Prova escrita de Representação de grupos 1, semestre 2020-2.

30 de abril de 2021.

Respostas não motivadas não serão consideradas.

The base field is always $\mathbb{C}$.

(1) (1 point) Find all the group homomorphisms $S_3 \times S_3 \to \mathbb{C}^*$, where S_3 is the symmetric group of degree 3.

(2) (2.5 points) Let χ be an irreducible character of a finite group A and let ψ be an irreducible character of a finite group B.

(a) (1 point) Prove that the map

$$\eta = \eta(\chi, \psi) : A \times B \to \mathbb{C},$$
$$(a, b) \mapsto \chi(a)\psi(b)$$

is an irreducible character of $A \times B$.

(b) (1 point) Prove that if χ_1, χ_2 are irreducible characters of A, ψ_1, ψ_2 are irreducible characters of B and the pair (χ_1, ψ_1) is distinct from the pair (χ_2, ψ_2), then

$$\eta(\chi_1, \psi_1) \neq \eta(\chi_2, \psi_2).$$

(c) (0.5 points) Using a counting argument, prove that every irreducible character of $A \times B$ is equal to $\eta(\chi, \psi)$ for some $\chi \in Irr(A)$ and some $\psi \in Irr(B)$.

(3) (1 point) Using the previous item, compute the irreducible character degrees of $Q_8 \times Q_8$, where Q_8 is the quaternion group of order 8. [Attention: I am not asking for the whole character table, just the irreducible character degrees!]

(4) (1.5 points) The symmetric group $G = S_5$ has the following character table.

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0

Decompose the following class functions and determine whether they are characters or not.

$f_3(x) = |\{g \in G \;:\; g^4 = x\}|$.

$f_2(x) = |\{g \in G \;:\; g^5 = x^4\}|$.

(5) (1 point) Complete the following character table of the group G, where the top line contains the sizes of the conjugacy classes of the conjugacy class representatives $x_1, \dots, x_9$.

	1	18	8	2	3	18	8	6	8
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	1	1	1
χ_3	2	0	-1	2	2	0	-1	2	-1
χ_4	2	0	2	-1	2	0	-1	-1	-1
χ_5	2	0	-1	-1	2	0	-1	-1	2
χ_6	2	0	-1	-1	2	0	2	-1	-1
χ_7	3	-1	0	3	-1	1	0	-1	0
χ_8									
χ_9									

(6) (1 point) Count the normal subgroups of the group in the previous item and compute their sizes.

(7) (1 point) Let G be a finite group of even order and assume that there exists an irreducible G-module of dimension $n = |G|/3$. Prove that $G \cong S_3$.

(8) (1 point) G acts on itself by conjugation. Let χ be the corresponding permutation character and let 1_G be the trivial character of G. Prove that $[\chi, 1_G]$ equals the number of conjugacy classes of G. Is χ irreducible?