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THE ASSOCIATED FAMILY

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Abstract

Minimal surfaces in euclidean 3-space, i.e. conformal harmonic maps, enjoy two important properties: They allow a circle of isometric deformations rotating the principal curvature directions, the so called associated family, and they are obtained as the real part of holomorphic functions into $\mathbb{C}^3$. These properties are shared by arbitrary (pluri-)harmonic maps into euclidean n-space. Replacing $\mathbb{R}^n$ with an arbitrary symmetric space leads to similar results, but the rôle of $\mathbb{C}^n$ is played by an infinite dimensional complex homogeneous space acted on by a twisted loop group. We give a survey of the development of this theory from our view point, and we discuss applications to the construction of (pluri-)harmonic maps into symmetric spaces and their rank restrictions.

1. Introduction: Associated families in $\mathbb{R}^n$

Associated families are certain isometric deformations of surfaces in euclidean space. The best known example is the deformation of the catenoid into the helicoid by cutting the catenoid along a vertical meridian and move the two ends of the cut upwards and downwords, respectively, apart from each other.[1] Starting with a surface f, the associated family is an isometric deformation f_θ preserving the principal curvatures with three additional properties (visible in the pictures):

- At every point, the tangent plane and the Gauss map remain the same during the deformation,

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[1] http://page.mi.fu-berlin.de/polthier/Calendar/Kalender86/Kalender86.htm, http://en.wikipedia.org/wiki/Catenoid

- principal curvature lines rotate,
- the deformation is periodic, $f_{\theta+2\pi} = f_\theta$, and after a half period π we see the same object in opposite orientation.

In fact, denoting by R_θ the rotation by the angle θ in the tangent plane of the surface, we have the equation

$$df \circ \mathsf{R}_\theta = df_\theta. \tag{1}$$

For which (other) surfaces $f : M \to \mathbb{R}^n$ does there exist an associated family with (1)? It is enough to consider the 90^o rotation $J = \mathsf{R}_{\pi/2}$ since $\mathsf{R}_\theta = (\cos\theta)I + (\sin\theta)J$. We need to find a map $g : M \to \mathbb{R}^n$ with

$$df \circ J = dg.$$

If M is simply connected (which we will always assume), this is equivalent to

$$d(df \circ J) = 0.$$

From $df = f_x dx + f_y dy$ we see $df \circ J = f_y dx - f_x dy$ and hence

$$d(df \circ J) = f_{yy} dx \wedge dy - f_{xx} dy \wedge dx = \Delta f \, dx \wedge dy$$

where $\Delta f = \text{trace } ddf = f_{xx} + f_{yy}$ is the Laplacian. Hence (1) is equivalent to f being *harmonic*, i.e. $\Delta f = 0$. In particular this applies to minimal surfaces which are conformal harmonic maps.

Harmonic maps of surfaces are easy to describe in terms of holomorphic maps (convergent complex power series) h: We have

$$f = 2\,\mathrm{Re}\; h \tag{2}$$

for some holomorphic map h. This is easy to see: Since $\Delta f = 4f_{z\bar z}$ (where $z = x + iy$), a harmonic map f satisfies $f_{z\bar z} = 0$, hence f_z is holomorphic and thus it has a holomorphic principal function h, i.e. $h_z = f_z$, $h_{\bar z} = 0$. Now

$$(f - (h + \bar h))_z = 0,$$

and therefore $f - (h + \bar h)$ is antiholomorphic and real which is impossible unless $f - (h + \bar h) = const$. This shows (2).

Property (1) is beautiful, Property (2) is useful. In harmonic maps, both beauty and use come together.

Everything can be generalized immediately to several variables: The surface M (a complex curve) can be replaced with a complex manifold M of arbitrary dimension when f is not only harmonic but *pluriharmonic*: i.e. $f|_C$ is harmonic for each complex 1-dimensional submanifold (complex curve) $C \subset M$.

Question: Does (1) and (2) remain true when $\mathbb{R}^n$ is replaced with a (suitable) Riemannian manifold S?

The answer to this question is a main part of my joined work with Renato.

2. An integrability theorem

Our first common paper [EGT] (joined with I.V. Guadalupe) was a result of the lectures of Prof. S.S. Chern at Berkeley in the fall of 1981. The subject was the theory of differential forms and moving frames with an application to minimal surfaces in $\mathbb{C}P^n$. As an application of these ideas, we worked out the local invariants and their differential equations for minimal surfaces in $\mathbb{C}P^2$. These displayed a close similarity to minimal surfaces in the 4-sphere which had been investigated before by my co-authors [GT]. The methods we used were moving frames and the Frobenius integrability theorem. A consequence was the existence of associated families for minimal surfaces not only in space forms, but also in $\mathbb{C}P^2$.

How much farther could this method be applied? In some sense, we had extended the classical existence and uniqueness theorem for surfaces in euclidean space to $\mathbb{C}P^2$ as ambient space. Later [ET1] we investigated this question more systematically. Existence and uniqueness theorems for submanifolds, more generally for maps, exist precisely on homogeneous spaces. As long as we want to use only torsion free affine connections we must restrict our attention even to a subclass of homogeneous spaces, the symmetric ones.

Let $S = G/K$ be a symmetric space with parallel curvature tensor (*Lie triple product*) R^S and let $f : M \to S$ be any smooth map. Its differential, the 1-form $\omega = df : TM \to f^*TS =: E$ is easily seen to satisfy the following equations (*Cartan structure equations*):

$$d^\nabla \omega = 0, \quad \omega^* R^S = R^E, \tag{C}$$

where $d^\nabla \omega(X,Y) = (\nabla_X \omega)Y - (\nabla_Y \omega)X$ and R^E the curvature tensor of the vector bundle $E = f^*TS$ with its induced connection. The main result of [ET1] is the converse statement:

Theorem 1. *Given any vector bundle E with a metric connection ∇ and a parallel Lie triple product R^S on each fibre, isomorphic to the one on S, and a bundle map $\omega : TM \to E$ with (C), there exists a map $f : M \to S$, unique up to composition with some $g \in G$, and a parallel Lie triple bundle isomorphism $\Phi : E \to f^*TS$ such that*

$$df = \Phi \circ \omega. \tag{3}$$

The proof was again by moving frames and Frobenius, but in the statement moving frames had disappeared.

3. Generalizing the associated family

Now we were able to find associated families for pluriharmonic maps with values in any symmetric space, $f : M \to S$ (cf. [ET3]). We just applied Theorem 1 to the 1-form $\omega = \omega_\theta = df \circ \mathsf{R}_\theta : TM \to E = f^*TS$. In fact, pluriharmonicity of f is equivalent to (C) for all ω_θ, $\theta \in (0, 2\pi]$, and then by (3) there exists a map $f_\theta : M \to S$ and a parallel isomorphism $\Phi_\theta : f^*TS \to f_\theta^*TS$ with

$$df_\theta = \Phi_\theta \circ df \circ \mathsf{R}_\theta. \qquad (1')$$

This is what we call an *associated family* for f. Thus we have seen:

Theorem 2. *Let M be a complex manifold, S a symmetric space*[2] *and $f : M \to S$ a smooth map. Then f is pluriharmonic if and only if it has an associated family in the sense of* $(1')$.

4. Extended frames

So much on generalizations of the associated family (1). But what about (2), the relation to holomorphic maps? In [ET3] we could only do a special case, the so called *isotropic* pluriharmonic maps where the associated family happens to be trivial; these are projection of certain (so called *superhorizontal*) holomorphic maps ("*twistor lifts*") with values in a finite dimensional flag manifold Z fibering over S, the *twistor space*. The general case remained unsolved. However for surfaces ($\dim M = 2$), a complete answer was given by Dorfmeister

[2] We need a little restriction on S: the curvature operator of S needs to be semi-definite which means that the irreducible factors are either all of compact or euclidean type or they are all of noncompact or euclidean type.

et al. in [DPW], extending previous work by Uhlenbeck [U] and others. In a joined work with Dorfmeister, combining both methods, we got a complete answer [DE].

We came back to moving frames. A moving frame along $f : M \to S = G/K$ is just a map F into the group G which projects onto f, i.e. $\pi \circ F = f$ for the canonical projection $\pi : G \to G/K$. However, since π is a nontrivial fibration, the lift F can be defined only locally, on any contractible open subset $M_o \subset M$. Having chosen a frame for f, we define another frame for the associated f_θ as

$$F_\theta = \Phi_\theta F \tag{4}$$

where $\Phi_\theta(x)$ as defined in $(1')$ is now is considered as an element of G for any $x \in M_o$.[3] The family of these frames $(F_\theta)_{\theta\in[0,2\pi]}$ is considered as a new object, a map F from M_o into the set of smooth maps from the circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ into G. Using our freedom in the choice of Φ_θ and the equality $(1')$ for $\theta + \pi$ (remind $\mathsf{R}_{\theta+\pi} = -\mathsf{R}_\theta$) we may assume that the map $\mathsf{F} = (F_\theta)$ takes values in the *twisted loop group*

$$\Lambda = \Lambda_\sigma G = \{\gamma : S^1 \to G;\ \gamma(\theta + \pi) = \sigma(\gamma(\theta))\} \tag{5}$$

where $\sigma : G \to G$ is the involution with $K = \mathrm{Fix}(\sigma)$.[4] We call $\mathsf{F} : M_o \to \Lambda$ the *extended frame* of f. It is unique only up to right multiplication by elements of K which do not depend on θ. In order to make it unique and hence globally defined (by patching the local frames together) we work modulo K and define

$$\hat{h} := \mathsf{F} \bmod K : M \to \Lambda/K.$$

[3]Note that $\Phi_\theta(x) : T_{f(x)}S \to T_{f_\theta(x)}S$ is a linear isometry preserving R^S and hence it is the differential at the point $f(x)$ of a unique isometry $g_\theta(x) \in G$ sending $f(x)$ onto $f_\theta(x)$; whenever possible we will use the same symbol $\Phi_\theta(x)$ for $g_\theta(x)$.

[4]In general we only know that K lies between $\mathrm{Fix}(\sigma)$ and its identity component. But we will assume $K = \mathrm{Fix}(\sigma)$ which is no restriction up to finite coverings.

where $K \subset \Lambda$ is considered as the subgroup of constant loops. Clearly,

$$f = e \circ \hat{h} \tag{2'}$$

where $e : \Lambda/K \to G/K = S$ is the evaluation map

$$e : \Lambda/K \ni \gamma K \mapsto \gamma(0)K \in G/K.$$

Theorem 3. *Let M be a complex manifold and $S = G/K$ a symmetric space. Then the pluriharmonic maps $f : M \to S$ are precisely given by (2′) where $\hat{h} : M \to \Lambda/K$ is any holomorphic and superhorizontal map.*

We have to explain these terms. To define holomorphicity we need a complex structure on Λ/K. Let us assume that G is a matrix group, $G \subset \mathbb{R}^{n\times n}$, with Lie algebra $\mathfrak{g} \subset \mathbb{R}^{n\times n}$. Let $G^c \subset \mathbb{C}^{n\times n}$ the complex matrix group containing G with Lie algebra $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$, the *complexification* of G, and let $\Lambda^c = \Lambda_\sigma G^c$ like in (5) where σ is extended to a holomorphic involution of G^c.[5] Any $\gamma \in \Lambda^c$ can be decomposed into a matrix valued Fourier series:

$$\gamma(\lambda) = \sum_{j\in\mathbb{Z}} a_j \lambda^j$$

with $a_j \in \mathbb{C}^{n\times n}$ and $\lambda = e^{-i\theta} \in S^1$. Let Λ^+ be the subgroup of Λ^c which consists of those Fourier series in Λ^c with $a_j = 0$ for $j < 0$. It is known that each $\gamma^c \in \Lambda^c$ allows an (almost unique) decomposition

$$\gamma^c = \gamma\gamma^+ \tag{6}$$

with $\gamma \in \Lambda$ and $\gamma^+ \in \Lambda^+$, the so called *Iwasawa decomposition* of the loop group Λ^c (cf. [PS]). In particular, $\Lambda \subset \Lambda^c$ acts transitively on the homogeneous space

[5] Often the involution σ on G extends to an involution of the full matrix algebra $\mathbb{R}^{n\times n}$; in this case σ is complex linearly extended to $\mathbb{C}^{n\times n}$.

Λ^c/Λ^+, and the stabilizer subgroup of this action is $\Lambda \cap \Lambda^+$ which is the set of constant loops K.[6] Hence

$$\Lambda/K = \Lambda^c/\Lambda^+$$

which explains the complex structure of Λ/K since both Λ^c and Λ^+ are complex groups.

To show holomorphicity of $\hat{h}$ we have to prove that $d\hat{h}$ is complex linear. Recall that $\hat{h} = \mathsf{F} \bmod K$. Hence the differential of $\hat{h}$ is determined by the $\mathfrak{g}$-valued 1-form $\alpha^\theta = F_\theta^{-1} df_\theta$. Due to the choice (4) of the frame F_θ, the associated family property (1′) is equivalent to a simple dependence of α^θ on $e^{-i\theta} = \lambda$:

$$\alpha^\theta = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}} \tag{7}$$

where $\alpha_{\mathfrak{k}}$ and $\alpha_{\mathfrak{p}}$ are the components of $\alpha = \alpha^0 = F^{-1}dF$ with respect to the Cartan decompostion $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to the symmetric space G/K and where α' and α'' denote the restrictions of the (complexified) 1-form α to $T'M$ and $T''M$ respectively.[7] Considering $\hat{\alpha} = (\alpha^\theta)_{\theta\in[0,2\pi]}$ as a single 1-form with values in the Lie algebra L of Λ, we have

$$\hat{\alpha} \equiv \lambda^{-1}\alpha'_p \mod L^+$$

where L^+ denotes the Lie algebra of Λ^+; note that by (6) we have to work modulo Λ^+ in order to compute $d\hat{h}$. Since $\alpha'_{\mathfrak{p}}$ is complex linear (it is the complex linear part of $\alpha_{\mathfrak{p}}$), the holomorphicity of $\hat{h}$ is shown.

Superhorizontality just means that α^θ has the form (7), in particular its Fourier decomposition has lowest λ-power λ^{-1}; clearly this property is well

[6] Note that the Fourier coefficients a_j of a real loop γ (i.e. $\gamma = \overline{\gamma}$) satisfy $a_{-j} = \overline{a_j}$, hence $\gamma \in \Lambda \cap \Lambda^+$ implies $\gamma = a_0$ with $\sigma(a_0) = a_0$.

[7] Recall that $T'M, T''M \subset TM \otimes \mathbb{C}$ are the $\pm i$-eigenbundles of the complex structure J on M, i.e. $J = i$ on $T'M$ and $J = -i$ on $T''M$.

defined even modulo L^+. In other words, the space of such Fourier series, $\{\sum_{j\geq -1} a_j\lambda^j;\ a_j \in \mathfrak{g}^c\} \subset L^c$, is invariant under conjugation by Λ^+ and thus it determines a left invariant distribution on Λ^c/Λ^+, the *superhorizontal distribution* H_1, and $d\hat{h}$ takes values in H_1.

Our general theory is completely analogous to the isotropic case mentioned above, but now the space $\hat{Z} = \Lambda/K = \Lambda^c/\Lambda^+$ is infinite dimensional; it is called the *universal twistor space*. The finite dimensional twistor spaces allow a canonical embedding into $\hat{Z}$ which preserves the superhorizontal distributions (cf [E]).

When S is euclidean space, $S = \mathbb{R}^n$, the transvection group G is just the translation group, $G = \mathbb{R}^n$, and $\alpha_p' = d'f = \sum_j f_{z_j} dz_j$ takes values in $\mathfrak{g}^c = \mathbb{C}^n$. We find $\hat{h} : M \to \Lambda^c/\Lambda^+$ by solving $d\hat{h} = \lambda^{-1}\alpha_p'$, which yields $\hat{h} = \lambda^{-1}h$ where $h : M \to \mathbb{C}^n$ is holomorphic with $dh = \alpha_p'$. Hence modulo Λ^+ we have $\hat{h} = \lambda^{-1}h + \lambda\bar{h} = 2\,\mathrm{Re}\,(\lambda^{-1}h)$. Evaluation at $\lambda = 1$ (i.e. $\theta = 0$) yields $f = e \circ \hat{h} = 2\,\mathrm{Re}\ h$ as we have seen in (2).

5. Applications

1. On the homogeneous space $\hat{Z} = \Lambda^c/\Lambda^+$, the group Λ^c acts holomorphically by left translations, preserving the left invariant superhorizontal distribution H_1. Therefore, if h is the twistor lift of a pluriharmonic map $f : M \to S$, then $\gamma \circ h$ for any $\gamma \in \Lambda^c$ is the twistor lift of another pluriharmonic map $\tilde{f} : M \to S$. This is called the *dressing action* on the space of pluriharmonic maps. Thus from one single pluriharmonic map we obtain infinitely many others by applying the dressing action.

2. For loops γ^c belonging to some open dense subset $\Lambda_o^c \subset \Lambda^c$ (the "big cell"), there is another decomposition, the *Birkhoff decomposition* [PS]:

$$\gamma^c = \gamma_- \gamma_+ \tag{8}$$

with $\gamma_\pm \in \Lambda^\pm$ where $\Lambda_- = \{\sum_{j<0} a_j \lambda^j\} \cap \Lambda^c$. Thus we may split the extended frame F as $\mathsf{F} = \mathsf{F}_- \mathsf{F}_+$, and it turns out that F_- is meromorphic on M. Since $\mathsf{F}_- \equiv \mathsf{F} \mod \Lambda_+$, the Fourier series of $\mathsf{F}_-^{-1} d\mathsf{F}_-$ has lowest power λ^{-1} and takes values in L^-, hence only the λ^{-1}-term remains, i.e. $\mathsf{F}_-^{-1} d\mathsf{F}_- = \eta\lambda^{-1}$ for some closed meromorphic 1-form η on M with values in $\mathfrak{p}^c$.[8] Vice versa, given any such 1-form η we find F_- and hence F as the Λ-component in the Iwasawa splitting $\mathsf{F}_- = \mathsf{F}\mathsf{F}_+^{-1}$ (cf (6)). Thus we obtain pluriharmonic maps f from certain closed meromorphic $\mathfrak{p}^c$-valued 1-forms η, called *normalized potential* of f (see [DPW], [DE]). If M is a surface, no further restriction on η is needed, but in higher dimensions η must be a *curved flat*, i.e. for any $x \in M$ the linear map $\eta_x : T'_x M \to \mathfrak{p}^c$ takes values in a flat (abelian) subspace of $\mathfrak{p}^c$ which may depend on x.

3. The curved flat property can be rephrased by saying that for any pluriharmonic map $f : M \to S$ and any $x \in M$, the curvature tensor R^S vanishes on the subspace $df_x(T'_x M) \subset T^c_{f(x)} S$. This property was observed by Ohnita and Valli ([OV], see also [ET3]). It gives a restriction for the rank of a (non-holomorphic) pluriharmonic map $f : M \to S$ which cannot be bigger than the dimension of the largest flat subspace of $\mathfrak{p}^c$ (other than $\mathfrak{p}'$ and $\mathfrak{p}''$). In [ET2] we found examples of maximal rank in complex Grassmannians. In a recent joint paper with P. Kobak [EK] (dedicated to Renato Tribuzy), we have classified all isotropic pluriharmonic maps of maximal rank into complex Grassmannians.

[8] From the twisting condition $\gamma(-\lambda) = \sigma(\gamma(\lambda))$ for all $\gamma \in L$ we get $\sigma(a_j) = (-1)^j a_j$ for the Fourier coefficients, hence $a_j \in \mathfrak{k}^c$ for even j and $a_j \in \mathfrak{p}^c$ for odd j.

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