Prova escrita - Grupos profinitos - 2018-1
Resolução do exercício 1 (para o exercício 2 veja as notas de aula).

a) **Let G be a profinite group. Prove that G has a closed normal pronilpotent subgroup that contains all the closed normal pronilpotent subgroups of G. You can use the fact that this is true when G is finite**.

Let H be the subgroup of G abstractly generated by the closed normal pronilpotent subgroups of G. It is enough to prove that $\overline{H}$ is a closed normal pronilpotent subgroup of G. Let $\mathscr{N}$ be the family of closed normal pronilpotent subgroups of G and let $F = \overline{\langle \mathscr{N} \rangle}$, clearly F is a normal closed subgroup of G, we must show that F is pronilpotent. If U is an open normal subgroup of F then F/U is finite generated by $\{NU/U \ : \ N \in \mathscr{N}\}$ hence there are $N_1, \ldots, N_t \in \mathscr{N}$ with $F/U = N_1U/U \cdots N_tU/U$, so F/U is nilpotent.

b) **Let $f : A \to B$ be an abstract group homomorphism. Prove that when A and B are given the profinite topology f is continuous and it induces canonically a continuous homomorphism between the profinite completions $\widehat{A} \to \widehat{B}$**.

We need to show that if bN is a coset of a normal subgroup N of B of finite index then $f^{-1}(bN)$ is profinite open in A. We may assume $f^{-1}(bN)$ is nonempty, let $x \in f^{-1}(bN)$, then $f(x) \in bN$ so we may assume that $b = f(x)$. We have

$$\begin{aligned} f^{-1}(bN) &= \{a \in A \ : \ f(a) \in bN\} = \{a \in A \ : \ f(a) \in f(x)N\} \\ &= \{a \in A \ : \ f(ax^{-1}) \in N\} = \{a \in A \ : \ a \in f^{-1}(N)x\} = f^{-1}(N)x \end{aligned}$$

so it is enough to prove that $f^{-1}(N)$ has finite index in A. This is clear because $f^{-1}(N)$ is the kernel of $A \to B \to B/N$ hence by the isomorphism theorem $A/f^{-1}(N)$ is isomorphic to a subgroup of B/N, which is finite.

Now the composition of $f : A \to B$ with $B \to \widehat{B}$ gives a continuous homomorphism $A \to \widehat{B}$ hence by the universal property of the profinite completion we have a canonical continuous homomorphism $\widehat{A} \to \widehat{B}$.

c) **Prove that the profinite topology makes any group a topological group and that a group G is residually finite if and only if the profinite topology on G is Hausdorff**.

The profinite topology on G is the topology whose base is given by the cosets of the subgroups of finite index. Let G have the profinite topology and consider

$$v : G \times G \to G, \ (x, y) \mapsto xy^{-1}.$$

To show that v is continuous it is enough to show that $v^{-1}(Ng)$ is continuous whenever N is a normal subgroup of G of finite index and $g \in G$. This is because any subgroup of finite index is a union of cosets of its normal core, which also has finite index. Now

$$v^{-1}(Ng) = \{(x, y) \in G \times G \ : \ xy^{-1} \in Ng\} = \bigcup_{y \in G} Ngy \times Ny$$

is clearly open.

G is residually finite if and only if the intersection of the subgroups of finite index is $\{1\}$, and this implies that $\{1\}$ is closed in the profinite topology hence G is Hausdorff. Conversely if G is Hausdorff and $g \in G - \{1\}$ then we can separate 1 from g, so there is a subgroup H of finite index with $1 \notin Hg$, equivalently $g \notin H$, implying that g does not belong to all subgroups of finite index. This proves that the intersection of the subgroups of finite index is $\{1\}$ hence G is residually finite.

d) **Let G be a topological group and define an equivalence relation on G by saying that $x \sim y$ if and only if there is a connected subset of G containing x and y. Let $C = \{x \in G \ : \ x \sim 1\}$ be the equivalence class of 1. Prove that C is a connected subset of G that contains every connected subset of G containing 1, and prove that C is a closed normal subgroup of G.** [Use the fact that the cartesian product of two connected spaces is connected.]

C is connected because if $C = A \cup B$ with A, B open and $a \in A$, $b \in B$ let L be a connected subset containing a and b (it exists because $a \sim b$), then $L = (A \cap L) \cup (B \cap L)$, a contradiction. If Y is a connected subset of G containing 1 then $y \sim 1$ for every $y \in Y$ by definition, hence $Y \subseteq C$. To prove that C is closed it is enough to prove that $\overline{C}$ is connected (because then $\overline{C}$ is connected and it contains 1 hence $\overline{C} \subseteq C$), say $\overline{C} = U \cup V$ with U and V open and disjoint, then $C = (U \cap C) \cup (V \cap C)$ hence WLOG $U \cap C = \emptyset$ (being C connected) so $C = V \cap C$, in other words $C \subseteq V$, however V is closed in $\overline{C}$ (its complement being U) hence V is closed in G, so $\overline{C} \subseteq V$ hence $U = \emptyset$. The fact that C is a subgroup follows by observing that setting $v : G \times G \to G$ given by $v(x, y) = xy^{-1}$, the set $v(C \times C)$ is connected (because v is continuous and $C \times C$ is connected) and it contains 1, so it is contained in C. The fact that C is normal follows from the same argument considering $G \to G$, $x \mapsto g^{-1}xg$ for every $g \in G$.

e) **Using the previous item (and its notation) show that G/C (with the quotient topology) is a profinite group if G is compact**. [Hint: prove that the equivalence class of x is Cx. Let A be a nonempty connected subset of G/C, show that if $|A| > 1$ then its preimage in G is not connected and writing it as $U \cup V$ with U and V open disjoint subsets show that U and V are unions of cosets of C.]

G/C is compact because it is a continuous image (via the canonical projection) of G, which is compact, and we know that G/C is a topological group. We are left to show that G/C is totally disconnected. First observe that if $x \in G$ then Cx is connected (being C connected) so it is contained in the equivalence class of x, on the other hand if $x \sim y$ then letting D be a connected subset containing x and y, Dx^{-1} is a connected subset containing 1 and yx^{-1} so $1 \sim yx^{-1}$, hence $yx^{-1} \in C$ implying $y \in Cx$. So the equivalence class of x is Cx. This implies (similarly as for C) that Cx contains all the connected subsets of G containing x. Let A be a nonempty connected subset of G/C and let P be its preimage in G. Suppose $|A| > 1$ by contradiction. Then P contains two cosets Cx, Cy of C hence it is not connected (it is not contained in Cx) so we may write $P = U \cup V$ with U and V open disjoint nonempty subsets of P. If $z \in P$ then Cz is connected so intersecting it with U and V we see that Cz must be contained in one of them. So U and V are unions of cosets of C. Now apply the canonical projection $\pi : G \to G/C$ to find $A = \pi(P) = \pi(U \cup V) = \pi(U) \cup \pi(V)$, a union of two open subsets. Such union is disjoint because if $\pi(u) = \pi(v)$ for some $u \in U$, $v \in V$ then $Cu = Cv$ so since U is a union of cosets of C we deduce $Cu = Cv \subseteq U$ contradicting $U \cap V = \emptyset$. Since A is connected WLOG we have $\pi(U) = \emptyset$ that is $U = \emptyset$. A contradiction.

f) **Let G be a group with the profinite topology and let H be a normal subgroup of G. Prove that the quotient topology on G/H coincides with the profinite topology**.

By "q.t." we mean "quotient topology". Let U be a q.t. open subset of G/H, call V its preimage in G, which is open by definition. Since V is open we can write $V = \bigcup_i N_i x_i$ with each N_i normal subgroup of finite index and $x_i \in G$. Let $\pi : G \to G/H$ be the canonical projection. Then $U = \pi(V) = \bigcup_i \pi(N_i)\pi(x_i)$ so to conclude it is enough to show that $\pi(N_i) = N_i H/H$ is profinite open. This is clear because $|G/H : N_i H/H| = |G : N_i H|$, $N_i \leq N_i H \leq G$ and N_i has finite index.

Conversely let N/H be a normal subgroup of G/H of finite index, we need to show that N/H is q.t. open. Its preimage in G is N which has index $|G : N| = |G/H : N/H|$, finite, hence N is open, as we want.