Sequent Calculus for ‘Generally’

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Overview

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Logics of ‘Generally’

Examples:

1. “Birds *generally* fly”
2. “Metals *rarely* are liquid under normal conditions”
3. “A reply to a message will be received *almost always*”

Motivation:

- ‘generally’, ‘rarely’, ‘many’,
- ‘most’, ‘several’, ‘almost always’, etc.

appear often in

ordinary language

some branches of science.

Logics of 'Generally': Syntax

- $L^{\nabla} \supseteq L$
- FOL + generalized quantifier ∇
- Generalized Formula: $\nabla v \varphi$

Expressive power:

- "Birds *generally* fly": $\nabla x F(x)$
- "Metals *rarely* are liquid under normal conditions": $\neg \nabla x L(x)$

Logics of 'Generally': Semantics

- Modulated Structure: $\mathcal{A}^{\mathcal{K}} = \langle \mathcal{A}, \mathcal{K} \rangle$ (complex $\mathcal{K}$)

- Satisfaction (extension in $\mathcal{K}$)

  $\mathcal{A}^{\mathcal{K}} \models \nabla v \varphi[s]$ iff $\{b \in A$: $\mathcal{A}^{\mathcal{K}} \models \varphi[s(v \mapsto b)]\}$ is in complex $\mathcal{K}$.

- Model: $\mathcal{A}^{\mathcal{K}} \models \varphi$ &$\mathcal{A}^{\mathcal{K}} \models \Gamma$ (as usual).

Logics for 'generally': Semantics

Consequence relation:

Many boys love sports	$M \in \mathcal{K}$
Sports lovers watch SporTv channel	$M \subseteq W$
Many boys watch SporTv channel	$W \in \mathcal{K}$

Class of complexes $\mathcal{C}$:

$$\Gamma \models_{\mathcal{C}} F$$

iff

$$\mathcal{A}^{\mathcal{K}} \models \Gamma \Rightarrow \mathcal{A}^{\mathcal{K}} \models F \text{ for } \mathcal{K} \in \mathcal{C}$$

Axiomatic Systems

- Basic Logic: (complexes without restriction)

$$[\nabla\alpha] : \nabla x A(x) \leftrightarrow \nabla y A(y) \text{ (y is a new var)}$$

$$[\leftrightarrow \nabla] : \forall x(A \leftrightarrow B) \rightarrow (\nabla x A \rightarrow \nabla x B)$$

- Specific Logic: intersection-closure

$$[\nabla\wedge] : \nabla x A \wedge \nabla x B \rightarrow \nabla x(A \wedge B)$$

Sequent Calculus for 'Generally'

- Marked formulas: $\langle A[x/\_]\rangle$
  - '$\_$' represents a generic object;
  - '$\langle$' and '$\rangle$' emphazise that $A(x)$ is the scope of a generalized quantifier.
- Sequent: $A_1, ..., A_n \Rightarrow B_1, ..., B_m$

  $A_1, ..., A_n, B_1, ..., B_m$: unmarked or marked formulas.

- Meaning: $T(A_1) \wedge ... \wedge T(A_n) \rightarrow T(B_1) \vee ... \vee T(B_m)$
  - T: unmarking marked formulas;
  - $T(\langle A\rangle) = \nabla z A[\_/z]$ (z is the first variable such that $z \notin occ[A]$).

Sequent Calculus for 'Generally'

Sequent Calculus for Basic Logic: $SC(\mathcal{B})$

- Sequent Calculus for Classical Logic $(SC) + \{(\nabla a), (\nabla c), (\Updownarrow)\}$.

$$\frac{\langle A\rangle, \Gamma \Rightarrow \Delta}{\nabla v A[\_/v], \Gamma \Rightarrow \Delta}(\nabla a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle A\rangle}{\Gamma \Rightarrow \Delta, \nabla v A[\_/v]}(\nabla c)$$

(with $v \notin occ[A]$)

$$\frac{A, \Gamma \Rightarrow \Delta, B \qquad B, \Gamma \Rightarrow \Delta, A}{\langle A[v/\_]\rangle, \Gamma \Rightarrow \Delta, \langle B[v/\_]\rangle}(\Updownarrow)$$

(with $v \notin free(\Gamma \cup \Delta)$)

Sequent Calculus for ‘Generally’

Sequent Calculus for Basic Logic: $SC(\mathcal{B})$

- $SC(\mathcal{B})$ is equivalent to the basic logic for ‘generally’:

Proposition 1:

$$\vdash_{SC(\mathcal{B})} \Gamma \Rightarrow \Delta$$

$$\text{iff}$$

$$\vdash_{SC} T(\Gamma), B_1, ..., B_n \Rightarrow T(\Delta)$$

(with $\{B_1, ..., B_n\} \subseteq [\nabla\alpha] \cup [\leftrightarrow \nabla]$)

Sequent Calculus for 'Generally'

Sequent Calculi for Specific Logics

Correspondence: axiom/schemas $\Longleftrightarrow$ sequent rules

$[\nabla\wedge]$: $\nabla x A \wedge \nabla x B \to \nabla x(A \wedge B)$ can be formulated as:

$$\frac{\Gamma \Rightarrow \Delta, \nabla v A \qquad \Gamma \Rightarrow \Delta, \nabla v B}{\Gamma \Rightarrow \Delta, \nabla v(A \wedge B)}(\nabla\wedge).$$

$(\nabla\wedge)$ can be reformulated as:

$$\frac{\Gamma \Rightarrow \Delta, \langle A\rangle \qquad \Gamma \Rightarrow \Delta, \langle B\rangle}{\Gamma \Rightarrow \Delta, \langle A \wedge B\rangle}(\wedge^* c).$$

Sequent Calculus for 'Generally'

Sequent Calculi for Specific Logics: $SC(\Omega) = SC(\mathcal{B}) \cup \Omega^*$

$$\frac{\langle A \to A \rangle, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}(\top^* a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle \bot \rangle}{\Gamma \Rightarrow \Delta}(\bot^* c)$$

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta}{\langle A \wedge B \rangle, \Gamma \Rightarrow \Delta}(\wedge^* a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle A \rangle \quad \Gamma \Rightarrow \Delta, \langle B \rangle}{\Gamma \Rightarrow \Delta, \langle A \wedge B \rangle}(\wedge^* c)$$

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta \quad \langle B \rangle, \Gamma \Rightarrow \Delta}{\langle A \vee B \rangle, \Gamma \Rightarrow \Delta}(\vee^* a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle A \rangle \quad \Gamma \Rightarrow \Delta, \langle B \rangle}{\Gamma \Rightarrow \Delta, \langle A \vee B \rangle}(\vee^* c)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle A \rangle}{\langle \neg A \rangle, \Gamma \Rightarrow \Delta}(\neg^* a) \qquad \frac{\langle A \rangle, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \langle \neg A \rangle}(\neg^* c)$$

Cut Elimination

- Cut Elimination for Sequent Calculus for Basic Logic:
  - The cut formula is a generalized formula:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma \Rightarrow \Theta, \langle A\rangle\end{array}}{\Gamma \Rightarrow \Theta, \nabla x A[\_/x]}(\nabla c) \quad \dfrac{\begin{array}{c}\Sigma_2\\ \langle A\rangle, \Delta \Rightarrow \Lambda\end{array}}{\nabla x A[\_/x], \Delta \Rightarrow \Lambda}(\nabla a)}{\Gamma, \Delta \Rightarrow \Theta, \Lambda}(cut)$$

  - The cut formula is a marked formula:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ A, \Gamma \Rightarrow \Theta, B\end{array} \quad \begin{array}{c}\Sigma_2\\ B, \Gamma \Rightarrow \Theta, A\end{array}}{\langle A[x/\_]\rangle, \Gamma \Rightarrow \Theta, \langle B[x/\_]\rangle}(\Updownarrow) \quad \dfrac{\begin{array}{c}\Sigma_3\\ C, \Delta \Rightarrow \Lambda, D\end{array} \quad \begin{array}{c}\Sigma_4\\ D, \Delta \Rightarrow \Lambda, C\end{array}}{\langle C[y/\_]\rangle, \Delta \Rightarrow \Lambda, \langle D[y/\_]\rangle}(\Updownarrow)}{\langle A[x/\_]\rangle, \Gamma, \Delta \Rightarrow \Theta, \Lambda, \langle D[y/\_]\rangle}(cut)$$

where $\langle B[x/\_]\rangle = \langle C[y/\_]\rangle$, $x \notin free(\Gamma \cup \Theta)$ and $y \notin free(\Delta \cup \Lambda)$.

Cut Elimination

- Cut Elimination for Sequent Calculus for Basic Logic:

Example:

$$\dfrac{\dfrac{A\to B\Rightarrow\neg(A\wedge\neg B)\quad \neg(A\wedge\neg B)\Rightarrow A\to B}{\langle A\to B\rangle\Rightarrow\langle\neg(A\wedge\neg B)\rangle}(\Updownarrow)\qquad \dfrac{\neg(A\wedge\neg B)\Rightarrow\neg A\vee B\quad \neg A\vee B\Rightarrow\neg(A\wedge\neg B)}{\langle\neg(A\wedge\neg B)\rangle\Rightarrow\langle\neg A\vee B\rangle}(\Updownarrow)}{\langle A\to B\rangle\Rightarrow\langle\neg A\vee B\rangle}(cut)$$

$$\Downarrow$$

$$\dfrac{\dfrac{A\to B\Rightarrow\neg(A\wedge\neg B)\quad \neg(A\wedge\neg B)\Rightarrow\neg A\vee B}{A\to B\Rightarrow\neg A\vee B}(cut)\qquad \dfrac{\neg A\vee B\Rightarrow\neg(A\wedge\neg B)\quad \neg(A\wedge\neg B)\Rightarrow A\to B}{\neg A\vee B\Rightarrow A\to B}(cut)}{\langle A\to B\rangle\Rightarrow\langle\neg A\vee B\rangle}(\Updownarrow)$$

$\Downarrow$ (Classical Cut Elimination)

Cut Elimination

- Cut Elimination for Sequent Calculi for Specific Logics:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Theta, \langle A \rangle \quad \Gamma \Rightarrow \Theta, \langle B \rangle}{\Gamma \Rightarrow \Theta, \langle A \wedge B \rangle}(\wedge^* c) \quad \dfrac{\langle A \rangle, \Delta \Rightarrow \Lambda}{\langle A \wedge B \rangle, \Delta \Rightarrow \Lambda}(\wedge^* a)}{\Gamma, \Delta \Rightarrow \Theta, \Lambda}(cut)$$

$$\Downarrow$$

$$\dfrac{\Gamma \Rightarrow \Theta, \langle A \rangle \quad \langle A \rangle, \Delta \Rightarrow \Lambda}{\Gamma, \Delta \Rightarrow \Theta, \Lambda}(cut)$$

Cut Elimination

- Cut Elimination for Sequent Calculi for Specific Logics:

Warning: This cut rule application cannot be eliminated.

$$\dfrac{\dfrac{A, A \leftrightarrow C \wedge D \Rightarrow C \wedge D \quad C \wedge D, A \leftrightarrow C \wedge D \Rightarrow A}{\langle A \rangle, A \leftrightarrow C \wedge D \Rightarrow \langle C \wedge D \rangle}(\Updownarrow) \quad \dfrac{\langle D \rangle \Rightarrow \langle D \rangle}{\langle C \wedge D \rangle \Rightarrow \langle D \rangle}(\wedge^* a)}{\langle A \rangle, A \leftrightarrow C \wedge D \Rightarrow \langle D \rangle}(cut)$$

Conclusion and Future Work

- Sequent Calculus for Intuitionistic Logic + Basic Logic
- Sequent Calculus for Intuitionistic Logic + Specific Logics
- Characterization of the Derivation Structure
- Future work:

  Analytical Tableau Methods for Logics of 'Generally'

Logics for ‘generally’: Semantics

Module description: common properties of its complexes

Name	Property
universe	$A \in \mathcal{K}$
non-void	$\emptyset \notin \mathcal{K}$
superset	$T \cap S \in \mathcal{K} \Rightarrow T \in \mathcal{K}$ and $S \in \mathcal{K}$
intersection	$T \in \mathcal{K}$ and $S \in \mathcal{K} \Rightarrow T \cap S \in \mathcal{K}$
union	$T \in \mathcal{K}$ and $S \in \mathcal{K} \Rightarrow T \cup S \in \mathcal{K}$
prime	$S \cup T \in \mathcal{K} \Rightarrow S \in \mathcal{K}$ or $T \in \mathcal{K}$
rejection	$\overline{S} \in \mathcal{K} \Rightarrow S \notin \mathcal{K}$
absorption	$S \notin \mathcal{K} \Rightarrow \overline{S} \in \mathcal{K}$

Module: characteristic property & axioma/schema

Module Property	Schema	Notation
universe	$\nabla x(A \to A)$	$[\nabla\top]$
non-void	$\neg\nabla x\bot$	$[\bot\nabla]$
intersection	$\nabla xA \wedge \nabla xB \to \nabla x(A \wedge B)$	$[\nabla\wedge]$
union	$\nabla xA \wedge \nabla xB \to \nabla x(A \vee B)$	$[\nabla\vee]$
superset	$\nabla x(A \wedge B) \to (\nabla xA \wedge \nabla xB)$	$[\wedge\nabla]$
prime	$\nabla x(A \vee B) \to \nabla xA \vee \nabla xB$	$[\vee\nabla]$
rejection	$\nabla x\neg A \to \neg\nabla xA$	$[\neg\nabla]$
absortion	$\neg\nabla xA \to \nabla x\neg A$	$[\nabla\neg]$