u_0, \omega) = u_0$, *for all* $\omega \in \mathcal{M}$;

ii) $\|\varphi(t,u,\omega) - u_0\|_V \leqslant r$ *for all* $(t,u,\omega) \in [0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}$;

iii) $\varphi \in C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, \overline{B(u_0,r)})$.

Moreover, the function $u : [0,T] \to V$ defined by $u(t) = \varphi(t,u_0,\omega)$ is the unique mild solution of the system (2.8).

Proof. Let $\delta > 0$ and $T > 0$ be such that $[0,T] \subset J$. Given $\varphi \in C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, V)$, we define

$$S\varphi(t,u,\omega) = e^{-At}u + \int_0^t e^{-A(t-s)}g(s,\omega,\varphi(s))ds,$$

where $\varphi(s) = \varphi(s,u,\omega)$ and $g(s,\omega,\varphi(s)) = -\mathscr{B}(\sigma(s,\omega))(\varphi(s),\varphi(s)) + \mathscr{F}(s,\sigma(s,\omega),\varphi(s))$, for all $s \in [0,T]$, $u \in \overline{B(u_0,\delta)}$ and $\omega \in \mathcal{M}$. Since functions in $C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, \overline{B(u_0,r)})$ are bounded, we can consider the distance

$$d_\infty(\varphi_1,\varphi_2) = \sup\{\|\varphi_1(t,u,\omega) - \varphi_2(t,u,\omega)\|_V : 0 \leqslant t \leqslant T,\ u \in \overline{B(u_0,\delta)},\ \omega \in \mathcal{M}\},$$

for $\varphi_1, \varphi_2 \in C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, \overline{B(u_0,r)})$. Note that $(C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, \overline{B(u_0,r)}), d_\infty)$ is a complete metric space. For convenience, let us denote $\Gamma(\delta,T,r) = C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, \overline{B(u_0,r)})$ and $\Gamma(\delta,T) = C([0,T] \times \overline{B(u_0,\delta)} \times \mathcal{M}, V)$.

Assertion 1: $S \in C(\Gamma(\delta, T, r), \Gamma(\delta, T))$.

In fact, at first note that $S\varphi \in \Gamma(\delta, T)$ for all $\varphi \in \Gamma(\delta, T, r)$. Now, let $\varphi_1, \varphi_2 \in \Gamma(\delta, T, r)$ and $(t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$. By Condition (C4) there is a bounded function $L : \mathbb{R} \to \mathbb{R}_+$ such that

$$\int_0^t e^{-a(t-s)}(t-s)^{-\alpha_1} \|\mathscr{F}(s, \sigma(s, \omega), \varphi_1(s)) - \mathscr{F}(s, \sigma(s, \omega), \varphi_2(s))\|_E \, ds \leqslant$$

$$\leqslant \int_0^t L(s)(t-s)^{-\alpha_1} \|\varphi_1(s) - \varphi_2(s)\|_V \, ds \leqslant N d_\infty(\varphi_1, \varphi_2) \frac{T^{1-\alpha_1}}{1-\alpha_1}, \tag{2.9}$$

where $N = \sup_{s \in [0,T]} |L(s)|$.

Then, using (2.3), (2.4) and (2.9), we have

$$\|S\varphi_1(t, u, \omega) - S\varphi_2(t, u, \omega)\|_V \leqslant$$

$$\leqslant \int_0^t \left\| e^{-A(t-s)} \left[\mathscr{B}(\sigma(s, \omega))(\varphi_1(s), \varphi_1(s)) - \mathscr{B}(\sigma(s, \omega))(\varphi_2(s), \varphi_2(s)) \right] \right\|_V ds +$$

$$+ \int_0^t \left\| e^{-A(t-s)} \left[\mathscr{F}(s, \sigma(s, \omega), \varphi_1(s)) - \mathscr{F}(s, \sigma(s, \omega), \varphi_2(s)) \right] \right\|_V ds \leqslant$$

$$\leqslant 2\|\mathscr{B}\|_\infty K_1(r + \|u_0\|_V) d_\infty(\varphi_1, \varphi_2) \int_0^t (t-s)^{-\alpha_1} e^{-a(t-s)} ds + K_1 N \frac{T^{1-\alpha_1}}{1-\alpha_1} d_\infty(\varphi_1, \varphi_2) \leqslant$$

$$\leqslant \left(2\|\mathscr{B}\|_\infty K_1(r + \|u_0\|_V) \frac{T^{1-\alpha_1}}{1-\alpha_1} + K_1 N \frac{T^{1-\alpha_1}}{1-\alpha_1} \right) d_\infty(\varphi_1, \varphi_2). \tag{2.10}$$

Hence, $S \in C(\Gamma(\delta, T, r), \Gamma(\delta, T))$.

Assertion 2: There are $\delta_1 = \delta_1(u_0, r) \in (0, \delta)$ and $T_1 = T_1(u_0, r) \in (0, T)$ such that $S : \Gamma(\delta_1, T_1, r) \to \Gamma(\delta_1, T_1, r)$.

In fact, let $\varphi \in \Gamma(\delta, T, r)$ and $(t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$. By (2.3) and Condition $(C4)$, one can obtain a bounded function $L : \mathbb{R} \to \mathbb{R}_+$ such that

$$\left\| \int_0^t e^{-A(t-s)} \left[\mathscr{F}(s, \sigma(s, \omega), \varphi(s)) - \mathscr{F}(s, \sigma(s, \omega), 0) \right] ds \right\|_V \leqslant$$

$$\leqslant \int_0^t K_1 (t-s)^{-\alpha_1} e^{-a(t-s)} \|\mathscr{F}(s, \sigma(s, \omega), \varphi(s)) - \mathscr{F}(s, \sigma(s, \omega), 0)\|_E ds \leqslant$$

$$\leqslant K_1 \int_0^t L(s)(t-s)^{-\alpha_1} \|\varphi(s)\|_V ds \leqslant K_1 N(\|u_0\|_V + r) \frac{T^{1-\alpha_1}}{1-\alpha_1}, \tag{2.11}$$

where $N = \sup_{s \in [0,T]} |L(s)|$.

Let $m(\delta, T) = \sup\left\{\|e^{-At}u - u_0\|_V : t \in [0,T],\ u \in \overline{B(u_0,\delta)}\right\}$ and $\mathscr{M} = \sup_{s\in[0,T]} |M(s)|$, where M is the function given by Condition (C3). Then, using (2.3), (2.5), (2.11) and Condition (C3), we obtain

$$\|S\varphi(t,u,\omega) - u_0\|_V \leqslant \left\|e^{-At}u - u_0\right\|_V + \left\|\int_0^t e^{-A(t-s)}\mathscr{B}(\sigma(s,\omega))(\varphi(s),\varphi(s))ds\right\|_V +$$

$$+\left\|\int_0^t e^{-A(t-s)}\mathscr{F}(s,\sigma(s,\omega),\varphi(s))ds\right\|_V \leqslant m(\delta,T) + \int_0^t K_1 e^{-a(t-s)}(t-s)^{-\alpha_1}\|\mathscr{B}\|_\infty \|\varphi(s)\|_V^2 ds +$$

$$+\left\|\int_0^t e^{-A(t-s)}\left[\mathscr{F}(s,\sigma(s,\omega),\varphi(s)) - \mathscr{F}(s,\sigma(s,\omega),0)\right]ds\right\|_V + \left\|\int_0^t e^{-A(t-s)}\mathscr{F}(s,\sigma(s,\omega),0)ds\right\|_V \leqslant$$

$$\leqslant m(\delta,T) + K_1\|\mathscr{B}\|_\infty(\|u_0\|_V + r)^2\frac{T^{1-\alpha_1}}{1-\alpha_1} + K_1 N\frac{T^{1-\alpha_1}}{1-\alpha_1}(\|u_0\|_V + r) +$$

$$+\int_0^t K_1(t-s)^{-\alpha_1}e^{-a(t-s)}M(s)ds \leqslant m(\delta,T) + K_1\|\mathscr{B}\|_\infty(\|u_0\|_V + r)^2\frac{T^{1-\alpha_1}}{1-\alpha_1} +$$

$$K_1 N(\|u_0\|_V + r)\frac{T^{1-\alpha_1}}{1-\alpha_1} + K_1\mathscr{M}\frac{T^{1-\alpha_1}}{1-\alpha_1} := d_1(u_0,r,\delta,T).$$

Now, we note that $d_1(u_0,r,\delta,T) \to 0$ as $\delta \to 0$ and $T \to 0$. Therefore, there are $\delta_1 = \delta_1(u_0,r) > 0, \delta_1 < \delta$, and $T_1 = T_1(u_0,r) > 0, T_1 < T$, such that $d_1(u_0,r,\delta',T') \leqslant r$ for all $\delta' \in (0,\delta_1]$ and $T' \in (0,T_1]$.

Assertion 3: There exist $T_0 = T_0(u_0,r) > 0$ and $\delta_0 = \delta_0(u_0,r) > 0$ such that $S : \Gamma(\delta_0,T_0,r) \to \Gamma(\delta_0,T_0,r)$ is a contraction.

Indeed, we may obtain $T_2 > 0$ such that

$$2\|\mathscr{B}\|_\infty K_1(r + \|u_0\|_V)\frac{T_2^{1-\alpha_1}}{1-\alpha_1} + K_1 N\frac{T_2^{1-\alpha_1}}{1-\alpha_1} < 1.$$

It is enough to take $\delta_0 = \delta_1$, $T_0 = \min\{T_1,T_2\}$ and use (2.10) to conclude Assertion 3.

In conclusion, by Banach Fixed Point Theorem, there exists a unique function $\varphi \in \Gamma(\delta_0,T_0,r)$ satisfying the system (2.8) on the interval $[0,T_0]$ and the result follows. □

Since the assumptions (2.3), (2.4), (2.6) and conditions (C1)–(C5) hold, we have the next result on the boundedness of the solution of system (2.8).

Lemma 2.3. *The inequality*

$$\|\varphi(t,u_0,\omega)\|_V \leqslant \max\left\{\|u_0\|_V, \frac{\|\mathscr{F}\|_1}{a}\right\}$$

holds for all $t \in [0,\alpha_{(u_0,\omega)})$, $\omega \in \mathcal{M}$ *and* $u_0 \in V$, *where* a *is given by* (2.1) *and* $[0,\alpha_{(u_0,\omega)})$ *denotes the maximal interval of existence of the solution* $\varphi(t,u_0,\omega)$ *of* (2.8).

Proof. Let $u_0 \in V$ and $\omega \in \mathcal{M}$. Using the proof of [8, Example 11.1], we may conclude that

$$\|\varphi(t, u_0, \omega)\|_V \leqslant \left(\|u_0\|_V - \frac{\|\mathscr{F}\|_1}{a}\right) e^{-at} + \frac{\|\mathscr{F}\|_1}{a}, \quad t \in [0, \alpha_{(u_0,\omega)}). \tag{2.12}$$

Hence, we obtain the desired result. □

The next theorem shows that the mild solution of system (2.8) may be prolongated on $\mathbb{R}_+$.

Theorem 2.4. *Assume that* $J = \mathbb{R}_+$. *Then the mild solution of system* (2.8) *may be prolonged on* $\mathbb{R}_+$.

Proof. By Theorem 2.2, there is a unique mild solution $\varphi(t, u_0, \omega)$ of system (2.8) passing through the point $u_0 \in V$ at time $t = 0$. This solution is defined on some maximal interval $[0, \alpha_{(u_0,\omega)})$. Suppose that $\alpha_{(u_0,\omega)} < \infty$. Since $\varphi(t, u_0, \omega)$ is bounded (Lemma 2.3), we define

$$\varphi(\alpha_{(u_0,\omega)}, u_0, \omega) = \lim_{t \to \alpha_{(u_0,\omega)}-} \varphi(t, u_0, \omega).$$

Then $\varphi(t, u_0, \omega)$ may be extended on the interval $[0, \alpha_{(u_0,\omega)}]$ which is a contradiction. Hence, $\alpha_{(u_0,\omega)} = +\infty$. □

Another property that we may establish for the system (2.8) is that the set $\{\varphi(t, u_0, \omega) : t \in \mathbb{R}_+, u_0 \in V, \omega \in \mathcal{M}\}$ defines a cocycle (see [8] for more details), where $\varphi(t, u_0, \omega)$ (for each fixed $u_0 \in V$ and $\omega \in \mathcal{M}$) is the unique solution of (2.8) defined on $\mathbb{R}_+$ with initial condition $\varphi(0, u_0, \omega) = u_0$. In other words, the mapping φ satisfies the properties:

(*i*) $\varphi(0, u_0, \omega) = u_0$ for all $u_0 \in V$ and $\omega \in \mathcal{M}$,

(*ii*) $\varphi(t + s, u_0, \omega) = \varphi(t, \varphi(s, u_0, \omega), \sigma(s, \omega))$ for all $t, s \in \mathbb{R}_+$ and $\omega \in \mathcal{M}$,

(*iii*) the map $\mathbb{R}_+ \times V \times \mathcal{M} \ni (t, u_0, \omega) \mapsto \varphi(t, u_0, \omega) \in V$ is continuous.

From the proof of [8, Lemma 11.3], we have the following result.

Theorem 2.5. *Let* $J = \mathbb{R}_+$. *Then the abstract nonautonomous Navier-Stokes equation* (2.8) *generates a cocycle* φ.

Definition 2.6. We say that system (2.8) is *bounded dissipative* on V if there is a nonempty bounded set $B_0 \subset V$ such that for each bounded set $B \subset V$ there exists $T = T(B) > 0$ such that $\varphi(t, u_0, \omega) \in B_0$ for all $t \geqslant T$, $u_0 \in B$ and $\omega \in \mathcal{M}$. In this case, B_0 is called a *bounded attractor* for the system (2.8).

Let $J = \mathbb{R}_+$. By (2.12), we have

$$\lim_{t\to+\infty} \sup_{\|u_0\|_V \leqslant r,\, \omega\in\mathcal{M}} \|\varphi(t,u_0,\omega)\|_V \leqslant \frac{\|\mathscr{F}\|_1}{a},$$

for all $r > 0$. Thus, the set $B_0 = \left\{ u \in V : \|u\|_V \leqslant \frac{\|\mathscr{F}\|_1}{a} \right\}$ is a bounded attractor for the system (2.8) and we have the next result.

Theorem 2.7. *The system* (2.8) *is bounded dissipative.*

3 The 2D Navier-Stokes equations

In this section we translate the abstract nonautonomous formulation of Section 2 to obtain existence and uniqueness of global mild solutions for the following 2D NSE's without impulses

$$\begin{cases} \dfrac{\partial u}{\partial t} + q(t)(u\cdot\nabla)u - \nu\Delta u + \nabla p = f(t,u), & (t,x) \in (0,+\infty)\times\Omega, \\ \operatorname{div} u = 0, & (t,x) \in (0,+\infty)\times\Omega, \\ u = 0, & (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot), & x \in \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^2$.

As presented in the Introduction, we assume that the weight $q(t)$ is a bounded function. With respect to the external forcing term f we shall assume the following hypotheses:

(H1) $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded function such that for each fixed $t \in \mathbb{R}_+$, $f(t,\cdot)$ is continuous on $\mathbb{R}^2$.

(H2) For each $x \in \mathbb{R}^2$, $f(\cdot,x) \in G(\mathbb{R}_+,\mathbb{R}^2)$.

(H3) There is $C > 0$ such that $|f(s,x) - f(s,y)| \leqslant C|x-y|$ for all $s \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}^2$.

Using the notations exhibited in the Introduction, we can rewrite system (3.1) as the abstract evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au + B(t)(u,u) = F(t)(u), \quad \text{in } \mathbf{V}', \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \end{cases} \tag{3.2}$$

where $Au = -\nu\Pi\Delta u$, $F(t)(u) = \Pi f(t,u)$ and $B(t)(u,u) = q(t)\Pi\big((u\cdot\nabla)u\big)$.

It is well known [28] that the Stokes operator A is a positive self-adjoint operator with domain $D(A)$ dense in $\mathbf{H}$, $0 \in \rho(A)$ and its inverse A^{-1} is compact. In particular, A satisfies (2.1), that is, there exists $\alpha > 0$ such that

$$\langle Au, u\rangle_{\mathbf{V}',\mathbf{V}} \geqslant \alpha \|u\|^2_{\mathbf{H}}, \tag{3.3}$$

for all $u \in D(A)$, and $-A$ is the generator of an analytic semigroup $\{e^{-At}\}_{t\geqslant 0}$ on $\mathbf{H}$. Moreover,

$$\mathbf{V} = X^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{\mathbf{V}} = \|u\|_{\mathbb{H}^1_0}.$$

Setting $\mathbf{E} = X^{\gamma}$ for some $-\frac{1}{2} < \gamma < 0$, $\{e^{-At}\}_{t\geqslant 0}$ satisfies (2.3) with $\alpha_1 = \frac{1}{2} + \gamma < 1$. Note that $B \in G(\mathbb{R}_+, \mathscr{L}^2(\mathbf{V}, \mathbf{E})) \subset G(\mathbb{R}_+, \mathscr{L}^2(\mathbf{V}, \mathbf{V}'))$ and by integration by parts, we also have the orthogonality property of the convective term

$$(B(t)(u, v), v)_{\mathbf{E}} = 0, \tag{3.4}$$

for all $u, v \in \mathbf{V}$ and for all $t \in \mathbb{R}_+$, which expresses the conservation of energy of the inertial forces acting on the fluid, see [23].

Since we will use the results from Section 2 to obtain the existence and uniqueness of mild solutions to the system (3.1), we show in the next lines how to transform system (3.2) into a nonautonomous system of form (2.8).

Note that $F \in G(\mathbb{R}_+, C(\mathbf{V}, \mathbf{E}))$. Let us denote $\mathbf{Y} = G(\mathbb{R}_+, \mathscr{L}^2(\mathbf{V}, \mathbf{E})) \times G(\mathbb{R}_+, C(\mathbf{V}, \mathbf{E}))$ and let $(\mathbf{Y}, \mathbb{R}_+, \sigma)$ be the semidynamical system of translations, that is, $\sigma(\tau, g) = g_\tau = g(\tau + \cdot)$ for all $g \in \mathbf{Y}$ and $t \geqslant 0$. Now, set

$$\mathcal{M} = \mathcal{H}(B, F) = \overline{\{(B_\tau, F_\tau) : \tau \in \mathbb{R}_+\}} \subset \mathbf{Y},$$

where $B_\tau(t) = B(t + \tau)$ for all $t \in \mathbb{R}_+$ and $F_\tau(t)(u) = F(t + \tau)(u)$ for all $t \in \mathbb{R}_+$ and $u \in \mathbf{V}$ and by bar we denote the closure in the compact-open topology. We set $(\mathcal{M}, \mathbb{R}_+, \sigma|_{\mathcal{M}})$ the semidynamical system of translations on $\mathcal{M}$, where $\sigma|_{\mathcal{M}}(\tau, (\mathcal{B}, \mathcal{F})) = \sigma(\tau, (\mathcal{B}, \mathcal{F})) = (\mathcal{B}_\tau, \mathcal{F}_\tau)$, for $(\mathcal{B}, \mathcal{F}) \in \mathcal{M}$.

According to [8], the equation

$$\frac{du}{dt} + Au + \mathcal{B}(t)(u, u) = \mathcal{F}(t)(u), \tag{3.5}$$

where $(\mathcal{B}, \mathcal{F}) \in \mathcal{M}$, is called the $\mathcal{H}-$class along with the equation (3.2).

Now, we define the mappings $\mathbf{B} : \mathcal{M} \to \mathscr{L}^2(\mathbf{V}, \mathbf{E})$ by

$$\mathbf{B}(\sigma(t, \omega)) = \mathbf{B}(\mathcal{B}_t, \mathcal{F}_t) := \mathcal{B}_t(0), \quad \text{for all} \quad \omega = (\mathcal{B}, \mathcal{F}) \in \mathcal{M} \ \text{ and } \ t \geqslant 0,$$

and $\mathbf{F} : \mathbb{R}_+ \times \mathcal{M} \times \mathbf{V} \to \mathbf{E}$ by

$$\mathbf{F}(t, \sigma(s, \omega), u) = \mathbf{F}(t, (\mathcal{B}_s, \mathcal{F}_s), u) := \mathcal{F}_s(0)(u), \ \text{ for all } u \in \mathbf{V}, \omega = (\mathcal{B}, \mathcal{F}) \in \mathcal{M} \text{ and } t, s \geqslant 0.$$

Then equation (3.5) can be rewritten in the form

$$\frac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t,\sigma(t,\omega),u).$$

Therefore, associated with system (3.1), we have the following abstract system

$$\begin{cases} \dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t,\sigma(t,\omega),u), \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}. \end{cases} \tag{3.6}$$

In the sequel, we show that $\mathbf{F}$ and $\mathbf{B}$ satisfy the conditions presented in Section 2.

Since $L^2(\Omega) \overset{d}{\hookrightarrow} X^\gamma = \mathbf{E}$ $(\gamma < 0)$, there is a constant $\bar{c} > 0$ such that

$$\|\Pi f(t,u)\|_{\mathbf{E}} \leqslant \bar{c}\|\Pi f(t,u)\|_{\mathbb{L}^2} \leqslant \bar{c}\|f(t,u)\|_{\mathbb{L}^2},$$

for all $u \in \mathbf{V}$ and $t \geqslant 0$. Using condition (H1), there is $\eta > 0$ such that

$$\sup_{\substack{u \in \mathbf{V} \\ t \geqslant 0}} \|\Pi f(t,u)\|_{\mathbf{E}} \leqslant \eta. \tag{3.7}$$

On the other hand, by the boundedness of $q(t)$, there is $\ell > 0$ such that

$$\sup_{\|u\|_{\mathbf{V}} \leqslant 1, \|v\|_{\mathbf{V}} \leqslant 1} \|B(t)(u,v)\|_{\mathbf{E}} = \sup_{\|u\|_{\mathbf{V}} \leqslant 1, \|v\|_{\mathbf{V}} \leqslant 1} \|q(t)\Pi((u \cdot \nabla)v)\|_{\mathbf{E}} \leqslant \ell. \tag{3.8}$$

In this way, for $(\mathcal{B},\mathcal{F}) \in \mathcal{M}$, we may consider the norms

$$\|\mathcal{F}\|_G = \sup_{\substack{u \in \mathbf{V} \\ t \geqslant 0}} \|\mathcal{F}(t)(u)\|_{\mathbf{E}} \quad \text{and} \quad \|\mathcal{B}\|_* = \sup_{t \geqslant 0} \|\mathcal{B}(t)\|_{\mathscr{L}^2(\mathbf{V},\mathbf{E})},$$

which are well defined since $\|\mathcal{F}\|_G \leqslant \|F\|_G < \infty$ by (3.7) and $\|\mathcal{B}\|_* \leqslant \|B\|_* < \infty$ by (3.8). Moreover,

$$\|\mathcal{B}(t)(u,v)\|_{\mathbf{E}} \leqslant \|\mathcal{B}\|_* \|u\|_V \|v\|_V \quad \text{for all} \quad u,v \in \mathbf{V}.$$

Now, using the identity $\mathcal{B}(t)(u,u) - \mathcal{B}(t)(v,v) = \mathcal{B}(t)(u-v,v) - \mathcal{B}(t)(u,v-u)$, we obtain

$$\|\mathcal{B}(t)(u,u) - \mathcal{B}(t)(v,v)\|_{\mathbf{E}} \leqslant \|\mathcal{B}\|_*(\|u\|_{\mathbf{V}} + \|v\|_{\mathbf{V}})\|u-v\|_{\mathbf{V}}, \tag{3.9}$$

for all $u,v \in \mathbf{V}$.

Lemma 3.1. $\|\mathbf{B}\|_\infty = \sup_{\omega \in \mathcal{M}} \|\mathbf{B}(\omega)\|_{\mathscr{L}^2(\mathbf{V},\mathbf{E})} = \|B\|_*$.

Proof. Note that

$$\sup_{\omega \in \mathcal{M}} \|\mathbf{B}(\omega)\|_{\mathscr{L}^2(\mathbf{V},\mathbf{E})} = \sup_{\omega \in \{(B_\tau,F_\tau):\, \tau \in \mathbb{R}_+\}} \|\mathbf{B}(\omega)\|_{\mathscr{L}^2(\mathbf{V},\mathbf{E})} = \sup_{\tau \geqslant 0} \|B(\tau)\|_{\mathscr{L}^2(\mathbf{V},\mathbf{E})} = \|B\|_*.$$

□

Let us consider in $\mathcal{M}$ the metric $d_{\mathcal{M}}$ given by

$$d_{\mathcal{M}}(\omega_1, \omega_2) = d_{\mathcal{M}}((\mathcal{B}_1, \mathcal{F}_1), (\mathcal{B}_2, \mathcal{F}_2)) = \|\mathcal{B}_1 - \mathcal{B}_2\|_* + \|\mathcal{F}_1 - \mathcal{F}_2\|_G,$$

for all $\omega_1, \omega_2 \in \mathcal{M}$. Using the metric $d_{\mathcal{M}}$, (3.4), (3.9) and Lemma 3.1, we have the following immediate result.

Lemma 3.2. *The mapping* $\mathbf{B} : \mathcal{M} \to \mathscr{L}^2(\mathbf{V}, \mathbf{E})$ *is continuous and satisfies conditions* (2.4) *and* (2.6)*.*

Next, we show that $\mathbf{F}$ satisfies the conditions (C1)–(C4) presented in Section 2. This will help us to show that system (3.6) admits a unique global mild solution, see Theorem 3.4 in the sequel.

Lemma 3.3. *The mapping* $\mathbf{F} : \mathbb{R}_+ \times \mathcal{M} \times \mathbf{V} \to \mathbf{E}$ *satisfies the conditions (C1)–(C5).*

Proof. First, let us show that $\mathbf{F}$ satisfies condition (C1). Let $t \in \mathbb{R}_+$ be fixed. Take a sequence $(\omega_n, u_n), (\omega_0, u_0) \in \mathcal{M} \times \mathbf{V}$, $n = 1, 2, \ldots$, such that

$$d_{\mathcal{M}}(\omega_n, \omega_0) \to 0 \quad \text{and} \quad \|u_n - u_0\|_{\mathbf{V}} \to 0,$$

as $n \to +\infty$. Note that $\omega_0 = (\mathcal{B}, \mathcal{F})$ and $\omega_n = (\mathcal{B}_n, \mathcal{F}_n)$, $n = 1, 2, 3, \ldots$. Moreover,

$$\|\mathcal{F}_n - \mathcal{F}\|_G \to 0 \quad \text{as} \quad n \to +\infty.$$

Since $(\mathcal{B}_n, \mathcal{F}_n), (\mathcal{B}, \mathcal{F}) \in \mathcal{M}$, $n = 1, 2, 3, \ldots$, there exist sequences $\{\tau_k^n\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\mathcal{F}_n(r)(u) = \lim_{k \to +\infty} F_{\tau_k^n}(r)(u) \quad \text{and} \quad \mathcal{F}(r)(u) = \lim_{k \to +\infty} F_{s_k}(r)(u)$$

for each $(r, u) \in \mathbb{R}_+ \times \mathbf{V}$ and $n \in \mathbb{N}$.

By the inclusions $\mathbf{V} \overset{d}{\hookrightarrow} \mathbf{H} \overset{d}{\hookrightarrow} \mathbf{E}$, there are $c_1, c_2 > 0$ such that

$$\|\Pi f(t, v_1) - \Pi f(t, v_2)\|_{\mathbf{E}} \leqslant c_1 \|\Pi f(t, v_1) - \Pi f(t, v_2)\|_{\mathbf{H}} \leqslant$$

$$\leqslant c_1 C \|v_1 - v_2\|_{\mathbf{H}} \leqslant c_1 c_2 C \|v_1 - v_2\|_{\mathbf{V}},$$

for all $v_1, v_2 \in \mathbf{V}$ and $t \geqslant 0$, where C comes from condition (H3). Then,

$$\|\mathbf{F}(t, \omega_n, u_n) - \mathbf{F}(t, \omega_0, u_0)\|_{\mathbf{E}} = \|\mathcal{F}_n(0)(u_n) - \mathcal{F}(0)(u_0)\|_{\mathbf{E}} = \lim_{k \to +\infty} \|F(\tau_k^n)(u_n) - F(s_k)(u_0)\|_{\mathbf{E}}$$

$$\leqslant \lim_{k \to +\infty} \|F(\tau_k^n)(u_n) - F(\tau_k^n)(u_0)\|_{\mathbf{E}} + \lim_{k \to +\infty} \|F(\tau_k^n)(u_0) - F(s_k)(u_0)\|_{\mathbf{E}}$$

$$= \lim_{k \to +\infty} \|\Pi f(\tau_k^n, u_n) - \Pi f(\tau_k^n, u_0)\|_{\mathbf{E}} + \|\mathcal{F}_n(0)(u_0) - \mathcal{F}(0)(u_0)\|_{\mathbf{E}}$$

$$\leqslant c_1c_2C\|u_n - u_0\|_{\mathbf{V}} + \|\mathcal{F}_n - \mathcal{F}\|_G.$$

Therefore, we may conclude that $\|\mathbf{F}(t,\omega_n,u_n) - \mathbf{F}(t,\omega_0,u_0)\|_{\mathbf{E}} \to 0$ as $n \to +\infty$.

Condition (C2) is an immediate consequence of the definition of $\mathbf{F}$.

In order to show that Condition (C3) holds, we define the function $M : \mathbb{R} \to \mathbb{R}_+$ by $M(t) = \eta$ for all $t \in \mathbb{R}$, where η comes from (3.7). Given $\omega = (\mathcal{B},\mathcal{F}) \in \mathcal{M}$, there is a sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\mathbf{F}(t,\omega,u) = \mathcal{F}(0)(u) = \lim_{n\to+\infty} F_{r_n}(0)(u) = \lim_{n\to+\infty} \Pi f(r_n,u),$$

for all $t \geqslant 0$ and $u \in \mathbf{V}$. Then, for $[a,b] \subset \mathbb{R}_+$ and using (3.7), we have

$$\int_a^b |\phi(s)|\|\mathbf{F}(s,\omega,u)\|_{\mathbf{E}}ds \leqslant \int_a^b |\phi(s)|\eta ds = \int_a^b M(s)|\phi(s)|ds,$$

for all $\phi \in L^1[a,b]$, $\omega \in \mathcal{M}$ and $u \in \mathbf{V}$.

Now, let us verify the Condition (C4) holds. Define $L : \mathbb{R} \to \mathbb{R}_+$ by $L(t) = c_1c_2C + 1$, $t \in \mathbb{R}$. Then, given $[a,b] \subset \mathbb{R}_+$, we have

$$\int_a^b |\phi(s)|\|\mathbf{F}(s,\omega_1,u_1) - \mathbf{F}(s,\omega_2,u_2)\|_{\mathbf{E}}ds = \int_a^b |\phi(s)|\|\mathcal{F}_1(0)(u_1) - \mathcal{F}_2(0)(u_2)\|_{\mathbf{E}}ds \leqslant$$

$$\leqslant \int_a^b |\phi(s)| \left(\|\mathcal{F}_1(0)(u_1) - \mathcal{F}_1(0)(u_2)\|_{\mathbf{E}} + \|\mathcal{F}_1(0)(u_2) - \mathcal{F}_2(0)(u_2)\|_{\mathbf{E}}\right) ds \leqslant$$

$$\leqslant \int_a^b |\phi(s)| \left(c_1c_2C\|u_1 - u_2\|_{\mathbf{V}} + \|\mathcal{F}_1 - \mathcal{F}_2\|_G\right) ds \leqslant$$

$$\leqslant \int_a^b L(s)|\phi(s)|(d_{\mathcal{M}}(\omega_1,\omega_2) + \|u_1 - u_2\|_{\mathbf{V}})ds,$$

for all $\phi \in L^1[a,b]$, $\omega_1 = (\mathcal{F}_1,\mathcal{B}_1), \omega_2 = (\mathcal{F}_2,\mathcal{B}_2) \in \mathcal{M}$ and $u_1,u_2 \in \mathbf{V}$.

Finally, using (3.7), we conclude that

$$\|\mathbf{F}\|_1 = \sup\{\|\mathbf{F}(t,\omega,u)\|_{\mathbf{E}} : t \geqslant 0, \omega \in \mathcal{M}, u \in \mathbf{V}\} \leqslant \eta$$

and condition (C5) holds. □

In summary, the conditions (2.3), (2.4), (2.6), (C1), (C2), (C3), (C4) and (C5) hold, and $\|\mathbf{B}\|_\infty < \infty$ by Lemma 3.1. Consequently, Theorem 2.2 implies in Theorem 3.4 below and Lemma 2.3, Theorem 2.5 and Theorem 2.7 imply in Theorem 3.5.

Theorem 3.4. *Under conditions (H1)–(H3), the system* (3.6) *admits a unique mild solution* $\varphi(\cdot,u_0,\omega) : \mathbb{R}_+ \to \mathbf{V}$ *satisfying* $\varphi(0,u_0,\omega) = u_0$.

Theorem 3.5. *The mild solution $\varphi(t, u_0, \omega)$ of* (3.6) *satisfies the boundedness*

$$\|\varphi(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_0\|_{\mathbf{V}}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\},$$

for all $t \geqslant 0$, $\omega \in \mathcal{M}$ and $u_0 \in \mathbf{V}$, where α is given by (3.3)*. Moreover, system* (3.6) *is bounded dissipative and generates a cocycle.*

For $\omega = (B, F)$ we have $\varphi(t, u_0, \omega)$ is solution of system (3.2). Using the projection of Leray, we obtain the existence of a mild solution and a pressure to the system (3.1).

Theorem 3.6. *Assume that conditions (H1)–(H3) hold. Then there exist functions $p = p(t, x)$ and $u = u(t, x)$ on $[0, +\infty) \times \Omega$, satisfying system* (3.1)*. Moreover, $[0, +\infty) \ni t \to p(t, \cdot) \in H^1(\Omega)$ and $[0, +\infty) \ni t \to u(t, \cdot) \in \mathbb{H}_0^1(\Omega)$ are continuous functions and*

$$\|u(t, \cdot)\|^2_{\mathbb{H}_0^1(\Omega)} \leqslant \max\left\{\|u(0, \cdot)\|^2_{\mathbb{H}_0^1(\Omega)}, \left(\frac{\eta}{\alpha}\right)^2\right\} \quad \textit{for all} \quad t \geqslant 0,$$

where α is given by (3.3) *and η in* (3.7)*.*

4 The 2D Navier-Stokes equations with impulses at variable times

In this last section, we consider the following 2D Navier-Stokes equations with impulses

$$\begin{cases} \dfrac{\partial u}{\partial t} + q(t)(u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t, u), & (t, x) \in (0, +\infty) \times \Omega, \\ \operatorname{div} u = 0, & (t, x) \in (0, +\infty) \times \Omega, \\ u = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, \cdot) = u_0 \in \mathbf{V}, & x \in \Omega, \\ I : M \subset \mathbf{V} \to \mathbf{V}, & \end{cases} \tag{4.1}$$

where Ω, $\nu > 0$, p, $f(t, u)$ and $q(t)$ satisfy the conditions presented in Section 3.

Moreover, in Section 3, we constructed the abstract system associated to (3.1). Using this construction we have the following abstract system associated to (4.1)

$$\begin{cases} \dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t, \omega))(u, u) = \mathbf{F}(t, \sigma(t, \omega), u), \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \\ I : M \subset \mathbf{V} \to \mathbf{V}. \end{cases} \tag{4.2}$$

As presented in Section 3, the conditions (2.3), (2.4), (2.6), (C1), (C2), (C3), (C4) and (C5) hold.

In order for the solution of system (4.2) to have a good behavior, we need to impose some conditions on the structure of impulses. We shall use the theory presented in [5] to construct an impulsive cocycle associated to the system (4.2).

For each $D \subseteq \mathbf{V}$, $J \subseteq \mathbb{R}_+$ and $\omega \in \mathcal{M}$ we define

$$F_\varphi(D, J, \omega) = \{u_0 \in \mathbf{V} \colon \varphi(t, u_0, \omega) \in D, \text{ for some } t \in J\},$$

where $\varphi(t, u_0, \omega)$ is solution of system (3.6) (solution of system (4.2) without impulses).

We assume that the impulsive set is a nonempty closed subset $M \subset \mathbf{V}$ satisfying the property: for each $u_0 \in M$ and each $\omega \in \mathcal{M}$ there exists $\epsilon = \epsilon_{\omega,u_0} > 0$ with

$$\bigcup_{t \in (0,\epsilon)} F_\varphi(u_0, t, \sigma_{-t}\omega) \cap M = \varnothing \quad \text{and} \quad \{\varphi(s, u_0, \omega) \colon s \in (0, \epsilon)\} \cap M = \varnothing. \tag{4.3}$$

We use the notation $\sigma_s\omega = \sigma(s, \omega)$. Condition (4.3) means that the solution φ is, in some sense, transversal to M.

We also assume that the impulse function $I : M \to \mathbf{V}$ is a continuous map such that $I(M) \cap M = \emptyset$.

Theorem 3.5 shows that system (4.2), without the condition of impulses, generates a cocycle. Using this fact, we construct in the next lines an impulsive cocycle for the impulsive system (4.2), that is, a mapping $\psi : \mathbb{R}_+ \times \mathbf{V} \times \mathcal{M} \to \mathbf{V}$ satisfying the conditions:

(i) $\psi(0, u_0, \omega) = u_0$ for all $u_0 \in \mathbf{V}$ and $\omega \in \mathcal{M}$,

(ii) $\psi(t + s, u_0, \omega) = \psi(t, \psi(s, u_0, \omega), \sigma_s\omega)$ for all $t, s \in \mathbb{R}_+$, $u_0 \in \mathbf{V}$ and $\omega \in \mathcal{M}$.

For each $(u_0, \omega) \in \mathbf{V} \times \mathcal{M}$, define the set

$$M_\varphi^+(u_0, \omega) = \{\varphi(\tau, u_0, \omega) \colon \tau > 0\} \cap M.$$

If $M_\varphi^+(u_0, \omega) \neq \varnothing$, then there exists $t = t(u_0, \omega) > 0$ such that $\varphi(t, u_0, \omega) \in M$ and $\varphi(\tau, u_0, \omega) \notin M$ for $0 < \tau < t$, see [5, Proposition 2.3]. Thus, we are able to define a function $\Phi(\cdot, \omega) \colon \mathbf{V} \to (0, +\infty]$ by

$$\Phi(u_0, \omega) = \begin{cases} s, & \text{if } \varphi(s, u_0, \omega) \in M \text{ and } \varphi(t, u_0, \omega) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \varphi(t, u_0, \omega) \notin M \text{ for all } t > 0. \end{cases} \tag{4.4}$$

Now, we can construct an impulsive cocycle associated to the system (4.2) which we shall denote by $\tilde{\varphi}$. Given $u_0 \in \mathbf{V}$ and $\omega \in \mathcal{M}$, let $\varphi(t, u_0, \omega)$ be the solution of (3.6) defined on $\mathbb{R}_+$. We know that

$$\varphi(t + s, u_0, \omega) = \varphi(t, \varphi(s, u_0, \omega), \sigma_s\omega) \quad \text{for all} \quad t, s \in \mathbb{R}_+,$$

as φ is a cocylce of system (3.6).

If $M_{\varphi}^{+}(u_0,\omega)=\varnothing$, then we define $\tilde{\varphi}(\cdot,u_0,\omega)$ as

$$\tilde{\varphi}(t,u_0,\omega)=\varphi(t,u_0,\omega)$$

for all $t\in[0,+\infty)$ and in this case $\Phi(u_0,\omega)=+\infty$.

However, if $M_{\varphi}^{+}(u_0,\omega)\neq\varnothing$ then we denote $u_0=u_0^{+}$ and we define $\tilde{\varphi}(\cdot,u_0,\omega)$ on $[0,\Phi(u_0^{+},\omega)]$ by

$$\tilde{\varphi}(t,u_0,\omega)=\begin{cases}\varphi(t,u_0^{+},\omega), & \text{if} \quad 0\leqslant t<\Phi(u_0^{+},\omega),\\ I(\varphi(\Phi(u_0^{+},\omega),u_0^{+},\omega)), & \text{if} \quad t=\Phi(u_0^{+},\omega).\end{cases}$$

Note that $\varphi(t,u_0^{+},\omega)\notin M$ for $0<t<\Phi(u_0^{+},\omega)$ and $\varphi(\Phi(u_0^{+},\omega),u_0^{+},\omega)\in M$.

Now let $s_0=\Phi(u_0^{+},\omega)$, $u_1=\varphi(s_0,u_0^{+},\omega)$ and $u_1^{+}=I(\varphi(s_0,u_0^{+},\omega))$. In this case, since $s_0<+\infty$ the process can go on, but now starting at u_1^{+}. Assume that $\tilde{\varphi}(\cdot,u_0,\omega)$ is defined on the interval $[t_{n-1},t_n]$ and that $\tilde{\varphi}(t_n,u_0^{+},\omega)=u_n^{+}$, where $t_0=0$ and $t_n=t_n(u_0,\omega)=\sum_{i=0}^{n-1}s_i$ for $n=1,2,3,\ldots$, $(s_0=\Phi(u_0^{+},\omega),s_1=\Phi(u_1^{+},\sigma_{t_1}\omega),\ldots,s_{n-1}=\Phi(u_{n-1}^{+},\sigma_{t_{n-1}}\omega))$. If $M_{\varphi}^{+}(u_n^{+},\sigma_{t_n}\omega)=\varnothing$, then

$$\tilde{\varphi}(t,u_0,\omega)=\varphi(t-t_n,u_n^{+},\sigma_{t_n}\omega)$$

for $t_n\leqslant t<+\infty$ and $\Phi(u_n^{+},\sigma_{t_n}\omega)=+\infty$. However, if $M_{\varphi}^{+}(u_n^{+},\sigma_{t_n}\omega)\neq\varnothing$, then we define $\tilde{\varphi}(\cdot,u_0,\omega)$ on $[t_n,t_{n+1}]$ by

$$\tilde{\varphi}(t,u_0,\omega)=\begin{cases}\varphi(t-t_n,u_n^{+},\sigma_{t_n}\omega), & \text{if} \quad t_n\leqslant t<t_{n+1},\\ I(\varphi(\Phi(u_n^{+},\sigma_{t_n}\omega),u_n^{+},\sigma_{t_n}\omega)), & \text{if} \quad t=t_{n+1}.\end{cases}$$

Now let $s_n=\Phi(u_n^{+},\sigma_{t_n}\omega)$, $u_{n+1}=\varphi(s_n,u_n^{+},\sigma_{t_n}\omega)$ and $u_{n+1}^{+}=I(\varphi(s_n,u_n^{+},\sigma_{t_n}\omega))$. This process ends after a finite number of steps if $M_{\varphi}^{+}(u_n^{+},\sigma_{t_n}\omega)=\varnothing$ for some integer $n\in\{0,1,2,\ldots\}$, and here $\tilde{\varphi}(\cdot,u_0,\omega)$ is defined on $[0,+\infty)$. However, it may proceed indefinitely, if $M_{\varphi}^{+}(u_n^{+},\sigma_{t_n}\omega)\neq\varnothing$ for all integer $n\in\{0,1,2,\ldots\}$ and in this case $\tilde{\varphi}(\cdot,u_0,\omega)$ is defined in the interval $[0,T(u_0,\omega))$, where $T(u_0,\omega)=\sum_{i=0}^{+\infty}s_i$.

We will assume that $T(u_0,\omega)=+\infty$ for all $u_0\in\mathbf{V}$ and $\omega\in\mathcal{M}$. For instance, this condition holds, if for each $\omega\in\mathcal{M}$ there is $\mu=\mu(\omega)>0$ such that $\Phi(u,\bar{\omega})\geqslant\mu$ for all $u\in I(M)$ and $\bar{\omega}\in\{\sigma_t\omega : t\in\mathbb{R}_+\}$.

By the construction we have

$$\tilde{\varphi}(t+s,u_0,\omega)=\tilde{\varphi}(t,\tilde{\varphi}(s,u_0,\omega),\sigma_s\omega),$$

for all $u_0\in\mathbf{V}$, $\omega\in\mathcal{M}$ and $t,s\in\mathbb{R}_+$, see [5] for more details.

For each fixed $u_0\in\mathbf{V}$ and $\omega\in\mathcal{M}$, the map $\tilde{\varphi}(\cdot,u_0,\omega)$ is solution of the impulsive system (4.2).

The function Φ presented in (4.4) is not continuous in general. In [5] and [9], the authors studied the continuity of this function for dynamical systems in the autonomous and nonautonomous cases, respectively. It is presented some conditions that assure the continuity of Φ for points outside from the impulsive set. This is important since we may obtain results on convergence for the solutions of an impulsive system. In this way, we shall assume that

$$\Phi : \mathbf{V} \times \mathcal{M} \to (0, +\infty] \quad \text{is continuous at every point} \quad (u_0, \omega) \in (\mathbf{V} \setminus M) \times \mathcal{M}.$$

In the next results, we assume all the conditions mentioned in this section and we use the notations presented above.

Theorem 4.1. *Assume that there is $\mathcal{K} > 0$ such that $\|I(u)\|_{\mathbf{V}} \leqslant \mathcal{K}$ for all $u \in M$. Then $\|\tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_0\|_{\mathbf{V}}, \mathcal{K}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\}$, for all $t \geqslant 0$, $\omega \in \mathcal{M}$ and $u_0 \in \mathbf{V}$, where α is given by (3.3). Moreover, system (4.2) is bounded dissipative.*

Proof. Let $u_0 \in \mathbf{V}$, $\omega \in \mathcal{M}$ and $\varphi(t, u_0, \omega)$ be the solution of (3.6) defined on $\mathbb{R}_+$. By Theorem 3.5, we have

$$\|\varphi(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_0\|_{\mathbf{V}}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\}, \quad \text{for all} \quad t \geqslant 0. \tag{4.5}$$

Then

$$\|\tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} = \|\varphi(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_0\|_{\mathbf{V}}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\} \quad \text{for all} \quad 0 \leqslant t < \Phi(u_0^+, \omega).$$

Using the same ideas to obtain (4.5), we get

$$\|\varphi(t - t_n, u_n^+, \sigma_{t_n}\omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_n^+\|_{\mathbf{V}}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\} \leqslant \max\left\{\mathcal{K}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\} \tag{4.6}$$

for all $t_n \leqslant t < t_{n+1}$, $n = 0, 1, 2, \ldots$. Therefore,

$$\|\tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\|u_0\|_{\mathbf{V}}, \mathcal{K}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\},$$

for all $t \geqslant 0$, $\omega \in \mathcal{M}$ and $u_0 \in \mathbf{V}$.

At last, let $\mathbb{B} \subset \mathbf{V}$ be a bounded set. Then there is $\eta_{\mathbb{B}} > 0$ such that $\|u\|_{\mathbf{V}} \leqslant \eta_{\mathbb{B}}$ for all $u \in \mathbb{B}$. Set $T_{\mathbb{B}} = \max\left\{0, -\frac{1}{\alpha}\ln\left(\frac{\mathcal{K}}{\eta_{\mathbb{B}}}\right)\right\}$ and let $u_0 \in \mathbb{B}$ and $\omega \in \mathcal{M}$.

If $\Phi(u_0, \omega) \leqslant T_{\mathbb{B}}$ then by (4.6) we have

$$\|\tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} \leqslant \max\left\{\mathcal{K}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\} \leqslant \mathcal{K} + \frac{\|\mathbf{F}\|_1}{\alpha} \quad \text{for all} \quad t \geqslant T_{\mathbb{B}}.$$

If $\Phi(u_0,\omega) > T_{\mathbb{B}}$ then we can use (2.12) to obtain

$$\|\varphi(t,u_0,\omega)\|_{\mathbf{V}} \leqslant \|u_0\|_{\mathbf{V}} e^{-\alpha t} + \frac{\|\mathbf{F}\|_1}{\alpha} \leqslant \mathcal{K} + \frac{\|\mathbf{F}\|_1}{\alpha} \quad \text{for all} \quad T_{\mathbb{B}} \leqslant t < \Phi(u_0,\omega),$$

which implies by (4.6) that

$$\|\tilde{\varphi}(t,u_0,\omega)\|_{\mathbf{V}} \leqslant \mathcal{K} + \frac{\|\mathbf{F}\|_1}{\alpha} \quad \text{for all} \quad t \geqslant T_{\mathbb{B}}.$$

In conclusion, for all $t \geqslant T_{\mathbb{B}}$, $u_0 \in \mathbb{B}$ and $\omega \in \mathcal{M}$, we get

$$\|\tilde{\varphi}(t,u_0,\omega)\|_{\mathbf{V}} \leqslant \mathcal{K} + \frac{\|\mathbf{F}\|_1}{\alpha}$$

which shows that system (4.2) is bounded dissipative. □

Next, we present a convergence result for the impulsive cocycle $\tilde{\varphi}$. Since the moments of impulses depend on the state, we need some correction on time to establish the result.

Theorem 4.2. *Let $u_0 \in \mathbf{V} \setminus M$, $\omega \in \mathcal{M}$ and $\{v_n\}_{n\in\mathbb{N}} \subset \mathbf{V}$ be a sequence such that $\|v_n - u_0\|_{\mathbf{V}} \stackrel{n\to\infty}{\longrightarrow} 0$. Given $t \geqslant 0$, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\eta_n \stackrel{n\to\infty}{\longrightarrow} 0$ and*

$$\|\tilde{\varphi}(t+\eta_n,v_n,\omega) - \tilde{\varphi}(t,u_0,\omega)\|_{\mathbf{V}} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Proof. Let $k \in \{0,1,2,\ldots\}$ be such that $t_k \leqslant t < t_{k+1}$ ($t_k = t_k(u_0,\omega)$ and $t_{k+1} = t_{k+1}(u_0,\omega)$). Then

$$\tilde{\varphi}(t,u_0,\omega) = \varphi(t-t_k, u_k^+, \sigma_{t_k}\omega).$$

Recall that $t_0 = 0$, $u_0^+ = u_0$, $s_0 = \Phi(u_0^+,\omega)$, $u_1 = \varphi(s_0,u_0^+,\omega)$, $u_1^+ = I(u_1)$ and for $j = 1,2,\ldots$, we obtain $t_j = \sum_{i=0}^{j-1} s_i$, $s_j = \Phi(u_j^+,\sigma_{t_j}\omega)$, $u_{j+1} = \varphi(s_j,u_j^+,\sigma_{t_j}\omega)$ and $u_{j+1}^+ = I(u_{j+1})$.

Now, for each n, define $r_0^n = 0$, $(v_n)_0^+ = v_n$, $s_0^n = \Phi((v_n)_0^+,\omega)$, $(v_n)_1 = \varphi(s_0^n,(v_n)_0^+,\omega)$ and $(v_n)_1^+ = I((v_n)_1)$. Inductively, we obtain

$$r_j^n = \sum_{i=0}^{j-1} s_i^n, \quad s_j^n = \Phi((v_n)_j^+, \sigma_{r_j^n}\omega), \quad (v_n)_{j+1} = \varphi(s_j^n,(v_n)_j^+,\sigma_{r_j^n}\omega)$$

and

$$(v_n)_{j+1}^+ = I((v_n)_{j+1})$$

for all $j = 1,2,\ldots$.

By the continuity of the maps Φ, I and φ in their corresponding spaces, we have

$$s_0^n \stackrel{n\to\infty}{\longrightarrow} s_0,$$

which implies

$$(v_n)_1 \overset{n\to\infty}{\longrightarrow} u_1 \quad \text{and} \quad (v_n)_1^+ \overset{n\to\infty}{\longrightarrow} u_1^+.$$

We continue in this fashion obtaining

$$s_j^n \overset{n\to\infty}{\longrightarrow} s_j, \quad (v_n)_j \overset{n\to\infty}{\longrightarrow} u_j \quad \text{and} \quad (v_n)_j^+ \overset{n\to\infty}{\longrightarrow} u_j^+,$$

for all $j = 0, 1, 2, \ldots$. Then,

$$r_k^n = \sum_{i=0}^{k-1} s_i^n \overset{n\to\infty}{\longrightarrow} \sum_{i=0}^{k-1} s_i = t_k.$$

Now, let us define the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ by

$$\eta_n = r_k^n - t + \frac{s_k^n}{s_k}(t - t_k), \quad n = 1, 2, \ldots.$$

Note that $\eta_n \overset{n\to\infty}{\longrightarrow} 0$. Since $t - t_k < s_k$ we get $\frac{s_k^n}{s_k}(t - t_k) < s_k^n$ and

$$r_k^n \leqslant r_k^n + \frac{s_k^n}{s_k}(t - t_k) < r_{k+1}^n.$$

In this way we may write

$$\tilde{\varphi}(t + \eta_n, v_n, \omega) = \tilde{\varphi}\left(r_k^n + \frac{s_k^n}{s_k}(t - t_k), v_n, \omega\right) = \varphi\left(\frac{s_k^n}{s_k}(t - t_k), (v_n)_k^+, \sigma_{r_k^n}\omega\right), \quad n = 1, 2, \ldots.$$

Hence, using the continuity of φ we obtain

$$\tilde{\varphi}(t + \eta_n, v_n, \omega) \overset{n\to\infty}{\longrightarrow} \tilde{\varphi}(t, u_0, \omega)$$

and we conclude the result. $\square$

Remark 4.3. For $\omega = (B, F)$ we conclude that $\tilde{\varphi}(t, u_0, \omega)$ is solution of

$$\begin{cases} \dfrac{du}{dt} + Au + B(t)(u, u) = F(t)(u), & \text{in } \mathbf{V}', \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \\ I : M \subset \mathbf{V} \to \mathbf{V}. \end{cases}$$

Thus for each fixed $k \in \{0, 1, 2, \ldots\}$, we may use the projection of Leray in the solution restricted to the interval $[t_k(u_0, \omega), t_{k+1}(u_0, \omega))$, and we obtain a continuous pressure $p_k(\cdot, x) : [t_k(u_0, \omega), t_{k+1}(u_0, \omega)) \to H^1(\Omega)$. Thus, there exist piecewise continuous functions $\tilde{u}(t, x)$ and $\tilde{p}(t, x)$ on $[0, +\infty) \times \Omega$ satisfying system (4.1), where $\tilde{p}(\cdot, x) = p_k(\cdot, x)$ for $t \in [t_k(u_0, \omega), t_{k+1}(u_0, \omega))$.

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