Problem set 2. *Representação de grupos 1, 2020-2.* This counts as 15 percent of the grade, as did the first problem set. The written exam (online) will count as 70 percent of the grade.

The base field is always $\mathbb{C}$.

1. Problem 1 (5 points)

Let $A := S_5$ and let B be the subgroup of A consisting of the elements that fix the points 4 and 5, so that $B \cong S_3$. Let N be the alternating group of degree 5, so that $N \trianglelefteq A$. Consider the group

$$G := \{(a, b) \in A \times B \ : \ a \equiv b \mod N\}.$$

This group has order $|G| = |S_5| \cdot |S_3|/2 = 360 = 2^3 \cdot 3^2 \cdot 5$ and it has 12 conjugacy classes. Setting $x := (12)$, $y := (123)$, $z := (1234)$, $w := (12)(34)$, $t := (12345)$, $s := (123)(45)$, representatives of the conjugacy classes of G are $(1, 1)$, $(1, y)$, $(y, 1)$, $(w, 1)$, $(t, 1)$, (x, x), (y, y), (z, x), (s, x), (w, y), (t, y), (t, y^{-1}). Compute the character table of G.

Solution. An element $(a, b) \in G$ has centralizer equal to

$$\begin{aligned} C_G((a, b)) &= G \cap (C_A(a) \times C_B(b)) \\ &= \{(r, s) \in C_A(a) \times C_B(b) \ : \ r \equiv s \mod N\}. \end{aligned}$$

Set $H := C_A(a) \times C_B(b)$. If H is not contained in G then, since G is a normal subgroup of $A \times B$ (the index is 2), GH is a subgroup of $A \times B$ properly containing H hence $GH = A \times B$. This implies that

$$2|G| = |A \times B| = |GH| = |G||H|/|G \cap H|$$

hence $|G \cap H| = |H|/2$. The only case in which H is contained in G is when a is a 5-cycle and b is a 3-cycle. With this information, and doing inflation on the two factors, we can deduce the first 8 irreducible characters. Moreover $\chi_3\chi_4$ and $\chi_3\chi_6$ are irreducible, so we can choose them to be χ_9 and χ_{10}.

	1	2	20	15	24	30	40	90	60	30	24	24
G	1	$1y$	$y1$	$w1$	$t1$	xx	yy	zx	sx	wy	ty	ty^{-1}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	1	-1	-1	1	1	1
χ_3	2	-1	2	2	2	0	-1	0	0	-1	-1	-1
χ_4	4	4	1	0	-1	2	1	0	-1	0	-1	-1
χ_5	4	4	1	0	-1	-2	1	0	1	0	-1	-1
χ_6	5	5	-1	1	0	1	-1	-1	1	1	0	0
χ_7	5	5	-1	1	0	-1	-1	1	-1	1	0	0
χ_8	6	6	0	-2	1	0	0	0	0	-2	1	1
χ_9	8	-4	2	0	-2	0	-1	0	0	0	1	1
χ_{10}	10	-5	-2	2	0	0	1	0	0	-1	0	0
χ_{11}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	a	b
χ_{12}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	c	d
$\chi_3\chi_8$	12	-6	0	-4	2	0	0	0	0	2	-1	-1

We have $[\chi_3\chi_8, \chi_i] = 0$ for $i = 1, \ldots, 10$ and $[\chi_3\chi_8, \chi_3\chi_8] = 2$, therefore $\chi_3\chi_8 = \chi_{11} + \chi_{12}$. Using this and the second orthogonality relation, we find that $\chi_{11}(1) = \chi_{12}(1) = 6$. Now, the value of $\chi_{11}(g)$ and $\chi_{12}(g)$ for g corresponding to columns $1, \ldots, 10$ can be calculated in the same way, via the second orthogonality relation and the fact that $\chi_{11} + \chi_{12} = \chi_3\chi_8$, keeping in mind that such elements g are conjugate to their inverse, so their character values $\chi_i(g)$ are real numbers.

	1	2	20	15	24	30	40	90	60	30	24	24
G	1	$1y$	$y1$	$w1$	$t1$	xx	yy	zx	sx	wy	ty	ty^{-1}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	1	-1	-1	1	1	1
χ_3	2	-1	2	2	2	0	-1	0	0	-1	-1	-1
χ_4	4	4	1	0	-1	2	1	0	-1	0	-1	-1
χ_5	4	4	1	0	-1	-2	1	0	1	0	-1	-1
χ_6	5	5	-1	1	0	1	-1	-1	1	1	0	0
χ_7	5	5	-1	1	0	-1	-1	1	-1	1	0	0
χ_8	6	6	0	-2	1	0	0	0	0	-2	1	1
χ_9	8	-4	2	0	-2	0	-1	0	0	0	1	1
χ_{10}	10	-5	-2	2	0	0	1	0	0	-1	0	0
χ_{11}	6	-3	0	-2	1	0	0	0	0	1	a	b
χ_{12}	6	-3	0	-2	1	0	0	0	0	1	c	d
$\chi_3\chi_8$	12	-6	0	-4	2	0	0	0	0	2	-1	-1

We are left to calculate $\chi_i(ty)$ and $\chi_i(ty^{-1})$ for $i = 11, 12$. Note that ty^{-1} is conjugate to the inverse of ty, therefore $\chi_i(ty^{-1}) = \overline{\chi_i(ty)}$. This implies that $b = \overline{a}$ and $d = \overline{c}$. The fact that $[\chi_{11}, \chi_1] = [\chi_{12}, \chi_1] = 0$ implies that $a + b = -1 = c + d$, so since $\chi_3\chi_8 = \chi_{11} + \chi_{12}$, we obtain that $a = d$ and $b = c$. Finally, the fact that the two last columns are orthogonal implies that $7 + 2a\overline{a} = 0$, and since $a + \overline{a} = -1$, we find $7 + 2a(-1 - a) = 0$, hence $a = \frac{-1 \pm i\sqrt{15}}{2}$. Up to exchanging the roles of χ_{11} and χ_{12}, we can choose either of these to be a. The character table is the following.

	1	2	20	15	24	30	40	90	60	30	24	24
G	1	$1y$	$y1$	$w1$	$t1$	xx	yy	zx	sx	wy	ty	ty^{-1}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	1	-1	-1	1	1	1
χ_3	2	-1	2	2	2	0	-1	0	0	-1	-1	-1
χ_4	4	4	1	0	-1	2	1	0	-1	0	-1	-1
χ_5	4	4	1	0	-1	-2	1	0	1	0	-1	-1
χ_6	5	5	-1	1	0	1	-1	-1	1	1	0	0
χ_7	5	5	-1	1	0	-1	-1	1	-1	1	0	0
χ_8	6	6	0	-2	1	0	0	0	0	-2	1	1
χ_9	8	-4	2	0	-2	0	-1	0	0	0	1	1
χ_{10}	10	-5	-2	2	0	0	1	0	0	-1	0	0
χ_{11}	6	-3	0	-2	1	0	0	0	0	1	$\frac{-1+i\sqrt{15}}{2}$	$\frac{-1-i\sqrt{15}}{2}$
χ_{12}	6	-3	0	-2	1	0	0	0	0	1	$\frac{-1-i\sqrt{15}}{2}$	$\frac{-1+i\sqrt{15}}{2}$

2. Problem 2 (1 point)

Let M be the character table of a finite group G. After seeing M as a square matrix, compute the determinant of M.

Solution. By the second orthogonality relation, $\overline{M^T} \cdot M$ is a diagonal matrix with diagonal entries the sizes of the centralizers of conjugacy class representatives, $|C_G(x_i)|$, $i = 1, \ldots, k$. Now, recall that $\overline{\chi(g)} = \chi(g^{-1})$ for every $g \in G$ and every irreducible character χ, therefore the column of x_j^{-1} is the conjugate of the column of x_j. This implies that the matrix $\overline{M}$ is obtained by switching the column of x_j with the column of x_j^{-1}, for every $j = 1, \ldots, k$. This amounts to t swaps, where t is the number of 2-element subsets $\{x_i, x_j\}$ of $\{x_1, \ldots, x_k\}$ such that x_j is conjugate to x_i^{-1} but not to x_i. Therefore $\det(\overline{M}) = (-1)^t \det(M)$. It follows that

$$\prod_{j=1}^{k} |C_G(x_i)| = \det(\overline{M^T} \cdot M) = \det(\overline{M^T}) \cdot \det(M)$$

$$= \det(\overline{M}) \cdot \det(M) = (-1)^t \det(M)^2.$$

It follows that

$$\det(M) = \pm i^t \cdot \sqrt{\prod_{j=1}^{k} |C_G(x_j)|}.$$

The sign $\pm$ is not determined since when writing the character table we are free to choose the order in which to write the rows and the columns, and a permutation of the rows and/or the columns results in a potential change of sign of the determinant.

3. Problem 3 (1 point)

Let S_n be the symmetric group of degree n. Consider the wreath product $G := S_n \wr S_2$, that is, the semidirect product $(S_n \times S_n) \rtimes \langle \varepsilon \rangle$ where $\varepsilon = (12)$, $S_2 = \langle \varepsilon \rangle = \{1, \varepsilon\}$, and the action is given by $(x, y)^\varepsilon := (y, x)$. Prove that in the character table of G all the entries are integers.

Solution. When we proved that in the character tables of symmetric groups every entry is an integer, the only thing we used was that in the symmetric group every element g is conjugate to g^k whenever k is coprime to the order of g, since these two elements have the same cycle structure. Therefore it is enough to prove that for every element $g \in G$ and every integer k coprime to the order of g, the elements g and g^k are conjugate in G. There are two types of elements in G: the elements of $S_n \times S_n$ and the elements of the coset $(S_n \times S_n)\varepsilon$. Let $g \in G$.

If $g = (x, y) \in S_n \times S_n$ then the order of g is the least common multiple of the order of x and the order of y, so k is coprime to both. This implies that x and x^k have the same cycle structure in S_n, so they are conjugate in S_n, and the same holds for y and y^k. Choosing $g, h \in S_n$ with $g^{-1}xg = x^k$ and $h^{-1}yh = y^k$, we obtain that $(g, h)^{-1}(x, y)(g, h) = (g^{-1}xg, h^{-1}yh) = (x^k, y^k) = (x, y)^k$.

Now assume $g \in G$ lies in the coset $(S_n \times S_n)\varepsilon$, so that $g = (x, y)\varepsilon \in G$ for some $x, y \in S_n$. We claim that the conjugacy class of $g = (x, y)\varepsilon$ is precisely

$$C = \{(a, b)\varepsilon \in G \ : \ ab \text{ is conjugate to } xy \text{ in } S_n\}.$$

To prove this, note that a typical conjugate of $(x, y)\varepsilon$ is of the form $((x, y)\varepsilon)^h$ where either $h = (r, s) \in S_n \times S_n$ or $h = (r, s)\varepsilon \in (S_n \times S_n)\varepsilon$. We have

$$\begin{aligned} &((x, y)\varepsilon)^{(r,s)} = (r^{-1}xs, s^{-1}yr)\varepsilon, \\ &((x, y)\varepsilon)^{(r,s)\varepsilon} = ((r^{-1}xs, s^{-1}yr)\varepsilon)^\varepsilon = (s^{-1}yr, r^{-1}xs)\varepsilon. \end{aligned}$$

In both cases we obtain elements of type $(a, b)\varepsilon$ with ab conjugate to xy in S_n. Therefore the conjugacy class of g in G is contained in C.

Now consider $(a, b)\varepsilon \in C$. We need to prove that $(x, y)\varepsilon$ and $(a, b)\varepsilon$ are conjugate in G. The above computation shows that it is enough to find two elements $r, s \in S_n$ with the property that $a = r^{-1}xs$ and $b = s^{-1}yr$. Since ab conjugate to xy in S_n, there exists $r \in S_n$ with $ab = r^{-1}xyr$. Let $s := yrb^{-1} = x^{-1}ra$. We have

$$a = r^{-1}xyrb^{-1} = r^{-1}xs, \qquad b = a^{-1}r^{-1}xyr = s^{-1}yr.$$

We need to understand the power $g^k = ((x,y)\varepsilon)^k$. We have

$$g^k = \begin{cases} (xy,xy)^{k/2} & \text{if k is even,} \\ (xy,xy)^{(k-1)/2}(x,y)\varepsilon & \text{if k is odd.} \end{cases}$$

This implies that what we need to show is that xy and $(xy)^k$ are conjugate in S_n, i.e. that they have the same cycle structure. For this, it is enough to show that k is coprime to the order of xy. But this is clear since k is coprime to the order of $g = (x,y)\varepsilon$, so it is also coprime to the order of $g^2 = (xy,xy)$, which equals the order of $xy \in S_n$.

4. Problem 4 (1 point)

Let χ be a (not necessarily irreducible) character of a finite group G. Let $x_1,\dots,x_k$ be representatives for the k conjugacy classes of G. Prove that $\sum_{i=1}^k \chi(x_i)$ is an integer.

Solution. G acts on itself by conjugation, and $g \in G$ has precisely $|C_G(g)|$ fixed points in this action. Call ψ the permutation character of this action. Then $\psi(x_i) = |C_G(x_i)|$ for every $i = 1,\dots,k$. If χ is irreducible, then it appears $[\chi,\psi]$ times in the decomposition of ψ, in particular $[\chi,\psi]$ is an integer. On the other hand

$$[\chi,\psi] = \frac{1}{|G|}\sum_{i=1}^k |G : C_G(x_i)| \cdot \chi(x_i) \cdot \overline{\psi(x_i)} = \sum_{i=1}^k \chi(x_i).$$

Now assume that χ is not irreducible, and write $\chi = \sum_{j=1}^k m_j\chi_j$ with the m_i's non-negative integers, not all zero. Then we know that $\sum_{i=1}^k \chi_j(x_i)$ is an integer for every $j = 1,\dots,k$, so

$$\sum_{i=1}^k \chi(x_i) = \sum_{i=1}^k\sum_{j=1}^k m_j\chi_j(x_i) = \sum_{j=1}^k m_j \sum_{i=1}^k \chi_j(x_i)$$

is a sum of integers, so it is an integer.

5. Problem 5 (1 point)

Let χ be an irreducible character of a finite simple group. Show that $\chi(1) \neq 2$. Can it be $\chi(1) = 3$?

Solution. Assume by contradiction that $\rho : G \to GL(V)$ is an irreducible representation of G with character χ and $\chi(1) = \dim(V) = 2$. Note that ρ is not the trivial homomorphism because the trivial action is not irreducible, being $\dim(V) > 1$. This implies that $\ker(\rho) \neq G$ so, since G is simple, ρ is injective. Since G is a group with a nonlinear irreducible character, G is nonabelian. Since the irreducible character degrees divide the order of G and $\chi(1) = 2$, the order of G is even, so there exists an element $g \in G$ of order 2.

Note that $\rho_g^2 = \rho_{g^2} = \rho_1 = 1$. If $\lambda \in \mathbb{C}$ is an eigenvalue of ρ_g then there exists a nonzero vector $v \in V$ with $v\rho_g = \lambda v$, so $v = v\rho_g^2 = \lambda^2 v$ and this implies that $\lambda^2 = 1$, that is, $\lambda = \pm 1$. Since $\dim(V) = 2$, if ρ_g is not a scalar map then its eigenvalues are precisely 1 and -1, each counted once, in particular $\det(\rho_g) = -1$, so the homomorphism $G \to \mathbb{C}^*$, $g \mapsto \det(\rho_g)$ is non-trivial, contradicting the fact that G is a nonabelian simple group.

If instead ρ_g is a scalar map then, since $\rho_g \neq 1$ (being ρ injective and $\rho_1 = 1$, $g \neq 1$), ρ_g lies in the center of G^ρ. But ρ, being an injective group homomorphism, induces an isomorphism $G \cong G^\rho$, therefore g must lie in the center of G, contradicting the fact that G is a nonabelian simple group.

It can be $\chi(1) = 3$, for example the alternating group of degree 5 is a nonabelian simple group which has an irreducible character of degree 3.

6. Problem 6 (1 point)

Let χ be an irreducible character of a finite group G and let $g \in G$ be an element of order m. Prove that $m \cdot \chi(1) \leq |G|$.

Solution. Let $H := \langle g \rangle \leq G$. We need to show that $\chi(1) \leq |G|/m = |G : H|$. Consider the restriction $\chi|_H$. Being a character of H, it is a sum of irreducible characters of H. Let ψ be an irreducible character of H which appears in the decomposition of $\chi|_H$, so that $[\chi|_H, \psi] \geq 1$. Recall that ψ^G is a character of G of degree $\psi(1) \cdot |G : H| = |G : H|$, being ψ a linear character (every irreducible character of an abelian group is linear). Using Frobenius reciprocity we find that $1 \leq [\chi|_H, \psi] = [\chi, \psi^G]$, which means that χ appears in the decomposition of ψ^G, in particular $\chi(1) \leq \psi^G(1) = |G : H|$.

Note that the only thing we used in this proof is that H is abelian. So we obtain the following stronger statement: if H is an abelian subgroup of a finite group G then every irreducible character degree of G is at most $|G : H|$.