Wreath products in permutation group theory

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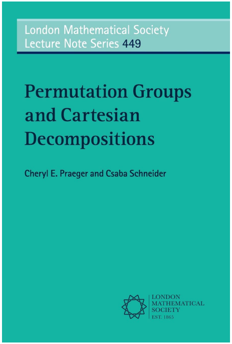
Universidade de Brasília, 19–21 February 2019

NOTES AND BOOKS

These notes: Available online

The book: Permutation Groups and Cartesian Decompositions



Other recomended literature:

1. Peter Cameron, Permutation groups.
2. Dixon & Mortimer, Permutation groups.

OUTLINE OF LECTURES

- I the wreath product construction;
- II the imprimitive action of a wreath product;
- III the primitive action of a wreath product;
- IV groups that preserve cartesian decompositions;
- V twisted wreath products.

The wreath product construction

Input:

1. A group G;
2. a permutation group $H \leqslant \mathsf{S}_\ell$.

Set $B = G^\ell$. An element $h \in H$ induces an $\alpha_h \in \mathsf{Aut}(B)$:

$$(g_1, \ldots, g_\ell)\alpha_h = (g_{1h^{-1}}, \ldots, g_{\ell h^{-1}}) \quad \text{for all} \quad g_i \in G,\ h \in H.$$

The map

$$\varphi : H \to \mathsf{Aut}(B), \quad h \mapsto \alpha_h$$

is a homomorphism.

Example

Let $\ell = 4$, $h = (1,2,3) \in \mathsf{S}_4$:

$$(g_1, g_2, g_3, g_4)\alpha_{(1,2,3)} = (g_1, g_2, g_3, g_4)^{(1,2,3)} = (g_3, g_1, g_2, g_4).$$

The wreath product

Define

$$W = G \wr H = B \rtimes_{\varphi} H = G^{\ell} \rtimes_{\varphi} H.$$

A generic element of W can be written as

$$(g_1, \ldots, g_\ell)h \quad \text{with} \quad g_i \in G,\ h \in H.$$

The multiplication in W:

$$[(g_1, \ldots, g_\ell)h][(g'_1, \ldots, g'_\ell)h'] = (g_1, \ldots, g_\ell)(g'_1, \ldots, g'_\ell)^{h^{-1}}hh' =$$
$$(g_1, \ldots, g_\ell)(g'_{1h}, \ldots, g'_{\ell h})hh' = (g_1 g'_{1h}, \ldots, g_\ell g'_{\ell h})hh'.$$

Terminology:

1. B is the base group of W and $B \trianglelefteq W$.
2. H is the top group of W.

The imprimitive action

Suppose from now on that G is also a permutation group: $G \leqslant \mathsf{Sym}\,\Gamma$.

Then $W = G \wr H$ is a permutation group:

1. on $\Gamma \times \underline{\ell}$ where $\underline{\ell} = \{1, \ldots, \ell\}$ (imprimitive action)
2. on Γ^ℓ (product action).

Set $\Omega = \Gamma \times \underline{\ell}$. For $(g_1, \ldots, g_\ell)h \in W$, and $(\gamma, i) \in \Omega$, define

$$(\gamma, i)(g_1, \ldots, g_\ell)h = (\gamma g_i, ih). \tag{1}$$

Example

Let $g = (a, b, c, d)$, $G = \langle g \rangle \leqslant \mathsf{Sym}\{a, b, c, d\}$ and $H = \mathsf{S}_3$.
Then $W = G \wr H$ acts on $\Omega = \{a, b, c, d\} \times \{1, 2, 3\}$.
For example

$$(c, 2)[(g, g^2, 1)(1, 2, 3)] = (cg^2, 2(1, 2, 3)) = (a, 3).$$

LEMMA

Observe the following W-invariant partition:

$$\{(a,1),(b,1),(c,1),(d,1)\} \cup \{(a,2),(b,2),(c,2),(d,2)\} \cup \{(a,3),(b,3),(c,3),(d,3)\}$$

Lemma

Eq. (1) *defines a W-action on $\Gamma \times \underline{\ell}$ such that:*

1. $(g_1, \ldots, g_\ell)h \in W_{(\gamma,i)}$ *iff* $g_i \in G_\gamma$ *and* $h \in H_i$;
2. *The W-action on $\Gamma \times \underline{\ell}$ is faithful.*
3. $W \leqslant \mathsf{Sym}\,(\Gamma \times \underline{\ell})$ *is transitive iff* $G \leqslant \mathsf{Sym}\,\Gamma$ *is transitive and* $H \leqslant \mathsf{S}_\ell$ *is transitive.*
4. *The partition* $\mathcal{P} = \{\Gamma \times \{i\} \mid i \in \underline{\ell}\}$ *is W-invariant.*

Permutational isomorphisms and embeddings

Let $G_1 \leqslant \mathsf{Sym}\,\Omega_1$ and $G_2 \leqslant \mathsf{Sym}\,\Omega_2$.
A pair (ϑ, α) is said to be a permutational embedding if

1. $\vartheta : \Omega_1 \to \Omega_2$ is a bijection;
2. $\alpha : G_1 \to G_2$ is an injective homomorphism;
3. $(\omega g)\vartheta = (\omega\vartheta)(g\alpha)$ for all $\omega \in \Omega_1$ and $g \in G_1$.

We say: G_1 is permutationally isomorphic to a subgroup of G_2.
We denote this by $G_1 \lesssim G_2$.

The pair (ϑ, α) is a permutational isomorphism if α is an isomorphism.

Inclusions into imprimitive wreath products

Suppose that Γ is a set and $\ell \geqslant 2$. Let $\Omega = \Gamma \times \underline{\ell}$.
Let

$$\mathcal{P} = \{\Gamma \times \{i\} \mid i \in \underline{\ell}\}$$

be the "natural" partition of Ω.

Theorem

The full stabiliser of $\mathcal{P}$ in $\mathsf{Sym}\,(\Gamma \times \underline{\ell})$ is $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.

Consequence: Given a permutation group $G \leqslant \mathsf{Sym}\,\Omega$, the following are equivalent:

1. a homogeneous partition $\mathcal{P} = \{\Delta_1, \ldots, \Delta_\ell\}$ of Ω is G-invariant (homogeneous: $|\Delta_i| = |\Delta_j|$).
2. $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ with some set Γ.

Decomposing an imprimitive permutation group

Suppose that $G \leqslant \mathsf{Sym}\,\Omega$ is a transitive group and let $\mathcal{P} = \{\Delta_1, \dots, \Delta_\ell\}$ be a G-invariant partition.
Then we decompose G:

1. $G^{\Delta_i} = (G_{\Delta_i})^{\Delta_i}$: the group induced by G_{Δ_i} on Δ_i. We have $G^{\Delta_i} \cong G^{\Delta_j}$ for all i, j.
2. $G^{\mathcal{P}} \leqslant \mathsf{S}_\ell$: the permutation group induced on $\mathcal{P}$.

Example

Let $G = \mathsf{D}_8 = \langle (1,2,3,4), (1,2)(3,4) \rangle$. Then

$$\mathcal{P} = \{\Delta_1 = \{1,3\}, \Delta_2 = \{2,4\}\}$$

is G-invariant. Further,

$$G^{\Delta_1} = \langle (1,3), (2,4) \rangle^{\{1,3\}} = \langle (1,3) \rangle \cong C_2 \quad \text{and} \quad G^{\mathcal{P}} \cong C_2.$$

Imprimitive embedding theorem

Theorem

Let $G \leqslant \mathsf{Sym}\,\Omega$ be transitive and let $\mathcal{P} = \{\Delta_1, \ldots, \Delta_\ell\}$ be a G-invariant partition of Ω. Then

$$G \lesssim G^{\Delta_1} \wr G^{\mathcal{P}} \leqslant \mathsf{Sym}\,(\Delta_1 \times \underline{\ell}).$$

Example

In the previous example, we have that $\mathsf{D}_8 \lesssim C_2 \wr C_2$. In fact, $\mathsf{D}_8 \cong C_2 \wr C_2$.

The proof: I

Let's define a permutational embedding (ϑ, α) where

$$\vartheta : \Omega \to \Delta_1 \times \underline{\ell} \quad \text{and} \quad \alpha : G \to G^{\Delta_1} \wr G^{\mathcal{P}}.$$

Step 1. For all $i \in \underline{\ell}$ fix $g_i \in G$ such that $\Delta_i g_i = \Delta_1$.

The definition of ϑ: Let $\omega \in \Omega$. There is a unique $\Delta_i \in \mathcal{P}$ such that $\omega \in \Delta_i$. Define

$$\vartheta : \Omega \to \Delta_1 \times \underline{\ell}, \quad \omega\vartheta = (\omega g_i, i)$$

The definition of α: Let $x \in G$. Define

$$\alpha : G \to G^{\Delta_1} \wr G^{\mathcal{P}}, \quad x \mapsto (x_1, \ldots, x_\ell)\pi_x$$

where π_x is the permutation induced by x on $\underline{\ell}$ and

$$x_i \in G^{\Delta_1} \quad \text{such that} \quad x_i = (g_i^{-1}|_{\Delta_1})(x|_{\Delta_i})(g_{i\pi_x}|_{\Delta_{i\pi_x}}).$$

The proof: II

Now let's compute for $\omega \in \Delta_i \subseteq \Omega$ and $x \in G$ such that $\Delta_i x = \Delta_j$ that

$$\omega x \vartheta = (\omega x g_j, j)$$

while

$$(\omega\vartheta)(x\alpha) = (\omega g_i, i)(x\alpha) = (\omega g_i(g_i^{-1}|_{\Delta_1})(x|_{\Delta_i})(g_j|_{\Delta_j}), j) = (\omega x g_j, j).$$

Hence (ϑ, α) is a permutational embedding as claimed.

Application I: A Kaluzhnin-Krasner Theorem

Suppose that G is a group and $H \leqslant G$ such that $|G : H| < \infty$. Then G acts on G faithfully by right multiplication and $[G : H] = \{Hg \mid g \in G\}$ is a G-invariant partition.

1. The stabiliser of $H \subseteq G$ is H. Hence $G^H = H$.
2. The group induced by G on $[G : H]$ is $G^{[G:H]}$ (right coset action).

Corollary

G is isomorphic to a subgroup of $H \wr G^{[G:H]}$. If $H \trianglelefteq G$, then G is isomorphic to a subgroup of $H \wr (G/H)$.

Applications II: ℓ copies of a complete graph

For $n \geqslant 2$, let K_n be the complete graph on the vertex set $\underline{n}$. For $\ell \geqslant 1$, let ℓK_n is ℓ copies of the complete graph. For example $2K_5$ is the graph

The graph $\mathfrak{G} = \ell K_n$ has ℓ connected components. These connected components form an $\mathsf{Aut}(\mathfrak{G})$-invariant partition of $\mathfrak{G}$. Hence $\mathsf{Aut}(\mathfrak{G}) \leqslant \mathsf{S}_n \wr \mathsf{S}_\ell$. In fact, $\mathsf{Aut}(\mathfrak{G}) = \mathsf{S}_n \wr \mathsf{S}_\ell$.

Application III: Complete bipartite graphs

Let $\mathfrak{G} = K_{m,m}$ be the complete bipartite graph on the vertex set $\Delta_1 \cup \Delta_2$ where

$$\Delta_1 = \{(1,1), \ldots, (m,1)\} \quad \text{and} \quad \Delta_2 = \{(1,2), \ldots, (m,2)\}.$$

That is, (i_1, j_1) and (i_2, j_2) are adjacent if and only if $j_1 \neq j_2$.

Δ_1 and Δ_2 are the maximal independent sets of $\mathfrak{G}$ and hence the partition $\Delta_1 \cup \Delta_2$ is preserved by $\mathsf{Aut}(\mathfrak{G})$.

Hence $\mathsf{Aut}(\mathfrak{G}) \leqslant \mathsf{S}_m \wr \mathsf{S}_2$. In fact $\mathsf{Aut}(\mathfrak{G}) = \mathsf{S}_m \wr \mathsf{S}_2$.

Maximal subgroups of alternating and symmetric groups

Theorem

Let Γ be a finite set of size at least 2 and let $\ell \geqslant 2$. Then

1. $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ *is a maximal subgroup of* $\mathsf{Sym}\,(\Gamma \times \underline{\ell})$;
2. $(\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell) \cap \mathsf{Alt}\,(\Gamma \times \underline{\ell})$ *is a maximal subgroup of* $\mathsf{Alt}\,(\Gamma \times \underline{\ell})$ *unless* $|\Gamma| = 2$ *and* $\ell = 4$.

These maximal subgroups of $\mathsf{Sym}\,\Omega$ give rise to primitive actions of $\mathsf{Sym}\,\Omega$.

Notes:

1. Maximal subgroups of $\mathsf{Sym}\,\Omega$ for finite Ω were described by Jordan (1870), O'Nan & Scott (1979), Liebeck–Praeger–Saxl (1987).
2. For Ω infinite, maximal subgroups related to partitions were constructed by Richman (1967) and Brazil et al. (1994).

The base group is usually characteristic

Theorem (Neumann (1964), Bodnarchuk (1984), Gross (1992), Wilcox (2010), Brewster et al. (2011))

Let G be a group, let $H \leqslant \mathsf{S}_\ell$ be a permutation group, set $W = G \wr H$, and let B be the base group of W.

1. *If H is regular on $\underline{\ell}$, then the following are equivalent:*
   1.1 *B is not a characteristic subgroup of W;*
   1.2 *$H \cong C_2$ and G is a special dihedral group.*
2. *If G is finite and H acts faithfully on its orbits in $\underline{\ell}$, then the following are equivalent:*
   2.1 *B is not a characteristic subgroup of W;*
   2.2 *G is a finite special dihedral group, ℓ is even, and H is permutationally isomorphic to $\mathsf{S}_2 \wr Y$ where $Y \leqslant \mathsf{S}_{n/2}$ and $\mathsf{S}_2 \wr Y$ is considered as a permutation group on $\underline{2} \times \underline{\ell/2}$ in imprimitive action.*

The product action

We define the product action of $W = G \wr H = G^\ell \rtimes H$ on Γ^ℓ:

$$(\gamma_1, \ldots, \gamma_\ell)[(g_1, \ldots, g_\ell)h] = (\gamma_1 g_1, \ldots, \gamma_\ell g_\ell)h = (\gamma_{1h^{-1}} g_{1h^{-1}}, \ldots, \gamma_{\ell h^{-1}} g_{\ell h^{-1}}).$$

Example

Let $\Gamma = \{a, b, c, d\}$ and $\ell = 3$. Then

$$(a, c, b)[((a, b, c), (a, c), (a, d))(2, 3)] = (a(a, b, c), c(a, c), b(a, d))(2, 3) = (b, a, b)(2, 3) = (b, b, a).$$

Lemma

The product action is a faithful $(G \wr H)$-action on Γ^ℓ.
$(G \wr H)_{(\gamma, \ldots, \gamma)} = G_\gamma \wr H.$

CARTESIAN STRUCTURES

Let $\Delta_1, \ldots, \Delta_\ell$ be sets and set

$$\Omega = \Delta_1 \times \cdots \times \Delta_\ell.$$

For $i \in \underline{\ell}$ and $\delta \in \Delta_i$, let

$$B_{i,\delta} = \{(\delta_1, \ldots, \delta_\ell) \in \Omega \mid \delta_i = \delta\}.$$

Then clearly $B_{i,\delta} \subseteq \Omega$. Let

$$\Gamma_i = \{B_{i,\delta} \mid \delta \in \Delta_i\}.$$

Clearly Γ_i is a partition of Ω. Finally set

$$\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}.$$

$\mathcal{E}$ is a collection of partitions of Ω.
Observe: If $B_{1,\delta_1} \in \Gamma_1, \ldots, B_{\ell,\delta_\ell} \in \Gamma_\ell$, then

$$|B_{1,\delta_1} \cap \cdots \cap B_{\ell,\delta_\ell}| = 1.$$

Example

Let $\Delta_1 = \{a, b\}$ and $\Delta_2 = \{1, 2, 3\}$ $\Delta_3 = \{\alpha, \beta\}$. Then

$$B_{1,a} = \{(a, 1, \alpha), (a, 2, \alpha), (a, 3, \alpha), (a, 1, \beta), (a, 2, \beta), (a, 3, \beta)\}.$$

Further,

$$\begin{aligned}
\Gamma_1 &= \{\{(a, 1, \alpha), (a, 2, \alpha), (a, 3, \alpha), (a, 1, \beta), (a, 2, \beta), (a, 3, \beta)\},\\
&\quad \{(b, 1, \alpha), (b, 2, \alpha), (b, 3, \alpha), (b, 1, \beta), (b, 2, \beta), (b, 3, \beta)\}\};\\
\Gamma_2 &= \{\{(a, 1, \alpha), (a, 1, \beta), (b, 1, \alpha), (b, 1, \beta)\},\\
&\quad \{(a, 2, \alpha), (a, 2, \beta), (b, 2, \alpha), (b, 2, \beta)\},\\
&\quad \{(a, 3, \alpha), (a, 3, \beta), (b, 3, \alpha), (b, 3, \beta)\}\};\\
\Gamma_3 &= \{\{(a, 1, \alpha), (a, 2, \alpha), (a, 3, \alpha), (b, 1, \alpha), (b, 2, \alpha), (b, 3, \alpha)\},\\
&\quad \{(a, 1, \beta), (a, 2, \beta), (a, 3, \beta), (b, 1, \beta), (b, 2, \beta), (b, 3, \beta)\}\}.
\end{aligned}$$

Notice: If $B_1 \in \Gamma_1$, $B_2 \in \Gamma_2$, and $B_3 \in \Gamma_3$, then $|B_1 \cap B_2 \cap B_3| = 1$.

Cartesian decompositions

Suppose that Ω is a set. A cartesian decomposition of Ω is a set $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ of partitions of Ω such that

$$|B_1 \cap \cdots \cap B_\ell| = 1 \quad \text{whenever} \quad B_1 \in \Gamma_1, \ldots, B_\ell \in \Gamma_\ell.$$

$\mathcal{E}$ is homogeneous of $|\Gamma_i| = |\Gamma_j|$ for all $i,\ j$.
$\mathcal{E}$ is non-trivial if $|\mathcal{E}| \geqslant 2$.
We will assume that cartesian decompositions are non-trivial.
The cartesian decomposition $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ defined before is the natural cartesian decomposition of $\Delta_1 \times \cdots \times \Delta_\ell$.

If Ω is a set with cartesian decomposition $\{\Gamma_1, \ldots, \Gamma_\ell\}$, then there is a bijection

$$\vartheta : \Omega \to \Gamma_1 \times \cdots \times \Gamma_\ell,\ \omega \mapsto (B_1, \ldots, B_\ell) \text{ where } \omega \in B_i \in \Gamma_i.$$

The invariant cartesian decomposition

Theorem

Let Γ be a set with $|\Gamma| \geqslant 2$ and let $\ell \geqslant 2$.

1. *The stabiliser in $\mathsf{Sym}\,(\Gamma^\ell)$ of the natural cartesian decomposition $\mathcal{E}$ of Γ^ℓ is $W = \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.*
2. *If $5 \leqslant |\Gamma| < \infty$ and $\ell \geqslant 2$, then $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is a maximal subgroup of $\mathsf{Sym}\,(\Gamma^\ell)$ or of $\mathsf{Alt}\,(\Gamma^\ell)$.*

Exercise: Give a necessary and sufficient condition for the containment $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell \leqslant \mathsf{Alt}\,(\Gamma^\ell)$.

Problem: Find maximal subgroups in infinite symmetric groups that correspond to cartesian decompositions. Covington, Macpherson & Mekler (1996) solve this when Ω is countable.

The component of G

Suppose that $G \leqslant \mathsf{Sym}\,\Omega$ is a permutation group and $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ is a G-invariant cartesian decomposition of Ω.

Decomposing G:

1. for $\Gamma_i \in \mathcal{E}$, the stabiliser G_{Γ_i} induces a permutation group $G^{\Gamma_i} = (G_{\Gamma_i})^{\Gamma_i}$ on Γ_i; G^{Γ_i} is the Γ_i-component of G;
2. G induces a subgroup $G^{\mathcal{E}} \leqslant \mathsf{S}_\ell$.

Example

Consider the cube with vertex set $V = \{a, b, c, d, e, f, g, h\}$. Let G be its automorphism group.
If

$$\begin{aligned}
\Gamma_1 &= \{\{a, b, c, d\}, \{e, f, g, h\}\}; \\
\Gamma_2 &= \{\{b, f, c, g\}, \{a, e, d, h\}\}; \\
\Gamma_2 &= \{\{a, b, e, f\}, \{d, c, h, g\}\},
\end{aligned}$$

then $\mathcal{E} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ is a G-invariant cartesian decomposition of V. It is easy to see that

1. $G^{\Gamma_i} = C_2$ and
2. $G^{\mathcal{E}} = \mathsf{S}_3$.

Wreath Embedding Theorem

Theorem

Let $G \leqslant \mathsf{Sym}\,\Omega$ and assume that $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ is a G-invariant cartesian decomposition of Ω.

(1) *If G is transitive on $\mathcal{E}$, then $G \lesssim G^{\Gamma_1} \wr G^{\mathcal{E}}$ acting in product action on Γ_1^ℓ.*

(2) *If G is transitive on Ω then G^{Γ_1} is transitive.*

Consequence: The automorphism group of the cube $\lesssim C_2 \wr \mathsf{S}_3$. In fact it is equal to $C_2 \wr \mathsf{S}_3$.

Proof of (1): see the imprimitive embedding theorem.
Proof of (2): Assume wlog that G is transitive on $\mathcal{E}$ and $G \leqslant W = G^{\Gamma_1} \wr \mathsf{S}_\ell$. If G^{Γ_1} is intransitive, then so is W and so is G.

Primitive and quasiprimitive groups

Let $G \leqslant \mathsf{Sym}\,\Omega$ be transitive.
G is said to be primitive if $\{\{\omega\} \mid \omega \in \Omega\}$ and $\{\Omega\}$ are the only G-invariant partitions of Ω.
G is said to be quasiprimitive if all non-trivial normal subgroups of G are transitive.

Lemma

(1) If G is primitive, then it is quasiprimitive.
(2) A finite permutation group is quasiprimitive iff all minimal normal subgroups are transitive.

Praeger (1993) proved an O'Nan–Scott Theorem for finite quasiprimitive permutation groups.

Primitive and quasiprimitive wreath products

Assume that $2 \leqslant |\Gamma| \leqslant \infty$ and $\ell \geqslant 2$. Let $G \leqslant \mathsf{Sym}\,\Gamma$ and $H \leqslant \mathsf{S}_\ell$ and set $\Omega = \Gamma^\ell$, $W = G \wr H$.

Theorem

The following are equivalent.

1. *W is (quasi)primitive on Ω;*
2. 2.1 *G is (quasi)primitive Γ;*
   2.2 *G is not cyclic of prime order;*
   2.3 *H is transitive on $\underline{\ell}$.*

Sketch of proof for quasiprimitivity

Suppose that W is quasiprimitive.

Claim 1.: $G = W^{\Gamma} = G$ is transitive.
By Theorem above, since W is transitive.

Claim 2.: H is transitive on $\underline{\ell}$.
Let $\{1, \ldots, s\}$ be an H-orbit with $s < \ell$ and set $X = G^s$. Then $X \leqslant (G^{\ell}) \rtimes H = W$ and X is an intransitive normal subgroup of W: a contradiction.

Claim 4.: G is not cyclic of prime order.
If yes, then set $X = \{(g, \ldots, g) \mid g \in G\} \leqslant (G^{\ell}) \rtimes H$. Then X is an intransitive normal subgroup of W: a contradiction.

Claim 3.: G is quasiprimitive on Γ.
If not, then let $1 < N < G$ be an intransitive normal subgroup. Then $X = N^{\ell} \leqslant G^{\ell}$ is an intransitive normal subgroup of W: a contradiction.

Sketch of proof for primitivity

Assume now that W is primitive. We only need to show that G is primitive on Γ. Let $\omega = (\gamma, \ldots, \gamma) \in \Omega$.

Claim 1.: $W_\omega = G_\gamma \wr H$ is a maximal subgroup of W.

Claim 2.: G_γ is a maximal subgroup of G.
Suppose not and let $G_\gamma < Y < G$. Set $X = Y \wr H = Y^\ell \rtimes H$. Then $W_\omega < Y < W$: a contradiction.

Corollary

Suppose that $G \leqslant \Omega$ such that G preserves a homogeneous cartesian decomposition $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ of Ω. If G is primitive on Ω, then G^{Γ_1} is primitive on Γ_1, it is not C_p, and $G^{\mathcal{E}}$ is transitive.

Proof.

By primitivity, $G^{\mathcal{E}}$ is transitive. By Embedding Theorem, $G \leqslant G^{\Gamma_1} \wr G^{\mathcal{E}}$. Now apply previous theorem. $\square$

Application I: Complete bipartite graph

Let $K_{m,m}$ be a complete bipartite graph as above with bipartition $\Delta_1 = \{1_1, \ldots, m_1\}$ and $\Delta_2 = \{1_2, \ldots, m_2\}$.
Set $\Delta = \{1, \ldots, m\}$.
Then the edge set of $K_{m,m}$ is

$$E = \{\{i_1, j_2\} \mid 1 \leqslant i,\ j \leqslant m\} \cong \Delta_1 \times \Delta_2 \equiv \Delta^2.$$

The group $\mathsf{Aut}(K_{m,m}) = \mathsf{S}_m \wr \mathsf{S}_2$ acts in product action on E.

Corollary

Let $G \leqslant \mathsf{Aut}(K_{m,m}) = \mathsf{S}_m \wr \mathsf{S}_2$ and let $G_0 = G \cap (\mathsf{S}_m)^2$.

1. *If G is transitive on E, then G_0 is transitive on Δ_1 and on Δ_2.*
2. *If G is primitive on E, G projects onto S_2 and G_0 is primitive on Δ_1 and on Δ_2.*

Application II: Hamming graphs

Let Γ be a set and $\Omega = \Gamma^\ell$.
The Hamming graph $H(\ell, \Gamma)$:

1. vertex set Γ^ℓ;
2. $(\gamma_1, \ldots, \gamma_\ell)$ and $(\delta_1, \ldots, \delta_\ell)$ are adjacent iff $|\{i \mid \gamma_i \neq \delta_i\}| = 1$.

The following follows from more general results of Sabidussi (1960) and Vizing (1963).

Theorem
$\mathsf{Aut}(H(\ell, \Gamma))$ *coincides with the stabiliser* $\mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ *of the natural cartesian decomposition of* Γ^ℓ.

Arc-transitive groups on Hamming graphs

Theorem

Let $G \leqslant \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ act arc-transitively on $\mathfrak{G} = H(\ell, \Gamma)$. Then

1. *G projects onto a transitive subgroup of S_ℓ;*
2. *G^Γ is a 2-transitive permutation group on Γ.*

Proof.

Let $\omega = (\gamma, \ldots, \gamma)$ be a vertex. If G is arc-transitive, then G is vertex transitive and G_ω is transitive on the neighbourhood

$$\mathfrak{G}(\omega) = \{(\gamma' \neq \gamma, \gamma, \ldots, \gamma)\} \cup \{(\gamma, \gamma' \neq \gamma, \gamma, \ldots, \gamma)\} \cup \cdots \cup \{(\gamma, \gamma, \ldots, \gamma, \gamma' \neq \gamma)\} = \Sigma_1 \cup \cdots \cup \Sigma_\ell.$$

Hence G_ω is transitive but imprimitive on $\mathfrak{G}(\omega) = \Sigma_1 \cup \cdots \cup \Sigma_\ell$. This forces

1. G_ω to project onto a transitive subgroup of S_ℓ;
2. $(G_\omega)^{\Sigma_1}$ to be transitive on $\Gamma \setminus \{\gamma\}$; i.e. G^Γ to be 2-transitive

Arc-transitive subgroups of finite Hamming graphs

Theorem (CFSG)

Let $G \leqslant \mathsf{Aut}(H(\ell,\Gamma))$ be arc-transitive as above. If G^Γ is a finite almost simple 2-transitive group with socle T, then either

(a) $T^\ell \leqslant G$*; or*

(b) $T \in \{\mathsf{A}_6, \mathsf{M}_{12}\}$*,* ℓ *is even, and* $T^{\ell/2} \leqslant G$*.*

Using the terminology of Kovács (1989), in case (a), G is a blow-up of G^Γ.

Basic or non-basic?

Definition (Cameron)

A (quasi)primitive permutation group on Ω is basic if it does not preserve a non-trivial cartesian decomposition of Ω.

The following are equivalent for $G \leqslant \mathsf{Sym}\,\Omega$:

1. G is non-basic;
2. G preserves a non-trivial cartesian decomposition of Ω;
3. $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ with some set Γ and $\ell \geqslant 2$.

Question 1: Given $G \leqslant \mathsf{Sym}\,\Omega$. How do we decide if G is basic?

Question 2: Once we know that G is non-basic, construct all embeddings $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.

Example

Let $T \trianglelefteq G \leqslant \mathsf{Aut}(T)$ be a primitive group of type AS on Γ and let $H \leqslant \mathsf{S}_\ell$ be transitive.

Then $W = G \wr H \leqslant \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is primitive and non-basic.

Question: How is this visible by looking at W?

We have

$$M = T^\ell = T_1 \times \cdots \times T_\ell$$

is a transitive minimal normal subgroup of W such that

1. $M_\omega = (T_1)_\omega \times \cdots \times (T_\ell)_\omega$;
2. the set $\{T_1, \ldots, T_\ell\}$ is invariant under conjugation.

Hence we may identify

$$\Gamma^\ell \equiv [M : M_\omega] = [T_1 : (T_1)_\omega] \times \cdots \times [T_\ell : (T_\ell)_\omega] \equiv [T_1 : (T_1)_\omega]^\ell.$$

For example, primitive groups of type PA and CD are not basic.

Normal cartesian decompositions and normal embeddings

Suppose that N is a transitive normal subgroup of $G \leqslant \mathsf{Sym}\,\Omega$ such that

1. $N = N_1 \times \cdots \times N_\ell$;
2. $N_\omega = (N_\omega \cap N_1) \times \cdots \times (N_\omega \cap N_\ell)$;
3. the set $\{N_1, \ldots, N_\ell\}$ is invariant under conjugation by G_ω.

Then

$$\Omega \equiv [N : N_\omega] \equiv [N_1 : (N_1 \cap N_\omega)] \times \cdots \times [N_\ell : (N_\ell \cap N_\omega)] = \Delta_1 \times \cdots \times \Delta_\ell.$$

The natural cartesian decomposition of $\Delta_1 \times \cdots \times \Delta_\ell$ is a G-invariant cartesian decomposition.
If $|\Delta_i| = |\Delta_j|$ then $G \lesssim \mathsf{Sym}\,\Delta_1 \wr \mathsf{S}_\ell$.

Normal cartesian decompositions and normal embeddings

Definition

A G-invariant cartesian decomposition that arises this way is said to be normal. The corresponding embedding $G \lesssim \mathsf{Sym}\,\Delta_1 \wr \mathsf{S}_\ell$ is a normal embedding.

Example: Suppose that $k \geqslant 2$ and $M = T_1 \times \cdots \times T_k$ is a non-abelian regular minimal normal subgroup of a group G with the T_i simple. Then conditions (1)–(3) are satisfied for M and so G is non-basic.

In particular primitive groups of type HC, TW are non-basic and they admit a normal embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.

Non-basic affine groups

Let $G = V \rtimes G_\omega$ be a primitive affine group such that $V = (\mathbb{F}_p)^d$ and $G_\omega \leqslant \mathsf{GL}_d(p)$ is an irreducible subgroup.
Suppose that $V = V_1 \oplus \cdots \oplus V_\ell$ is a G_ω-invariant decomposition of V with $\ell \geqslant 2$.
In this case we say that G_ω is an imprimitive linear group.
Then

1. $V = V_1 \times \cdots \times V_\ell$;
2. $1 = V_\omega = (V_1 \cap V_\omega) \times \cdots \times (V_\ell \cap V_\omega)$;
3. the set $\{V_1, \ldots, V_\ell\}$ is a G_ω-conjugacy class.

Hence G is non-basic.

Theorem

A finite primitive group G of type HA (affine) is non-basic iff G_ω is an irreducible, but imprimitive linear group. In this case the embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is normal.

Other inclusions

Let $T = \mathsf{A}_6$, $G = \mathsf{Aut}(T) = \mathsf{P\Gamma L}_2(9)$, and let $H \leqslant G$ be a maximal subgroup with index 36. Then G is primitive on $\Omega = [G : H]$. The following hold:

1. T has two non-conjugate subgroups $A,\ B \cong \mathsf{A}_5$;
2. $\{A, B\}$ is H-invariant;
3. $AB = T$;
4. $A \cap B = H \cap T = T_\omega$.

These properties imply that there is a bijection

$$\Omega \equiv [T : A \cap B] \to [T : A] \times [T : B], \quad (A \cap B)t \mapsto (At, Bt).$$

Hence

$$G \leqslant \mathsf{S}_6 \wr \mathsf{S}_2$$

and G is a non-basic primitive group of type AS.

Let's generelise

Suppose that $G \leqslant \mathsf{Sym}\,\Omega$ has a transitive normal subgroup T such that

1. there are $A,\ B \leqslant T$ such that $|T : A| = |T : B|$;
2. $AB = T$;
3. $A \cap B = T_\omega$;
4. the set $\{A, B\}$ is invariant under conjugation by G_ω.

Then

$$\Omega \equiv [T : A \cap B] \equiv [T : A] \times [T : B] \equiv [T : A]^2$$

and

$$G \leqslant \mathsf{Sym}\,\Delta \wr \mathsf{S}_2 \quad \text{where} \quad \Delta = [T : A].$$

Such a G is non-basic.

Apply this to almost simple groups

Let's find non-basic finite primitive groups with AS type that admit an embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_2$.

We need to understand factorisations $T = AB$ of finite simple groups with $|A| = |B|$.

Lemma (CFSG)

Suppose that T is a finite simple group and $A,\ B < T$ such that $AB = T$ and $|A| = |B|$. Then one of the following is valid:

1. $T \cong \mathsf{A}_6$ *and* $A,\ B \cong \mathsf{A}_5$;
2. $T \cong \mathsf{M}_{12}$ *and* $A,\ B \cong \mathsf{M}_{11}$;
3. $T \cong \mathsf{Sp}(4, 2^a)$ *with* $a \geqslant 2$ *and* $A,\ B \cong \mathsf{Sp}(2, 2^{2a}) \cdot 2$.
4. $T \cong \mathsf{P}\Omega_8^+(q)$ *and* $A,\ B \cong \mathsf{P}\Omega_7^+(q)$.

Corollary

Let G be a finite non-basic primitive group of type AS with socle T such that $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_2$. Then $T \in \{\mathsf{A}_6, \mathsf{M}_{12}, \mathsf{Sp}(4, 2^a)\}$.

Proof.

Step 1: $T \leqslant (\mathsf{Sym}\,\Gamma)^2$.

Step 2. Let $\gamma \in \Gamma$ and $\omega = (\gamma, \gamma) \in \Gamma^2$. Set

$$\Delta_1 = \{(\gamma, \gamma') \mid \gamma' \in \Gamma\} \quad \text{and} \quad \Delta_2 = \{(\gamma', \gamma) \mid \gamma' \in \Gamma\}.$$

Then $\Delta_1,\ \Delta_2 \subseteq \Omega$ are blocks of imprimitivity for T.

Step 3. Let $A = T_{\Delta_1}$ and $B = T_{\Delta_2}$ be the block stabilisers. Then A and B satisfy the conditions of the factorisation lemma.
Step 4. Prove that $T \neq \mathsf{P}\Omega_8^+(q)$. $\square$

Let's generalise further

Question: How to detect if $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ for $\ell > 2$?

Suppose that $M \leqslant \mathsf{Sym}\,\Omega$ is a transitive group and $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is a set of proper subgroups of M such that

1. $K_1 \cap \cdots \cap K_\ell = M_\omega$;
2. $K_i(\bigcap_{j \neq i} K_j) = M$ for all i;

Then $\mathcal{K}$ is said to be a cartesian factorisation of M.

Theorem

Let $G \leqslant \mathsf{Sym}\,\Omega$, let M be a transitive minimal normal subgroup of G, and let $\omega \in \Omega$. The following are equivalent:

1. *G is non-basic, and so $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ with $\ell \geqslant 2$;*
2. *M admits a G_ω-invariant cartesian factorisation with ℓ subgroups such that $|M : K_i| = |M : K_j|$ for all i, j.*

Proof

$\Leftarrow$: If $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ is a G_ω-invariant cartesian factorisation, then

$$\Omega \equiv [M : M_\omega] \equiv [M : K_1] \times \cdots \times [M : K_\ell] \equiv [M : K_1]^\ell.$$

Hence we obtain an embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ with $\Gamma = [M : K_1]$.

$\Rightarrow$: Suppose that $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$.
Step 1. $M \leqslant (\mathsf{Sym}\,\Gamma)^\ell$.
If not, then $M \lesssim \mathsf{S}_\ell$ and so $|M| \mid \ell!$. As M is transitive on Γ^ℓ, $\Gamma^\ell \mid |M|$.
Therefore $|\Gamma|^\ell \mid \ell!$: a contradiction (easy fact from number theory).

Step 2. Fix $\gamma \in \Gamma$, set $\omega = (\gamma, \ldots, \gamma) \in \Gamma^\ell$ and define

$$\Delta_1 = \{(\gamma, \gamma_2, \ldots, \gamma_\ell) \mid \gamma_i \in \Gamma\} \quad \ldots \quad \Delta_\ell = \{(\gamma_1, \ldots, \gamma_{\ell-1}, \gamma) \mid \gamma_i \in \Gamma\}.$$

The Δ_i are blocks of imprimitivity for M. Now let $K_i = M_{\Delta_i}$.

Step 3. Then $\mathcal{K} = \{K_i\}_i$ is a G_ω-invariant cartesian factorisation of M.

NON-BASIC FINITE PRIMITIVE GROUPS I

Let's see how finite primitive groups can be non-basic:

HA: Non-basic if and only if G_ω is an imprimitive irreducible subgroup of $\mathsf{GL}_d(p)$. Further, every embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is normal.

HS: Always basic.

HC: Always non-basic and every embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is normal.

SD: Always basic.

Non-basic finite primitive groups II

CD: Always non-basic and every embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is normal.

TW: Always non-basic and every embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ is normal.

PA: Always non-basic. Assume that G admits a non-normal embedding $G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_\ell$ and let T be the simple direct factor of $\mathsf{Soc}\,G$. Then $T \in \{\mathsf{A}_6, \mathsf{M}_{12}, \mathsf{Sp}_4(2^a)\}$.

AS: If non-basic, then $\mathsf{Soc}\,G \in \{\mathsf{A}_6, \mathsf{M}_{12}, \mathsf{Sp}_4(2^a)\}$ and admits a non-normal embedding
$G \lesssim \mathsf{Sym}\,\Gamma \wr \mathsf{S}_2$.

This was proved by Cheryl Praeger (1990).

Primitive permutation groups with a non-abelian regular socle

Question: Is there such a thing?

The original theorem by Scott claims:

We can use the above results to get a very useful picture of the general primitive finite permutation group. Recall that such a group has a socle which is either elementary abelian (an irreducible representation for the point stabilizer) or a direct product of isomorphic nonabelian simple groups.[9] Any nontrivial normal subgroup of a primitive group of course acts transitively.

THEOREM. *Let H be a subgroup of a finite direct product $G = \prod_{i \in I} G_i$ of isomorphic nonabelian simple groups. Then the transitive permutation representation of G on G/H extends to a primitive permutation representation of some group in which G is the socle if and only if either*

(a) *there is a partition $\mathcal{P}$ of I into subsets of equal prime cardinality with H the direct product $\prod_{S \in \mathcal{P}} \Delta_S$ of full diagonal subgroups Δ_S of the subproducts $\prod_{i \in S} G_i$, or*

(b) *the subgroup H is a direct product $\prod_{i \in I} H_i$ where H_i is a subgroup of G_i which is an intersection $G_i \cap \tilde{H}_i$ for some maximal subgroup $\tilde{H}_i$ of a group $\tilde{G}_i$ with $G_i \subseteq \tilde{G}_i \subseteq \operatorname{Aut} G_i$. Also, for each pair i, j of indices there must be an isomorphism of G_i with G_j carrying H_i to H_j.*

Primitive permutation groups with non-abelian regular socle

Let's look at the proof:

The details of the proof are fairly straightforward from the previous results, and we leave them to the reader. We now aim at describing the possible maximal subgroups of the symmetric and alternating groups.

If N is a regular non-abelian minimal normal subgroup of a finite primitive group G then

$$G = N \rtimes G_\omega = (T_1 \times \cdots \times T_k) \rtimes G_\omega$$

where the T_i are non-abelian finite simple groups.
Further, G_ω is a maximal subgroup of G.
Pablo showed that G_ω is non-solvable. Combining this with Schreier's Conjecture, we obtain that $k > 1$.

Recognising twisted wreath products

Lemma (Bercov 1967, Lafuente 1987)

Let G be a group such that

1. $N = N_1 \times \cdots \times N_k \trianglelefteq G$;
2. $G = N \rtimes H$;
3. $\{N_1, \ldots, N_k\}$ *is a G-conjugacy class.*

Then $G \cong N_1 \operatorname{twr} H$ (twisted wreath product).

Hence our hypothetical group G would be a twisted wreath product.

Twisted wreath products were introduced by B. H. Neumann (1963).

The observation that these permutation groups are twisted wreath products was made by Gross & Kovács (1984).

The twisted wreath product

Input:

1. a group T;
2. a group P;
3. a subgroup $Q < P$ with $|P : Q| < \infty$;
4. and a homomorphism $\varphi : Q \to \mathsf{Aut}(T)$.

Consider $\mathsf{Func}(P, T) \cong T^{|P|}$ and set

$$B = \{f \in \mathsf{Func}(P, T) \mid (pq)f = (pf)(q\varphi) \text{ for all } p \in P,\ q \in Q\}.$$

It is easy to see that $B \leqslant \mathsf{Func}(P, T)$.

The base group

Let $\mathcal{T}$ be a left transversal of Q in P, such that $1 \in \mathcal{T}$. Then

$$P = \{rq \mid r \in \mathcal{T} \text{ and } q \in Q\} \quad \text{(unique representation).}$$

For each $c \in \mathcal{T}$ and $t \in T$, define $f_{c,t} \in \mathsf{Func}(P, T)$ by

$$f_{c,t} : rq \mapsto \begin{cases} 1 & \text{if } r \neq c \text{ (that is, } rq \notin cQ) \\ t(q\varphi) & \text{if } r = c \text{ (that is, } rq \in cQ) \end{cases} \tag{2}$$

and set

$$T_c = \{f_{c,t} \mid t \in T\}.$$

Lemma

1. *$T_c \cong T$ under the isom $t \mapsto f_{c,t}$;*
2. *$B = \prod_{c \in \mathcal{T}} T_c$, and so $B \cong T^{|P:Q|}$.*

THE P-ACTION ON B

For $p \in P$, define $\hat{p} : B \to B$ by

$$x(f\hat{p}) = (px)f \quad \text{for all} \quad x \in P \text{ and } f \in B. \tag{3}$$

Lemma

The map $\vartheta : p \mapsto \hat{p}$ is a homomorphism $\vartheta : P \to \mathsf{Aut}(B)$ with $\ker \vartheta = \mathsf{Core}_P(\ker \varphi)$.

The group

$$T \operatorname{twr} P = T \operatorname{twr}_\varphi P = B \rtimes_\vartheta P$$

is the twisted wreath product of T by P.

Terminology:

1. B: base group;
2. P: top group;
3. Q: twisting subgroup;
4. φ: twisting homomorphism.

Twisted wreath products as permutation groups

Let $W = T \operatorname{twr} P = B \rtimes P$ as above. We define an action of W on B (the base group action):

1. if $f \in B$, then f acts on B by right multiplication:

$$f : g \mapsto gf;$$

2. if $p \in P$, then p acts on B via the automorphism $\hat{p}$:

$$p : g \mapsto g\hat{p}.$$

Lemma

1. *The rule above gives a W-action on B with kernel $\mathsf{Core}_P(\ker\varphi)$.*
2. *B is a regular normal subgroup of W.*

Primitive twisted wreath products

Lemma

Assume that $\mathsf{Core}_P(\ker\varphi) = 1$ *and set* $W = T\operatorname{twr}_\varphi P$*. The group* W *is primitive iff no proper, non-trivial subgroup of* B *is normalised by* P*.*

Proof.

Step 1. Let $1 \in \Delta \subseteq B$ be a block of imprimitivity for W.

Step 2. B is regular $\Rightarrow \Delta \leqslant B$.

Step 3. P acts via automorphisms: Δ is P-invariant.

Hence a $1 \in \Delta \subseteq B$ is a W-block of imprimitivity iff Δ is a P-invariant subgroup of B. $\square$

A criterion for primitivity

Theorem (Aschbacher–Scott 1985, Kovács 1986)

Suppose that T, P, Q, and φ are as above and let $W = T \operatorname{twr}_{\varphi} P$.

1. *If W is primitive in its base group action then*
   1.1 *no proper and non-trivial subgroup of T is invariant under $Q\varphi$; and*
   1.2 *φ cannot be extended to a strictly larger subgroup of P.*
2. *If T is a non-abelian simple group (not necessarily finite), then (1.1) and (1.2) imply that W is primitive.*

Proof

W is primitive $\Rightarrow$ 1.1:
Assume that $1 < L < T$ is invariant under $Q\varphi$. Then $1 < L^{|P:Q|} < T^{|P:Q|} = B$ is normalised by P: W is not primitive by previous lemma.

W is primitive $\Rightarrow$ 1.2:
Assume that $Q < \overline{Q} \leqslant P$ is such that $\varphi : Q \to \mathsf{Aut}(T)$ can be extended to $\overline{\varphi} : \overline{Q} \to \mathsf{Aut}(T)$.
Set

$$\overline{B} = \{f \in \mathsf{Func}(P, T) \mid (pq)f = (pf)(q\overline{\varphi}) \text{ for all } p \in P,\ q \in \overline{Q}\}.$$

Then $\overline{B} < B$ and $P < \overline{B} \rtimes P < B \rtimes P$. Hence P is not a maximal subgroup of W and W is not primitive.

Criterion for primitivity

Lemma

Let T be a non-abelian finite simple group and $X \leqslant \mathsf{Aut}(T)$. Then no proper non-trivial subgroup of T is invariant under X iff $\mathsf{Inn}(T) \leqslant X$.

Corollary

Suppose that T is a non-abelian finite simple group. Then $W = T \operatorname{twr}_{\varphi} P$ is primitive in its base group action if and only if both

1. $\mathsf{Inn}(T) \leqslant Q\varphi$; *and*
2. φ *cannot be extended to a strictly larger subgroup of* P.

Example

Smallest example, $T = \mathsf{A}_5$, $P = \mathsf{A}_6$, $Q = \mathsf{A}_5$, and $\varphi : Q \to \mathsf{Aut}(T)$ is conjugation.
Then

$$W = T \operatorname{twr} P = T^6 \rtimes P$$

is primitive on T^6.
Then $|W| = 60^6 \cdot 360 = 16796160000000$ and
$|\Omega| = 60^6 = 46656000000$.

Theorem of Förster–Kovács

The following was proved in an Australian National University Research Report (1989).

Theorem

Let T, P, Q, and φ be as in the input such that $\mathsf{Core}_P(\ker\varphi) = 1$ *and W is primitive. Then P contains a unique minimal normal subgroup that is non-abelian. In particular P cannot contain a solvable normal subgroup.*

Corollary

Let T, P, Q, and φ be as in the input such that $\mathsf{Core}_P(\ker\varphi) = 1$ *and W is primitive. Then the P-action on the simple direct factors of $B = T^{|P:Q|}$ is faithful.*