Primeira prova **IAL Turma Z Tema A** **15/04/2016**

Nome e matricula:

Justificar todas as respostas.
Respostas não justificadas não serão consideradas.

1. **Exercício 1 [3 pontos]**

   Resolva os sistemas lineares seguintes usando a redução de Gauss (1 ponto para cada sistema).

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ -x_1 - x_2 + x_3 = 1 \\ x_1 - 4x_2 + 2x_3 = 2 \end{cases} \quad \begin{cases} x_1 - 2x_2 - 3x_3 + x_4 - x_5 = 2 \\ x_2 + x_3 - x_5 = 1 \\ x_1 - x_2 - x_3 = 0 \\ x_3 - x_4 + 2x_5 = -3 \end{cases} \quad \begin{cases} x_1 + 2x_2 = 0 \\ x_2 + x_3 + x_4 = -1 \\ x_1 + x_2 + 2x_3 + 2x_4 = 2 \\ x_2 + 4x_3 + x_4 = 3 \end{cases}$$

2. **Exercício 2 [3 pontos]**

   Calcule A^{-1} onde A é uma das matrizes seguintes (1 ponto para cada matriz).

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}.$$

3. **Exercício 3 [3 pontos]**

   Calcule os determinantes (1 ponto para cada determinante)

$$\begin{vmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & -1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 9 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

4. **Exercício 4 [1 ponto]**

   Sejam A, B duas matrizes 3×3 inversíveis e seja $C = AB$.

   Quantas soluções tem o sistema $Cx = 0$?

Primeira prova **IAL Turma Z Tema B** **15/04/2016**

Nome e matricula:

Justificar todas as respostas.
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1. **Exercício 1 [3 pontos]**

   Resolva os sistemas lineares seguintes usando a redução de Gauss (1 ponto para cada sistema).

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 - 4x_2 + 2x_3 = 2 \\ -x_1 - x_2 + x_3 = 1 \end{cases} \quad \begin{cases} x_1 - 2x_2 - 3x_3 + x_4 - x_5 = 2 \\ x_2 + x_3 - x_5 = 1 \\ x_3 - x_4 + 2x_5 = -3 \\ x_1 - x_2 - x_3 = 0 \end{cases} \quad \begin{cases} x_1 + 2x_2 = 0 \\ x_2 + x_3 + x_4 = -1 \\ x_2 + 4x_3 + x_4 = 3 \\ x_1 + x_2 + 2x_3 + 2x_4 = 2 \end{cases}$$

2. **Exercício 2 [3 pontos]**

   Calcule A^{-1} onde A é uma das matrizes seguintes (1 ponto para cada matriz).

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

3. **Exercício 3 [3 pontos]**

   Calcule os determinantes (1 ponto para cada determinante)

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & -1 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 7 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

4. **Exercício 4 [1 ponto]**

   Sejam A, B duas matrizes 3×3 inversíveis e seja $C = AB$.

   Quantas soluções tem o sistema $Cx = 0$?

I^a PROVA IAL TURMA Z 15/04/2016 (TEMA A)

(1) (no tema B tem os mesmos sistemas, com algumas linhas trocadas)

(a)

Matriz completa: $\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & -4 & 2 & 2 \end{array}\right) \xrightarrow[I \leftrightarrow III]{\substack{II+III \\ I+2II}} \left(\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -5 & 3 & 3 \\ 0 & -5 & 3 & 2 \end{array}\right) \xrightarrow{III-II} \left(\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -5 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right)$

$\Rightarrow$ A 3^a equação é impossível $\Rightarrow$ não tem solução.

(b)

Matriz completa: $\left(\begin{array}{ccccc|c} 1 & -2 & -3 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & -3 \end{array}\right) \xrightarrow{III-I} \left(\begin{array}{ccccc|c} 1 & -2 & -3 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 2 & -3 \end{array}\right)$

$\xrightarrow{III-II} \left(\begin{array}{ccccc|c} 1 & -2 & -3 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 2 & -3 \end{array}\right) \xrightarrow{IV-III} \left(\begin{array}{ccccc|c} 1 & -2 & -3 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Variaveis livres: $x_4 = t$, $x_5 = s$.

$\Rightarrow x_3 = t - 2s - 3$, $x_2 = -(t-2s-3) + s + 1 =$
$= 3s - t + 4$

$x_1 = 2(3s-t+4) + 3(t-2s-3) - t + s + 2 = s+1$

Solução:

$\Rightarrow (x_1, x_2, x_3, x_4) = (s+1, 3s-t+4, t-2s-3, t, s)$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s+1 \\ 3s-t+4 \\ t-2s-3 \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$

(c)

Matriz completa: $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 4 & 1 & 3 \end{array}\right) \xrightarrow{III-I} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 2 & 2 & 2 \\ 0 & 1 & 4 & 1 & 3 \end{array}\right) \xrightarrow[IV-II]{III+II} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 3 & 3 & 1 \\ 0 & 0 & 3 & 0 & 4 \end{array}\right)$

$\xrightarrow{IV-III} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & -3 & 3 \end{array}\right)$

$x_4 = -1$

$\to x_3 = \frac{1}{3} + 1 = \frac{4}{3}$

$x_2 = -\frac{4}{3} + 1 - 1 = -\frac{4}{3}$

$x_1 = -2\left(-\frac{4}{3}\right) = 8/3$

$\Rightarrow$ Sol: $(x_1, x_2, x_3, x_4) = (8/3, -4/3, 4/3, -1)$

(2) (No tema B as primeiras duas matrizes são as mesmas)

(a)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mathrm{II}-\mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mathrm{III}-\mathrm{II}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{I}-\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \Longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

(b)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{III}-2\mathrm{I}]{\mathrm{II}-\mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right) \xrightarrow[-\mathrm{II},\ \mathrm{II}\leftrightarrow\mathrm{III}]{\mathrm{I}+\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & -2 & -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{II}+3\mathrm{III}]{\mathrm{I}+\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -2 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{\mathrm{II}/-2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1/2 & 3/2 & -1/2 \\ 1 & -1 & 0 \end{pmatrix}$$

(c)

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{I}\leftrightarrow\mathrm{II}]{\mathrm{III}-2\mathrm{II}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{IV}-2\mathrm{II}]{\mathrm{III}\leftrightarrow\mathrm{IV}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{II}+\mathrm{III},\ -\mathrm{III},\ -\mathrm{IV}]{\mathrm{I}+\mathrm{IV}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 \end{array}\right) \Longrightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \end{pmatrix}$$

(c do tema B)

$$\left(\begin{array}{cccc|cccc} 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{I}\leftrightarrow\mathrm{II}]{\mathrm{III}-2\mathrm{II}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{III}\leftrightarrow\mathrm{IV}]{\mathrm{II}-2\mathrm{IV}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow[-\mathrm{II},\ -\mathrm{IV},\ \mathrm{II}\leftrightarrow\mathrm{III}]{\mathrm{III}+\mathrm{II}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 \end{array}\right) \xrightarrow{\mathrm{I}-\mathrm{IV}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \\ 0 & 2 & -1 & 0 \end{pmatrix}.$$

(3) (No tema B os primeiros dois determinantes são os mesmos)

(a)
$$\begin{vmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & -1 & -1 \end{vmatrix} \underset{IV-I}{=} \begin{vmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & -1 & -1 \end{vmatrix} = 0$$
pois tem uma linha nula.

(b)
$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{vmatrix} \underset{III-IV}{=} \begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{vmatrix} \underset{\text{(exp 3ª l.)}}{=} -\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} \underset{III-2II}{=} -\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} =$$

$$\underset{\text{(exp 3ª l.)}}{=} +\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

(c)
$$\begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 9 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix} \underset{\text{(exp 5ª l.)}}{=} -\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 9 & 2 & 2 \\ 1 & 1 & 2 & 1 \end{vmatrix} \underset{\text{(exp 1ª l.)}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 9 & 2 \\ 1 & 1 & 2 \end{vmatrix} \underset{III-I}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 9 & 2 \\ 0 & 0 & 1 \end{vmatrix} \underset{\text{(exp III l.)}}{=} \begin{vmatrix} 1 & 1 \\ 0 & 9 \end{vmatrix} = 9.$$

A solução do (c) do tema B é a mesma com 7 no lugar de 9.

(4)

A, B inversíveis $\Rightarrow \det(A) \neq 0, \det(B) \neq 0$.

$\det(C) = \det(AB) = \det(A)\det(B) \neq 0$ pois $\det(A) \neq 0$, $\det(B) \neq 0$.

Logo C tem inversa C^{-1}.

$Cx = 0 \rightarrow$ multiplico os dois lados por C^{-1} à esquerda

$C^{-1}Cx = C^{-1}0 \Rightarrow x = C^{-1}0 = 0$

$\Rightarrow$ a única solução é $x = 0$. Tem uma solução.