Problem set 2. *Representação de grupos 1, 2020-2.* This counts as 15 percent of the grade, as did the first problem set. The written exam (online) will count as 70 percent of the grade.

The base field is always $\mathbb{C}$.

1. Problem 1 (5 points)

Let $A := S_5$ and let B be the subgroup of A consisting of the elements that fix the points 4 and 5, so that $B \cong S_3$. Let N be the alternating group of degree 5, so that $N \trianglelefteq A$. Consider the group

$$G := \{(a,b) \in A \times B \ : \ a \equiv b \mod N\}.$$

This group has order $|G| = |S_5| \cdot |S_3|/2 = 360 = 2^3 \cdot 3^2 \cdot 5$ and it has 12 conjugacy classes. Setting $x := (12)$, $y := (123)$, $z := (1234)$, $w := (12)(34)$, $t := (12345)$, $s := (123)(45)$, representatives of the conjugacy classes of G are $(1,1)$, $(1,y)$, $(y,1)$, $(w,1)$, $(t,1)$, (x,x), (y,y), (z,x), (s,x), (w,y), (t,y), (t,y^{-1}). Compute the character table of G.

2. Problem 2 (1 point)

Let M be the character table of a finite group G. After seeing M as a square matrix, compute the determinant of M.

3. Problem 3 (1 point)

Let S_n be the symmetric group of degree n. Consider the wreath product $G := S_n \wr S_2$, that is, the semidirect product $(S_n \times S_n) \rtimes \langle \varepsilon \rangle$ where $\varepsilon = (12)$, $S_2 = \langle \varepsilon \rangle = \{1, \varepsilon\}$, and the action is given by $(x,y)^\varepsilon := (y,x)$. Prove that in the character table of G all the entries are integers.

4. Problem 4 (1 point)

Let χ be a (not necessarily irreducible) character of a finite group G. Let $x_1, \ldots, x_k$ be representatives for the k conjugacy classes of G. Prove that $\sum_{i=1}^k \chi(x_i)$ is an integer.

5. Problem 5 (1 point)

Let χ be an irreducible character of a finite simple group. Show that $\chi(1) \neq 2$. Can it be $\chi(1) = 3$?

6. Problem 6 (1 point)

Let χ be an irreducible character of a finite group G and let $g \in G$ be an element of order m. Prove that $m \cdot \chi(1) \leq |G|$.