Prova 1 IAL - Tema A - 12/07/2022 - Gabarito

1. Resolva os sistemas

$$\begin{cases} 2x+3y+z+3w=1 \\ x+3y+2w=1 \\ x+z+w=1 \end{cases} \qquad \begin{cases} x-y=1 \\ -z+2w=1 \\ 2x-y-w=-1 \\ 3x-2y-z+w=1 \end{cases}$$

Primeiro sistema.

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 3 & 1 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow[\text{I}-2\text{III}]{\text{II}-\text{III}} \left(\begin{array}{cccc|c} 0 & 3 & -1 & 1 & -1 \\ 0 & 3 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow[\text{I}\leftrightarrow\text{III}]{\text{I}-\text{II}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

A terceira equação é $0=-1$, logo o sistema não tem solução.

Segundo sistema.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 2 & -1 & 0 & -1 & -1 \\ 3 & -2 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\text{IV}-3\text{I},\ \text{II}\leftrightarrow\text{III}]{\text{III}-2\text{I}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow[\text{IV}-\text{II}]{-\text{III}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right) \xrightarrow{\text{IV}+\text{III}} \left(\begin{array}{cccc|c} \mathbf{1} & -1 & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -1 & -3 \\ 0 & 0 & \mathbf{1} & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Escolhendo $w=t$ obtemos $z=2t-1$, $y=t-3$, $x=y+1=t-2$.

$$(x,y,z,w)=(t-2,t-3,2t-1,t)=(-2,-3,-1,0)+t(1,1,2,1).$$

2. Calcule o determinante de

$$A_1 := \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

Usando a regra de Sarrus (que pode ser aplicada apenas para matrizes 3×3) temos

$\det(A_1) = 1\cdot1\cdot1+2\cdot(-1)\cdot0+0\cdot(-1)\cdot3-0\cdot1\cdot0-2\cdot(-1)\cdot1-1\cdot(-1)\cdot3 = 1+2+3=6.$

Para calcular $\det(A_2)$ podemos fazer operações de linha

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{I}-\text{II}} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{I}-\text{III}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

obtendo assim uma matriz com uma linha nula. Segue que $\det(A_2)=0$.

3. Encontre a inversa de

$$B_1 := \begin{pmatrix} 8 & 7 \\ 1 & 1 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_3 := \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\left(\begin{array}{cc|cc} 8 & 7 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\text{I}-7\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -7 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}-\text{I}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -7 \\ 0 & 1 & -1 & 8 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 7 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{\text{II}+7\text{III}}{\xrightarrow{\text{I}-2\text{II}}} \left(\begin{array}{ccc|ccc} 2 & 0 & -12 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{-\text{III}}{\xrightarrow{\text{I}-12\text{III}}}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -2 & -12 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \xrightarrow{\text{I}/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & -6 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\text{I}\leftrightarrow\text{III}}{\xrightarrow{\text{I}-2\text{III}}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\text{II}\leftrightarrow\text{III}}{\xrightarrow{\text{III}-3\text{II}}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}-2\text{IV}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 & -2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$B_1^{-1} = \begin{pmatrix} 1 & -7 \\ -1 & 8 \end{pmatrix}, \quad B_2^{-1} = \begin{pmatrix} 1/2 & -1 & -6 \\ 0 & 1 & 7 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_3^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -3 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Calcule $C := A_2 \cdot B_3$.

$$C = A_2 \cdot B_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 5 & 6 \\ 3 & 1 & 4 & 3 \\ 2 & 0 & 1 & 3 \\ -1 & -1 & -3 & -1 \end{pmatrix}$$

5. Calcule o determinante de C.

   Como $\det(A_2) = 0$, temos

$$\det(C) = \det(A_2 \cdot B_3) = \det(A_2) \cdot \det(B_3) = 0 \cdot \det(B_3) = 0.$$

6. Indicaremos com 0_n a matriz nula $n \times n$. Seja A uma matriz $n \times n$. É sempre verdade que se $A \neq 0_n$ então $A \cdot A \neq 0_n$? Se a resposta for sim, dê uma demonstração. Se a resposta for não, dê um contra-exemplo.

   A resposta é não, por exemplo se $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ então $A \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Prova 1 IAL - Tema B - 12/07/2022 - Gabarito

1. Resolva os sistemas

$$\begin{cases} 2x+3y+z+3w=1 \\ 3y-z+w=0 \\ x+z+w=1 \end{cases} \qquad \begin{cases} x-y=1 \\ -z+2w=1 \\ 2x-y-w=-1 \\ 4x-3y-z+w=2 \end{cases}$$

Primeiro sistema.

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 3 & 1 \\ 0 & 3 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{\text{I}-2\text{III}} \left(\begin{array}{cccc|c} 0 & 3 & -1 & 1 & -1 \\ 0 & 3 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{\text{I}-\text{II}} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -1 \\ 0 & 3 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{array}\right)$$

A primeira equação é $0=-1$, logo o sistema não tem solução.

Segundo sistema.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 2 & -1 & 0 & -1 & -1 \\ 4 & -3 & -1 & 1 & 2 \end{array}\right) \xrightarrow[\text{IV}-4\text{I},\ \text{II}\leftrightarrow\text{III}]{\text{III}-2\text{I}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow[\text{IV}-\text{II}]{-\text{III}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right) \xrightarrow{\text{IV}+\text{III}} \left(\begin{array}{cccc|c} \mathbf{1} & -1 & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -1 & -3 \\ 0 & 0 & \mathbf{1} & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Escolhendo $w=t$ obtemos $z=2t-1$, $y=t-3$, $x=y+1=t-2$.

$$(x,y,z,w)=(t-2,t-3,2t-1,t)=(-2,-3,-1,0)+t(1,1,2,1).$$

2. Calcule o determinante de

$$A_1 := \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 4 & 1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

Usando a regra de Sarrus (que pode ser aplicada apenas para matrizes 3×3) temos

$\det(A_1) = 1\cdot1\cdot1+2\cdot(-1)\cdot0+0\cdot(-1)\cdot4-0\cdot1\cdot0-2\cdot(-1)\cdot1-1\cdot(-1)\cdot4 = 1+2+4 = 7.$

Para calcular $\det(A_2)$ podemos fazer operações de linha

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{I}-\text{II}} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{I}-\text{III}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

obtendo assim uma matriz com uma linha nula. Segue que $\det(A_2)=0$.

3. Encontre a inversa de

$$B_1 := \begin{pmatrix} 8 & 1 \\ 7 & 1 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_3 := \begin{pmatrix} 2 & 1 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\left(\begin{array}{cc|cc} 8 & 1 & 1 & 0 \\ 7 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\text{I}-\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 7 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}-7\text{I}} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -7 & 8 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 7 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{\text{II}+7\text{III}}{\xrightarrow{\text{I}-2\text{II}}} \left(\begin{array}{ccc|ccc} 2 & 0 & -11 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{-\text{III}}{\xrightarrow{\text{I}-11\text{III}}}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -2 & -11 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \xrightarrow{\text{I}/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & -11/2 \\ 0 & 1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\text{I}\leftrightarrow\text{III}}{\xrightarrow{\text{I}-2\text{III}}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\text{II}\leftrightarrow\text{III}}{\xrightarrow{\text{III}-4\text{II}}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -4 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}-2\text{IV}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & -2 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$B_1^{-1} = \begin{pmatrix} 1 & -1 \\ -7 & 8 \end{pmatrix}, \quad B_2^{-1} = \begin{pmatrix} 1/2 & -1 & -11/2 \\ 0 & 1 & 7 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_3^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -4 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Calcule $C := A_2 \cdot B_3$.

$$C = A_2 \cdot B_3 = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 6 & 4 \\ 3 & 1 & 5 & 3 \\ 2 & 0 & 1 & 1 \\ -1 & -1 & -4 & -1 \end{pmatrix}$$

5. Calcule o determinante de C.

   Como $\det(A_2) = 0$, temos

$$\det(C) = \det(A_2 \cdot B_3) = \det(A_2) \cdot \det(B_3) = 0 \cdot \det(B_3) = 0.$$

6. Indicaremos com 0_n a matriz nula $n \times n$. Seja A uma matriz $n \times n$. É sempre verdade que se $A \neq 0_n$ então $A \cdot A \neq 0_n$? Se a resposta for sim, dê uma demonstração. Se a resposta for não, dê um contra-exemplo.

   A resposta é não, por exemplo se $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ então $A \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.