Prova escrita - Grupos profinitos - 2018-1

I will grade each item. Let $l_1, \ldots, l_6$ be the grades of exercise 1 and let $r_1, \ldots, r_4$ be the grades of exercise 2. The final grade will be
$\max\{l_i + l_j \;:\; 1 \leq i,j \leq 6,\ i \neq j\} + \max\{r_i + r_j \;:\; 1 \leq i,j \leq 4,\ i \neq j\}$.

1. Exercise 1. **Parte prática**. Prove two of the following (3 points each).
   (a) Let G be a profinite group. Prove that G has a closed normal pronilpotent subgroup that contains all the closed normal pronilpotent subgroups of G. Use the fact that this is true when G is finite.
   (b) Let $f : A \to B$ be an abstract group homomorphism. Prove that when A and B are given the profinite topology f is continuous and it induces canonically a continuous homomorphism between the profinite completions $\widehat{A} \to \widehat{B}$.
   (c) $(*)$ Prove that the profinite topology makes any group a topological group and that a group G is residually finite if and only if the profinite topology on G is Hausdorff.
   (d) $(*)$ Let G be a topological group and define an equivalence relation on G by saying that $x \sim y$ if and only if there is a connected subset of G containing x and y. Let $C = \{x \in G \;:\; x \sim 1\}$ be the equivalence class of 1. Prove that C is a connected subset of G that contains every connected subset of G containing 1, and that C is a closed normal subgroup of G. [Use the fact that the cartesian product of two connected spaces is connected.] - Don't prove that $\sim$ is an equivalence relation.
   (e) $(*)$ Using the previous item (and its notation) show that G/C (with the quotient topology) is a profinite group if G is compact. [Hint: prove that the equivalence class of $x \in G$ is Cx. Let A be a nonempty connected subset of G/C, show that if $|A| > 1$ then its preimage in G is not connected and writing it as $U \cup V$ with U and V open disjoint subsets show that U and V are unions of cosets of C.]
   (f) $(*)$ Let G be a group with the profinite topology and let H be a normal subgroup of G. Prove that the quotient topology on G/H coincides with the profinite topology.

2. Exercise 2. **Parte teórica**. Prove two of the following (2 points each):
   (a) The open subgroups of $G = \mathbb{Z}_p$ are the subgroups $p^n G$ with $n \geq 0$.
   (b) If G is a 1-generated profinite group every subgroup of G of finite index is open.
   (c) Let $G = \prod_{n \geq 5} A_n$ where A_n is the alternating group of degree n. Prove that G is topologically generated by 2 elements.
   (d) Let G be a profinite group. If G/N can be generated by d elements for every open normal subgroup N of G then G can be topologically generated by d elements.