Solution of the written exam.
Representação de grupos 1 - 2020-2.
April 30th, 2021.

The two types are merged in this solution.

(1) Find all the group homomorphisms $Q_8 \to \mathbb{C}^*$.

Solution. $G = Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$. The derived subgroup of G is $N = G' = \langle -1 \rangle = \{-1, 1\}$ and $G/G' \cong C_2 \times C_2$. Every homomorphism $\varphi : G \to \mathbb{C}^*$ is a composition $G \to G/G' \to \mathbb{C}^*$ where the first map $G \to G/G'$ is the canonical projection. Since elements in the same coset modulo G' have the same image via φ, we may list the homomorphisms as follows.

	N	iN	jN	kN
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	1	-1	-1
ρ_4	1	-1	-1	1

(2) Find all the group homomorphisms $S_3 \times S_3 \to \mathbb{C}^*$.

Solution. $G = S_3 \times S_3$. The derived subgroup of G is $N = G' = A_3 \times A_3$ and $G/G' \cong C_2 \times C_2$. Every homomorphism $\varphi : G \to \mathbb{C}^*$ is a composition $G \to G/G' \to \mathbb{C}^*$ where the first map $G \to G/G'$ is the canonical projection. Since elements in the same coset modulo G' have the same image via φ, we may list the homomorphisms as follows.

	N	$((12), 1)N$	$(1, (12))N$	$((12), (12))N$
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	1	-1	-1
ρ_4	1	-1	-1	1

(3) Let χ be an irreducible character of a finite group A and let ψ be an irreducible character of a finite group B.

(a) Prove that the map

$$\eta = \eta(\chi, \psi) : A \times B \to \mathbb{C},$$

$$(a, b) \mapsto \chi(a)\psi(b)$$

is an irreducible character of $A \times B$.

Solution. Note that

$$\widetilde{\chi} : A \times B \to \mathbb{C}, \qquad \widetilde{\chi}((a, b)) := \chi(a),$$

$$\widetilde{\psi} : A \times B \to \mathbb{C}, \qquad \widetilde{\psi}((a, b)) := \psi(b),$$

are irreducible characters of $A \times B$ by inflation. Therefore $\eta = \widetilde{\chi} \cdot \widetilde{\psi}$ is a character of G, being a product of characters. To prove that η is irreducible we need to show that $[\eta, \eta] = 1$.

$$\begin{aligned}
[\eta, \eta] &= \frac{1}{|G|} \sum_{g \in G} \eta(g) \overline{\eta(g)} = \frac{1}{|A \times B|} \sum_{(a,b) \in G} \chi(a)\psi(b)\overline{\chi(a)\psi(b)} \\
&= \frac{1}{|A|} \frac{1}{|B|} \sum_{a \in A, b \in B} \chi(a)\overline{\chi(a)}\psi(b)\overline{\psi(b)} \\
&= \frac{1}{|A|} \sum_{a \in A} \chi(a)\overline{\chi(a)} \cdot \frac{1}{|B|} \sum_{b \in B} \psi(b)\overline{\psi(b)} \\
&= [\chi, \chi] \cdot [\psi, \psi] = 1 \cdot 1 = 1.
\end{aligned}$$

(b) Prove that if χ_1, χ_2 are irreducible characters of A, ψ_1, ψ_2 are irreducible characters of B and the pair (χ_1, ψ_1) is distinct from the pair (χ_2, ψ_2), then

$$\eta(\chi_1, \psi_1) \neq \eta(\chi_2, \psi_2).$$

Solution. Let $\eta_i = \eta(\chi_i, \psi_i)$ for $i = 1, 2$. We have

$$\begin{aligned}
[\eta_1, \eta_2] &= \frac{1}{|G|} \sum_{g \in G} \eta_1(g) \overline{\eta_2(g)} = \frac{1}{|A \times B|} \sum_{(a,b) \in G} \chi_1(a)\psi_1(b)\overline{\chi_2(a)\psi_2(b)} \\
&= \frac{1}{|A|} \frac{1}{|B|} \sum_{a \in A, b \in B} \chi_1(a)\overline{\chi_2(a)}\psi_1(b)\overline{\psi_2(b)} \\
&= \frac{1}{|A|} \sum_{a \in A} \chi_1(a)\overline{\chi_2(a)} \cdot \frac{1}{|B|} \sum_{b \in B} \psi_1(b)\overline{\psi_2(b)} \\
&= [\chi_1, \chi_2] \cdot [\psi_1, \psi_2] = 0,
\end{aligned}$$

being at least one of $[\chi_1, \chi_2]$ and $[\psi_1, \psi_2]$ equal to zero. This implies that $\eta_1 \neq \eta_2$.

(c) Using a counting argument, prove that every irreducible character of $A \times B$ is equal to $\eta(\chi, \psi)$ for some $\chi \in Irr(A)$ and some $\psi \in Irr(B)$.

Solution. Denote by k_X the number of irreducible characters of the group X, which also equals the number of conjugacy classes of X. The characters $\eta(\chi_1, \chi_2)$ constructed above are $k_A k_B$ irreducible characters of $G = A \times B$. On the other hand we know that $k_{A \times B} = k_A k_B$, since the number of irreducible characters of a finite group equals the number of its conjugacy classes and, if $a \in A$ and $b \in B$, the conjugacy class of (a, b) in G is precisely $a^A \times b^B$. Therefore the characters constructed above are all the irreducible characters of G.

(4) Compute the irreducible character degrees of $A_4 \times A_4$. [Attention: I am not asking for the whole character table, just the irreducible character degrees!]

Solution. The irreducible character degrees of A_4 are $1, 1, 1, 3$. By the above item, the irreducible characters of $A_4 \times A_4$ are all the products $\chi_i(1)\chi_j(1)$ where χ_i, χ_j are irreducible characters of A_4, therefore they are 1 (9 times), 3 (6 times) and 9 (1 time).

(5) Compute the irreducible character degrees of $Q_8 \times Q_8$. [Attention: I am not asking for the whole character table, just the irreducible character degrees!]

Solution. The irreducible character degrees of Q_8 are $1, 1, 1, 1, 2$. By the above item, the irreducible character degrees of $Q_8 \times Q_8$ are all the products $\chi_i(1)\chi_j(1)$ where χ_i, χ_j are irreducible characters of Q_8, therefore they are 1 (16 times), 2 (8 times) and 4 (1 time).

(6) The group $G = S_5$ has the following character table.

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0

Decompose the following class functions and determine whether they are characters or not.

$f_1(x) = |\{g \in G \ : \ g^3 = x\}|$.

$f_2(x) = |\{g \in G \ : \ g^5 = x^4\}|$.

$f_3(x) = |\{g \in G \ : \ g^4 = x\}|$.

$f_4(x) = |\{g \in G \ : \ g^3 = x^3\}|$.

Solution. The values of the functions $f_1, \ldots, f_4$ are easily calculated by counting.

	1	10	20	30	24	15	20
S_5	1	(12)	(123)	(1234)	(12345)	(12)(34)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0
$--$	$--$	$--$	$--$	$--$	$--$	$--$	$--$
f_1	21	3	0	1	1	1	0
f_2	25	25	1	25	0	25	1
f_3	56	0	2	0	1	0	0
f_4	21	3	21	1	1	1	3

We compute their decompositions using the first orthogonality relation. Specifically, if $f_i = \sum_{j=1}^{7} m_j \chi_j$ for $i = 1, 2, 3, 4$, then $m_\ell = [f_i, \chi_\ell]$ for $\ell = 1, \ldots, 7$.

(a) $f_1 = \chi_1 + \chi_3 + \chi_5 + \chi_6 + \chi_7$. It is a character, since it is a linear combination of irreducible characters for which all the coefficients are non-negative integers and they are not all zero.

(b) $f_2 = 12\chi_1 - 5\chi_2 + 5\chi_3 - 3\chi_4 + 8\chi_6 - 5\chi_7$. It is not a character, since it is a linear combination of irreducible characters for which not all the coefficients are non-negative.

(c) $f_3 = \chi_1 + \chi_2 + 2\chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 3\chi_7$. It is a character, since it is a linear combination of irreducible characters for which all the coefficients are non-negative integers and they are not all zero.

(d) $f_4 = 5\chi_1 + 3\chi_2 + 4\chi_3 + 4\chi_4 - 2\chi_5 - 3\chi_6 + \chi_7$. It is not a character, since it is a linear combination of irreducible characters for which not all the coefficients are non-negative.

(7) Complete the following character table of the group G, where the top line contains the sizes of the conjugacy classes of the conjugacy class representatives $x_1, \ldots, x_{10}$.

	1	12	32	3	3	12	12	6	3	12
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	-1	1	1	-1
χ_3	2	0	-1	2	2	0	0	2	2	0
χ_4	3	-1	0	-1	3	1	-1	-1	-1	1
χ_5	3	-1	0	3	-1	-1	1	-1	-1	1
χ_6	3	1	0	-1	3	-1	1	-1	-1	-1
χ_7	3	1	0	3	-1	1	-1	-1	-1	-1
χ_8	3	-1	0	-1	-1	1	1	-1	3	-1
χ_9										
χ_{10}										

Solution. The irreducible character $\chi_8\chi_2$ is different from χ_i, $i = 1, \ldots, 8$, therefore we can choose $\chi_9 := \chi_8\chi_2$.

	1	12	32	3	3	12	12	6	3	12
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	-1	1	1	-1
χ_3	2	0	-1	2	2	0	0	2	2	0
χ_4	3	-1	0	-1	3	1	-1	-1	-1	1
χ_5	3	-1	0	3	-1	-1	1	-1	-1	1
χ_6	3	1	0	-1	3	-1	1	-1	-1	-1
χ_7	3	1	0	3	-1	1	-1	-1	-1	-1
χ_8	3	-1	0	-1	-1	1	1	-1	3	-1
χ_9	3	1	0	-1	-1	-1	-1	-1	3	1
χ_{10}										

The order of G is the sum of the sizes of the conjugacy classes,

$$|G| = 1 + 12 + 32 + 3 + 3 + 12 + 12 + 6 + 3 + 12 = 96.$$

The equality $\sum_{i=1}^{10} \chi_i(1)^2 = |G| = 96$ implies that $\chi_{10}(1) = 6$. We can now use the orthogonality between the first column and the other columns (which is much easier than using the orthogonality of rows) to deduce the value of $\chi_{10}(g)$ for all $g \in G$.

	1	12	32	3	3	12	12	6	3	12
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	-1	1	1	-1
χ_3	2	0	-1	2	2	0	0	2	2	0
χ_4	3	-1	0	-1	3	1	-1	-1	-1	1
χ_5	3	-1	0	3	-1	-1	1	-1	-1	1
χ_6	3	1	0	-1	3	-1	1	-1	-1	-1
χ_7	3	1	0	3	-1	1	-1	-1	-1	-1
χ_8	3	-1	0	-1	-1	1	1	-1	3	-1
χ_9	3	1	0	-1	-1	-1	-1	-1	3	1
χ_{10}	6	0	0	-2	-2	0	0	2	-2	0

(8) Count the normal subgroups of the group in the previous item and compute their sizes.

Solution. We know that every normal subgroup of G is an intersection of kernels of irreducible characters of G. Recall that if χ is an irreducible character of G then its kernel is

$$\ker(\chi) = \{g \in G \ : \ \chi(g) = \chi(1)\}.$$

Note that $x_1 = 1$ being the only element with 1 conjugate (as seen from the table). We have

- $\ker(\chi_1) = G$, its size is 96,
- $\ker(\chi_2) = \{1\} \cup x_3^G \cup x_4^G \cup x_5^G \cup x_8^G \cup x_9^G$, its size is $1 + 32 + 3 + 3 + 6 + 3 = 48$,
- $\ker(\chi_3) = \{1\} \cup x_4^G \cup x_5^G \cup x_8^G \cup x_9^G$, its size is $1+3+3+6+3 = 16$,
- $\ker(\chi_4) = \{1\} \cup x_5^G$, its size is $1 + 3 = 4$,
- $\ker(\chi_5) = \{1\} \cup x_4^G$, its size is $1 + 3 = 4$,
- $\ker(\chi_6) = \{1\} \cup x_5^G = \ker(\chi_4)$,
- $\ker(\chi_7) = \{1\} \cup x_4^G = \ker(\chi_5)$,
- $\ker(\chi_8) = \{1\} \cup x_9^G$, its size is $1 + 3 = 4$,
- $\ker(\chi_9) = \{1\} \cup x_9^G = \ker(\chi_8)$,
- $\ker(\chi_{10}) = \{1\}$, its size is 1.

It is clear from the table that the set $C = \{\ker(\chi_i) \ : \ i = 1, \ldots, 10\}$ is closed under intersection. It follows from the above analysis that $|C| = 7$, therefore G has precisely 7 normal subgroups.

(9) Complete the following character table of the group G, where the top line contains the sizes of the conjugacy classes of the conjugacy class representatives $x_1, \ldots, x_9$.

	1	18	8	2	3	18	8	6	8
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	1	1	1
χ_3	2	0	-1	2	2	0	-1	2	-1
χ_4	2	0	2	-1	2	0	-1	-1	-1
χ_5	2	0	-1	-1	2	0	-1	-1	2
χ_6	2	0	-1	-1	2	0	2	-1	-1
χ_7	3	-1	0	3	-1	1	0	-1	0
χ_8									
χ_9									

Solution. The irreducible character $\chi_7\chi_2$ is different from χ_i, $i = 1, \ldots, 7$, therefore we can choose $\chi_8 := \chi_7\chi_2$.

	1	18	8	2	3	18	8	6	8
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	1	1	1
χ_3	2	0	-1	2	2	0	-1	2	-1
χ_4	2	0	2	-1	2	0	-1	-1	-1
χ_5	2	0	-1	-1	2	0	-1	-1	2
χ_6	2	0	-1	-1	2	0	2	-1	-1
χ_7	3	-1	0	3	-1	1	0	-1	0
χ_8	3	1	0	3	-1	-1	0	-1	0
χ_9									

The order of G is the sum of the sizes of the conjugacy classes,

$$|G| = 1 + 18 + 8 + 2 + 3 + 18 + 8 + 6 + 8 = 72.$$

The equality $\sum_{i=1}^{9} \chi_i(1)^2 = |G| = 72$ implies that $\chi_9(1) = 6$. We can now use the orthogonality between the first column and the other columns (which is much easier than using the orthogonality of rows) to deduce the value of $\chi_9(g)$ for all $g \in G$.

	1	18	8	2	3	18	8	6	8
G	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1	1	1	1
χ_3	2	0	-1	2	2	0	-1	2	-1
χ_4	2	0	2	-1	2	0	-1	-1	-1
χ_5	2	0	-1	-1	2	0	-1	-1	2
χ_6	2	0	-1	-1	2	0	2	-1	-1
χ_7	3	-1	0	3	-1	1	0	-1	0
χ_8	3	1	0	3	-1	-1	0	-1	0
χ_9	6	0	0	-3	-2	0	0	1	0

(10) Count the normal subgroups of the group in the previous item and compute their sizes.

Solution. We know that every normal subgroup of G is an intersection of kernels of irreducible characters of G. Recall that if χ is an irreducible character of G then its kernel is

$$\ker(\chi) = \{g \in G \ : \ \chi(g) = \chi(1)\}.$$

Note that $x_1 = 1$ being the only element with 1 conjugate (as seen from the table). We have

- $\ker(\chi_1) = G$, its size is 72,
- $\ker(\chi_2) = \{1\} \cup x_3^G \cup x_4^G \cup x_5^G \cup x_7^G \cup x_8^G \cup x_9^G$, its size is $1+8+2+3+8+6+8 = 36$,
- $\ker(\chi_3) = \{1\} \cup x_4^G \cup x_5^G \cup x_8^G$, its size is $1+2+3+6 = 12$,
- $\ker(\chi_4) = \{1\} \cup x_3^G \cup x_5^G$, its size is $1+8+3 = 12$,
- $\ker(\chi_5) = \{1\} \cup x_5^G \cup x_9^G$, its size is $1+3+8 = 12$,
- $\ker(\chi_6) = \{1\} \cup x_5^G \cup x_7^G$, its size is $1+3+8 = 12$,
- $\ker(\chi_7) = \{1\} \cup x_4^G$, its size is $1+2 = 3$,
- $\ker(\chi_8) = \{1\} \cup x_4^G = \ker(\chi_7)$,
- $\ker(\chi_9) = \{1\}$, its size is 1.

It is clear from the above analysis that the set $C = \{\ker(\chi_i) : i = 1, \dots, 9\}$ has size 8. The intersections between kernels of irreducible characters belong to C except for $\ker(\chi_3) \cap \ker(\chi_4) = \{1\} \cup x_5^G$, therefore G has precisely 9 normal subgroups.

(11) Let G be a finite group of even order and assume that there exists an irreducible G-module of dimension $n = |G|/3$. Prove that $G \cong S_3$.

Solution. We know that $\sum_{i=1}^k \chi_i(1)^2 = |G|$, where $\chi_1, \dots, \chi_k$ are the distinct irreducible characters of G. Note that n is equal to $\chi_i(1)$ for some $i \in \{1, \dots, k\}$. Therefore $|G|^2/9 = n^2 \leq |G|$, so $|G| \leq 9$. But $|G|$ is even and n is an integer, so $|G|$ is divisible by 6, therefore $|G| = 6$ and $n = 2$. There are only two groups of order

6 up to isomorphism: C_6 and S_3. However G has an irreducible character of degree $n = 2$, so it cannot be abelian, hence $G \cong S_3$.

(12) Let χ be a character of the finite group G and assume that $\chi(g) = 0$ for every $1 \neq g \in G$. Prove that $|G|$ divides $\chi(1)$.

Solution. Let 1_G be the trivial character of G. We know that $m := [\chi, 1_G]$ is an integer: it is the number of times that 1_G appears in the decomposition of χ. However,

$$m = [\chi, 1_G] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{1_G(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{\chi(1)}{|G|},$$

therefore $\chi(1) = m|G|$.

(13) G acts on itself by conjugation. Let χ be the corresponding permutation character and let 1_G be the trivial character of G. Prove that $[\chi, 1_G]$ equals the number of conjugacy classes of G. Is χ irreducible?

Solution. The number of fixed points of $g \in G$ acting by conjugation on G is the order of its centralizer, $|C_G(g)|$. Therefore $\chi(g) = |C_G(g)|$. We compute

$$\begin{aligned} [\chi, 1_G] &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{1_G(g)} = \frac{1}{|G|} \sum_{g \in G} |C_G(g)| \\ &= \frac{1}{|G|} \sum_{i=1}^{k} |G : C_G(x_i)| \cdot |C_G(x_i)| = k, \end{aligned}$$

where $x_1, \ldots, x_k$ are conjugacy class representatives. In particular $[\chi, 1_G] = k > 0$, so χ is irreducible if and only if $\chi = 1_G$, and this happens if and only if $k = 1$, i.e. $G = \{1\}$.