Problem set 1. *Representação de grupos 1, 2020-2.* This counts as 15 percent of the grade. The second problem set will count as 15 percent of the grade. The written exam (online) will count as 70 percent of the grade.

1. Problem 1 (7 points)

Consider the following group: $G = S_3 \wr S_2$. This is the wreath product of S_3 and S_2, that is the semidirect product $(S_3 \times S_3) \rtimes S_2$ where, writing $S_2 = \langle \varepsilon \rangle$, where $\varepsilon = (12)$, the action is given by

$$(x, y)^\varepsilon := (y, x)$$

for every $x, y \in S_3$. This group has order $|G| = 2 \cdot 6^2 = 72 = 2^3 \cdot 3^2$ and it has 9 conjugacy classes. Find representatives for them and compute the character table of G.

Hint: G acts naturally on $\{1, 2, 3\} \times \{1, 2\}$ and also on $\{1, 2, 3\} \times \{1, 2, 3\}$.

Solution. Set $x := (12)$, $y := (123)$ in S_3. The conjugates of an element of $N = S_3 \times S_3$ are obtained by either conjugating inside $S_3 \times S_3$ or by swapping. An element $(a, b)\varepsilon^i$ is fixed by ε if and only if $a = b$, therefore $|C_G(\varepsilon)| = 2 \cdot 6 = 12$ hence ε has $72/12 = 6$ conjugates in G. Its conjugates are of the form

$$\varepsilon^{(a,b)} = (a^{-1}, b^{-1})\varepsilon(a, b) = (a^{-1}b, b^{-1}a)\varepsilon,$$

therefore $(1, x)\varepsilon$ is not conjugate to ε. An element (a, b) is fixed by $(1, x)\varepsilon$ if and only if $x^{-1}bx = a$ and $a = b$, in other words $a = b \in C_{S_3}(x) = \langle x \rangle$. An element $(a, b)\varepsilon$ is fixed by $(1, x)\varepsilon$ if and only if

$$(a, b)\varepsilon = (x^{-1}bx, a)\varepsilon(1, x^{-1})\varepsilon(1, x)\varepsilon = (x^{-1}b, ax)\varepsilon,$$

in other words $a = x^{-1}b$, $b = ax$, so that $ax = b = xa$. Thus $a \in C_{S_3}(x) = \langle x \rangle$ and b is determined by a. This implies that $|C_G((1, x)\varepsilon)| = 2 + 2 = 4$ hence $(1, x)\varepsilon$ has $72/4 = 18$ conjugates in G. A similar argument applied to $(1, y)\varepsilon$ shows that $|C_G((1, y)\varepsilon)| = 3 + 3 = 6$, being $C_{S_3}(y) = \langle y \rangle$, so $(1, y)\varepsilon$ has $72/6 = 12$ conjugates in G. Since $|N| + 6 + 18 + 12 = |G|$, there are no more conjugacy classes, so representatives for the classes are

$$(1, 1),\ (1, x),\ (1, y),\ (x, y),\ (x, x),\ (y, y),\ \varepsilon,\ (1, x)\varepsilon,\ (1, y)\varepsilon.$$

Note that we have two important permutation actions of G.

(1) Action on $A = \{1, 2, 3\} \times \{1, 2\}$. This is given by

$$\begin{cases} (i, 1)^{(a,b)} := (i^a, 1), \\ (i, 2)^{(a,b)} := (i^b, 2), \\ (i, 1)^{(a,b)\varepsilon} := (i^a, 2), \\ (i, 2)^{(a,b)\varepsilon} := (i^b, 1). \end{cases}$$

Let us call χ_A the permutation character associated to this action.

(2) Action on $B = \{1,2,3\} \times \{1,2,3\}$. This is given by

$$\begin{cases} (i,j)^{(a,b)} := (i^a, j^b), \\ (i,j)^{(a,b)\varepsilon} := (j^b, i^a). \end{cases}$$

Let us call χ_B the permutation character associated to this action.

Note that $G/(A_3 \times A_3) \cong D_8$ (the dihedral group of order 8) hence we can do inflation to obtain four linear characters and an irreducible character of degree 2. We can start to build our character table.

	1	6	4	12	9	4	6	18	12
$G = S_3 \wr S_2$	1	$(1,x)$	$(1,y)$	(x,y)	(x,x)	(y,y)	ε	$(1,x)\varepsilon$	$(1,y)\varepsilon$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1
χ_3	1	1	1	1	1	1	-1	-1	-1
χ_4	1	-1	1	-1	1	1	1	-1	1
χ_5	2	0	2	0	-2	2	0	0	0
χ_6									
χ_7									
χ_8									
χ_9									
χ_A	6	4	3	1	2	0	0	0	0
χ_B	9	3	0	0	1	0	3	1	0

Decomposing χ_A and χ_B using the first orthogonality relation shows that χ_A is a sum $\chi_1 + \chi_3 + \chi$ for some irreducible character χ, different from χ_i, $i = 1, \dots, 5$, which we can choose to be χ_6, and χ_B is a sum $\chi_1 + \chi_6 + \psi$ for some irreducible character ψ different from χ_i, $i = 1, \dots, 6$ and from $\chi_6\chi_2$, so we can choose $\chi_8 = \psi$ and $\chi_7 = \chi_6\chi_2$, $\chi_9 = \chi_8\chi_2$. The character table is

	1	6	4	12	9	4	6	18	12
$G = S_3 \wr S_2$	1	$(1,x)$	$(1,y)$	(x,y)	(x,x)	(y,y)	ε	$(1,x)\varepsilon$	$(1,y)\varepsilon$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1
χ_3	1	1	1	1	1	1	-1	-1	-1
χ_4	1	-1	1	-1	1	1	1	-1	1
χ_5	2	0	2	0	-2	2	0	0	0
χ_6	4	2	1	-1	0	-2	0	0	0
χ_7	4	-2	1	1	0	-2	0	0	0
χ_8	4	0	-2	0	0	1	2	0	-1
χ_9	4	0	-2	0	0	1	-2	0	1

2. Problem 2 (2 points)

This exercise was inspired by N. Vavilov in a summer school in Perugia. A finite group G has exactly seven conjugacy classes, $C_1 = \{1\}$, $C_2, \dots, C_7$, such that $C_i^{-1} = C_i$ for all $i = 1, \dots, 7$ (in other words in G every element is conjugate to its inverse). The first five rows of the character table of G are

	$\{1\}$	C_2	C_3	C_4	C_5	C_6	C_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	-1	0	2	-1	0
χ_4	4	1	-1	0	-2	1	0
χ_5	5	-1	0	1	1	1	-1

Calculate the orders of the conjugacy classes of G and complete the character table.

Solution. Since every element of G is conjugate to its inverse, $\overline{\chi_i(g)} = \chi_i(g^{-1}) = \chi_i(g)$ for every $i \in \{1, \dots, 7\}$ and for every $g \in G$, so all the values $\chi_i(g)$ are real numbers. Note that the dual $\chi_6 := \chi_5\chi_2$ is a legitimate new irreducible character of G since it is not equal to χ_i for $i = 1, 2, 3, 4, 5$. If $\chi_7\chi_2$ was equal to χ_i for some $i \in \{1, \dots, 6\}$ then $\chi_i\chi_2 = \chi_7\chi_2^2 = \chi_7$, however this is false since $\chi_1\chi_2 = \chi_2$, $\chi_2\chi_2 = \chi_1$, $\chi_3\chi_2 = \chi_4$, $\chi_4\chi_2 = \chi_3$, $\chi_5\chi_2 = \chi_6$ and $\chi_6\chi_2 = \chi_5$. Therefore the last character χ_7 must be self-dual, in other words $\chi_7\chi_2 = \chi_7$. In particular, the value of χ_7 on the classes C_5, C_6 and C_7 must be zero, since it must be equal to its opposite. We can now say that the table looks like this, where $a, b, c, d \in \mathbb{R}$:

	$\{1\}$	C_2	C_3	C_4	C_5	C_6	C_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	-1	0	2	-1	0
χ_4	4	1	-1	0	-2	1	0
χ_5	5	-1	0	1	1	1	-1
χ_6	5	-1	0	1	-1	-1	1
χ_7	a	b	c	d	0	0	0

Since the third and the fourth columns are orthogonal, $cd + 2 = 0$, i.e. $cd = -2$. Since the second and the fourth columns are orthogonal, $bd = 0$, and since $d \neq 0$ (being $cd = -2$), we deduce that $b = 0$. Since the first and the third columns are orthogonal, $ac = 6$, and since the first and the fourth columns are orthogonal, $ad = -12$. We deduce that

$$a = -12/d = 6c = 36/a,$$

and since $a = \chi_7(1)$ is positive, we finally get $a = 6$. It follows that $c = 1$ and $d = -2$, so the character table is the following.

	1	20	24	15	10	20	30
	$\{1\}$	C_2	C_3	C_4	C_5	C_6	C_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	1	-1	0	2	-1	0
χ_4	4	1	-1	0	-2	1	0
χ_5	5	-1	0	1	1	1	-1
χ_6	5	-1	0	1	-1	-1	1
χ_7	6	0	1	-2	0	0	0

The sizes of the conjugacy classes were computed using the second orthogonality relation.

3. Problem 3 (1 point)

For each of the following lists $d_1, \ldots, d_k$, answer the question: is there a group G whose complex irreducible character degrees are precisely $d_1, \ldots, d_k$? If not, prove it. If yes, display such a group and prove that its irreducible character degrees are $d_1, \ldots, d_k$.

(1) $1, 1, 2, 2, 2, 2, 2, 2, 2$. Assume these are the irreducible character degrees of a finite group G. Then $|G| = 2 + 7 \cdot 2^2 = 30$ and $|G/G'| = 2$, implying that $|G'| = 15$. Consider the dihedral group $G = D_{15}$ of degree 15, $|G| = 30$. This group has abelianization of order 2 and it has 9 conjugacy classes, so denoting by $d_1 \leq \ldots \leq d_9$ its irreducible character degrees, we obtain that $1 = d_1 = d_2 < d_3$ and $\sum_{i=1}^{9} d_i^2 = 30$. This implies that $28 = \sum_{i=3}^{9} d_i^2 \geq \sum_{i=3}^{9} 2^2 = 7 \cdot 4 = 28$ so we must have equality everywhere, and this implies that $d_i = 2$ for $i = 3, \ldots, 9$.

(2) $1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3$. Consider the semidirect product $G := (C_2 \times C_2) \rtimes C_9$ where the action of a generator of C_9 is given by cycling the three nontrivial elements of $C_2 \times C_2$:

$$(x^i, x^j)^h := (x^j, x^{i+j}).$$

Its elements are of type $(x^i, x^j)h^l$ where $\langle x \rangle = C_2$, $\langle h \rangle = C_9$, $i, j \in \{0, 1\}$, $l \in \{0, \ldots, 8\}$. Two conjugacy classes are $\{(1, 1)\}$ and $\{(x, 1), (1, x), (x, x)\}$. An element $g = (x^i, x^j)h^l$ is fixed by h if and only if $(x^j, x^{i+j}) = (x^i, x^j)$, that is, $i = j = 0$. This implies that $C_G(h) = \langle h \rangle$, hence h has 4 conjugates. Similarly, h^2, h^4, h^5, h^7, h^8 all have 4 conjugates. On the other hand, h^3 and h^6 fix all elements of G, so they lie in the center of G. It follows that, setting $N := \{(1, 1), (1, x), (x, 1), (x, x)\}$, the conjugacy classes of G are

$$\{1\},\ N - \{1\}, (N - \{1\})h^3,\ (N - \{1\})h^6,$$
$$Nh,\ Nh^2,\ \{h^3\},\ Nh^4,\ Nh^5,\ \{h^6\},\ Nh^7,\ Nh^8.$$

We deduce that G has 12 conjugacy classes. Moreover $G/N \cong C_9$, so $N \leq G'$. On the other hand, since h acts transitively on the nontrivial elements of N, there are no nontrivial normal subgroups of G properly contained in N, so $G' = N$. We deduce that G has precisely $|G/G'| = 9$ linear characters.

Denoting by $d_1 \leq \ldots \leq d_{12}$ the irreducible character degrees of G, we have $d_i = 1$ for $i \in \{1, \ldots, 9\}$, so $36 = |G| = 9+d_{10}^2+d_{11}^2+d_{12}^2$, implying that $27 \geq 4+4+d_{12}^2$, so $d_{12} \leq 4$, moreover $d_{10} \neq 4$ since otherwise $d_{10} = d_{11} = d_{12} = 4$, which is false. If $d_{10} = 2$ then we obtain $23 = d_{11}^2 + d_{12}^2$, which is easily seen to be impossible, so $d_{10} = 3$ and $d_{11}^2 + d_{12}^2 = 18$, which easily implies that $d_{11} = d_{12} = 3$.

(3) $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Assume these are the irreducible character degrees of a finite group G. Recall that each irreducible character degree must divide $|G|$. The sum of the squares of the character degrees is equal to $|G| = 385 = 5 \cdot 7 \cdot 11$, so the character degree 2 does not divide $|G|$, a contradiction. So in this case the answer is no.

(4) $1, 1, 5, 5, 5, 5, 9, 9, 10, 10, 16, 16$. Assume these are the irreducible character degrees of a finite group G. Recall that each irreducible character degree must divide $|G|$. We have $|G| = 2 + 4 \cdot 5^2 + 2 \cdot 9^2 + 2 \cdot 10^2 + 2 \cdot 16^2 = 976 = 2^4 \cdot 61$. So the character degree 5 does not divide $|G|$, a contradiction. So in this case the answer is no.