Automata, languages, and groups of automorphisms of rooted trees

Part I - Introduction to automata and languages

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Outline of the course

Contents

The course consists of the following (not equally divided) three parts:

- Part I - Introduction to Automata and Languages
- Part II - Groups and Automata: a perfect match
- Part III - Groups of automorphisms of rooted trees

Plan for today

1. Outline of the course
2. Why?
3. Basic notions in automata theory
4. Automata
5. Finite state automata
6. Regular Languages
7. The pumping lemma
8. Plan for tomorrow

Why?

Why Automata Theory and Formal Languages?

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- Auotmata Theory ranked high.
- Automata Theory is relevant in many areas of Mathematics and also in Computer science.

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- The theoretical models that were proposed in order to understand solvable and unsolvable problems led to the development of real computers.

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Automata Theory deals with definitions and properties of different types of "computation models". Examples of such models are:

- Finite Automata: used in text processing, compilers, and hardware design.
- Context-Free Grammars: used to define programming languages and in Artificial Intelligence.
- Turing Machines: form a simple abstract model of a "real" computer, such as your PC at home.

And now ...why groups of automorphisms of rooted trees? What is the relation?

No spoiler, we will see later. Let's say that in these lectures we will see some connection between Automata Theory and Group Theory.

Basic notions in automata theory

- Alphabets
- Strings
- Languages

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Example

- $A = \{0, 1\}$ is the binary alphabet; 0100 is a string of A of length 4.
- $A = \{a, b, \dots, z\}$ is the set of all lower-case letters.

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Note that $A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$.

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Example

Let $u = 000$, and $v = 111$. Then $uv = 000111$.

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Remark

Recall that all alphabets are finite. Languages may have an infinite number of strings, but these strings consist of strings drawn from one finite fixed alphabet.

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Example

The problem of testing whether an integer is a prime, can be expressed by the language L consisting of all binary strings whose value as a binary number is a prime.

Automata

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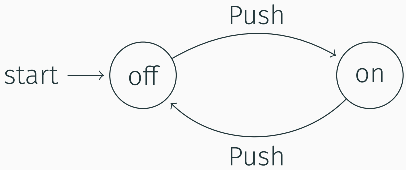
From Wikipedia

An automaton is a relatively self-operating machine, or a machine or control mechanism designed to automatically follow a predetermined sequence of operations, or to respond to predetermined instructions.

An easy example



(Still) Informal definition of automata

- An automaton $\mathcal{A}$ over an alphabet A is a device (machine) that reads input strings over A and accepts some of them (that is, given a string w, $\mathcal{A}$ accepts w by halting in an accepting state).

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- The language of $\mathcal{A}$, denoted by $L(\mathcal{A})$ is the set of all strings that $\mathcal{A}$ accepts.

Types of automata

Types of automata II

- finite state automaton ✓
  - deterministic
  - nondeterministic
- pushdown automaton ✓
- linear-bounded automaton
- Turing machine

Finite state automata

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- Transition diagrams
- Transition tables

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- Arc: labeled with input symbol from A (or letter from the alphabet A), they show the transition.

Let $\mathcal{A} = (Q, A, \delta, q_0, F)$ be a DFA.

- The language accepted by $\mathcal{A}$, that we denote with $L(\mathcal{A})$, is the set of labels of the paths in the transition diagram of $\mathcal{A}$ that start at the initial state q_0 and end at a final state in F.

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- The entry for the row corresponding to the state q and the column corresponding to the input a is the state $\delta(q, a)$.

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- Is this a good way to do it?

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- The transition table for δ is:

	0	1
q_E	q_E	q_O
q_O	q_O	q_E

Example

We have

- $\delta(q_E, 0) = q_E$
- $\delta(q_E, 1) = q_O$
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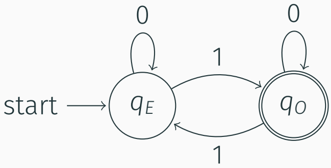

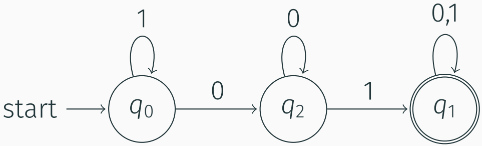

Can you guess what is this?

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$$\bar{\delta}(q, w) = \delta(\bar{\delta}(q, x), a).$$

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- In other words, a string w is accepted if $\bar{\delta}(q_0, w) \in F$.
- The language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is defined to be the set of all strings that are accepted by $\mathcal{A}$:

$$L(\mathcal{A}) = \{w \mid w \text{ is a string over } A \text{ and } \mathcal{A} \text{ accepts } w\}.$$

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- The state q_0 is the initial and the final state.

Summarizing:

$$\mathcal{A} = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \delta, q_0, \{q_0\}).$$

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q_1	q_3	q_0
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q_3	q_1	q_2

...continuing II

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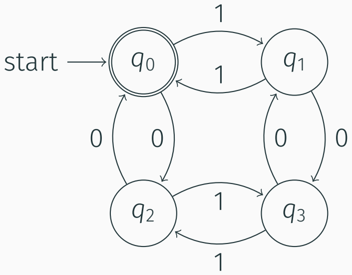

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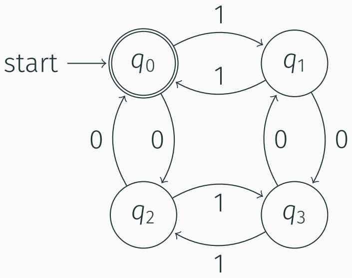
In other words, a transition function is a path in the transition diagram. Take again the string $w = 1010$.




Take a break

Let's start slowly and play a game

Let's play a game on a 3x3 chessboard.

1	2	3
4	5	6
7	8	9

- Goal: start at 1 and go to 9.
- Rules: move to an adjacent square.

- States: squares of the chessboard, that is $Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $A = \{b, p\}$, where
  - b = move to any adjacent blue square
  - p = move to any adjacent pink square
- Initial state: $q_0 = 1$
- Final state: $q_F = 9$

If there are choices where to go, we try all.

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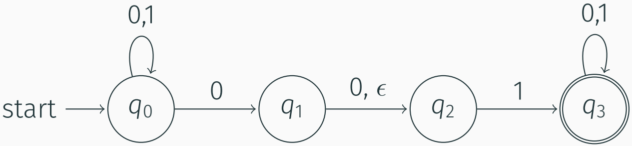
- A start state $q_0 \in Q$.
- A set of accepting states F from Q.

We will use the following notation:

$$\mathcal{A} = (Q, A, \delta, q_0, F).$$

Example



What happens if we take the string $w = 000110$?

Example II

Non-Deterministic Finite State Automata (NDFA)

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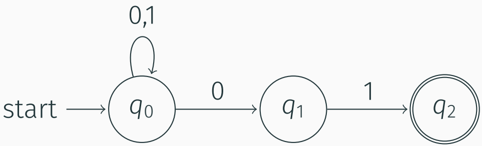
Similarly as before, the language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is defined to be the set of all strings that are accepted by $\mathcal{A}$.



- Can you guess what are the strings recognized by this automaton?
- Why is this NDFA and not DFA?

DFA vs NDFA

DFA	NDFA
The transition from a state is to a single particular next state for each input symbol.	The transition from a state can be to multiple next states for each input symbol.
Empty string transitions are not allowed.	NDFA permits empty string transitions.
Requires more space.	Requires less space.
A string is accepted by a DFA, if it transits to a final state.	A string is accepted by a NDFA, if at least one of all possible transitions.

Exercise for you

Formalise what we have said with the example of the 3x3 chessboard.

Question: It seems that NDFA are more powerful than DFA. Is this true?

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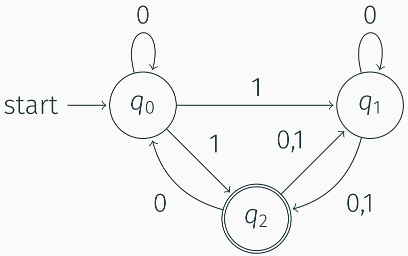
- Of course it is easy to convert a DFA to a NDFA.
- What about the converse?
- We will forget about the ϵ-transition.

From NDFA to DFA



From NDFA to DFA II

Transition table:

	0	1
q_0	q_1	q_1, q_2
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q_2	q_0, q_1	q_1

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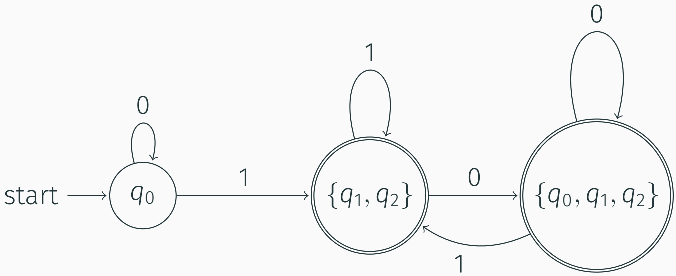
- The maximum number of states is at most $2^{|Q|}$
- In the resulting DFA, all those states that contain the final state(s) of NDFA are treated as final states

Let's now build this DFA

And the final result should be this ...



Finite state automata may have outputs corresponding to each transition. There are two types of finite state machines that generate output:

- Mealy Machine
- Moore machine

We will see this later (where *later* means tomorrow) ...

Regular Languages

Definition

A language K is called regular if there exists a finite state automaton $\mathcal{A}$ such that

$$K = L(\mathcal{A}).$$

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Regular operations

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- The *star* of L_1 is

$$L_1^* = \{u_1 u_2 \ldots u_k \mid k \geq 0 \text{ and } u_i \in L_1 \text{ for all } i = 1, 2, \ldots, k\}.$$

Let $L_1 = \{\text{empty}, \text{full}\}$ and $L_2 = \{\text{cup}, \text{bottle}\}$.

- What is $L_1 \cup L_2$?
- What is $L_1 L_2$?
- What is L_1^*?

Is the set of all regular languages closed under these operations?

Theorem

The set of regular languages is closed under the union operation, i.e., if L_1 and L_2 are regular languages over the same alphabet A, then $L_1 \cup L_2$ is also a regular language.

The proof works as follows:

Proof.

□

Is the set of all regular languages closed under the other operations seen before (concatenation and star)?

Theorem

Yes.

Proof.

Exercise ;) □

The pumping lemma

Therapist: The Pumping Lemma is not real, it can not hurt you

The Pumping Lemma:

Let L be a regular language. $\Rightarrow \exists$ constant $n \in \mathbb{N}$:

$$\underset{\substack{w \in L \\ |w| \geq n}}{\forall} \exists\, x, y, z \in \Sigma^* \text{ such that } w = xyz, y \neq \epsilon, |xy| \leq n, \underset{k \geq 0}{\forall} xy^k z \in L$$

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- Regular languages can be described by finite state automata.
- All these tools help to prove that a language is regular.
- What if one wants to prove that a language is *not* regular?
- Let $L = \{0^m 1^m \mid m \geq 0\}$. Can you establish if this language is regular?

The *pumping lemma* is a property that all regular languages must possess.

The pumping lemma II

Informal statement:

Theorem

If a language is regular, all sufficiently long string in the language can be pumped.

Formal statement:

Theorem

Let L be a regular language. Then there exists an integer $p \geq 1$ (called the pumping length) such that the following holds: Every string s in L, with $|s| \geq p$, can be written as $s = xyz$, such that

- $|y| \geq 1$
- $|xy| \leq p$
- *for all* $i \geq 0$, $xy^iz \in L$.

This means that by replacing the portion y in s by zero or more copies of it, the resulting string is still in the language L.

Any regular language L has a magic number p
and any long-enough word in L has the following property:
among its first p symbols is a segment you can find
whose repetition or omission leaves x among its kind.
So if you find a language L which fails this acid test,
and some long word you pump becomes distinct from all the rest,
by contradiction you have shown that language L is not
a regular guy, resilient to the damage you have wrought.
But if, upon the other hand, x stays within its L,
then either L is regular, or else you chose not well.
For w is xyz, and y cannot be null,
and y must come before p symbols have been read in full.

Consider the language $L = \{0^m 1^m \mid m \geq 0\}$.
Claim: this language is not regular.

Let $L = \{0^n \mid n \text{ is a prime number}\}$.
Is L a regular language?

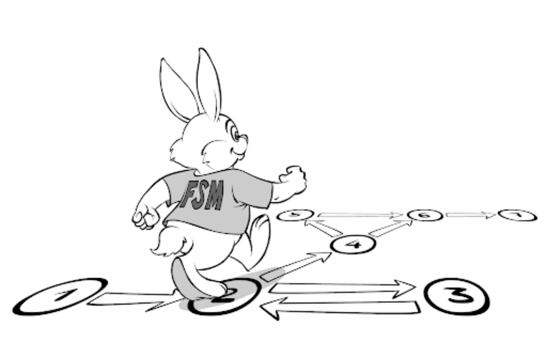
Plan for tomorrow

Plan for tomorrow

- Part I - Introduction to Automata and Languages:
  - Introduction to grammars
  - Context-free grammars
  - Pushdown automata
- Part II - Groups and Automata: what is this match?

Questions or answers?

Obrigada :)



(Picture of me trying to go to Brasilia with a finite state machine)