Automata, languages, and groups of automorphisms of rooted trees

Part III - Groups of automorphisms of rooted trees

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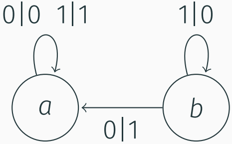
Previously, on Automata, languages, and groups of automorphisms of rooted trees

Previous lecture

- Groups and automata
- Automata groups
- Mealy machine
- How to generate automata groups

Example



How to generate automata groups

- We get a map ρ_q from A^* to A^* per state $q \in Q$.
- These maps have the same domain and codomain, hence we can compose them as we want.
- Note that the composition corresponds in the automaton to plugging in the output of a run to the input of another run.
- Given two states $p, q \in Q$, we define $\rho_{pq} = \rho_p \circ \rho_q$.
- The structure generated by $\{\rho_q \mid q \in Q\}$ is a semigroup.
- If the map ρ_q is bijective for all $q \in Q$, i.e. every state induces a function that can be inverted, then we obtain an automata group.

The Grigorchuk and the Gupta-Sidki automata are examples of *self-similar* group.

Self-similar groups are a subclass of the class of groups of automorphisms of rooted trees.

To Said Sidki, in honor of his 80th birthday.

Let's finally start with Part III

- Milnor's Problem $\Longrightarrow$ growth of a group.
- General Burnside Problem $\Longrightarrow$ finiteness properties of a group.

Let $G = \langle X \mid R \rangle$ be a presentation of a group G.

For each $g \in G$, let $|g|$ denote the smallest length of a word $w \in F(X)$ such that $w =_G g$.

The growth function of G is the map $\gamma : \mathbb{N} \to \mathbb{N}$:

$$\gamma(n) = |\{g \in G \mid |g| = n\}|.$$

It depends on the chosen presentation $\langle X \mid R \rangle$.

Some facts about growth of groups

- Free groups of finite rank $k > 1$: exponential growth.

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Milnor's question (1960):

Are there groups of intermediate growth between polynomial and exponential?

Grigorchuk's answer (1980):

Yes, the first . . . Grigorchuk group.

About the General Burnside Problem

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In modern terminology the general Burnside problem asks:

can a finitely generated periodic group be finite?

Recall that a group G is periodic if for any $g \in G$ there exists a positive integer n such that $g^n = 1$.

Are finitely generated periodic groups finite?

- Yes, for nilpotent groups.
- Yes, for finitely generated periodic subgroups of the general linear group of degree $n > 1$ over the complex numbers.
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- Yes, for nilpotent groups.
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- Yes, . . . for many other classes of groups.
- Counterexample: the first Grigorchuk group, the Gupta-Sidki p-groups.

To summarize

It seems that:

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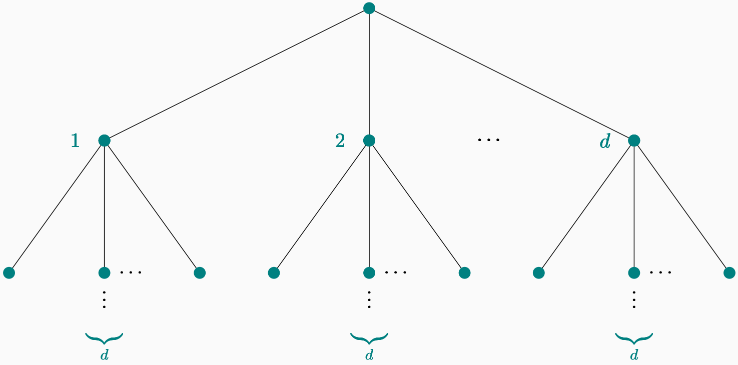
$$\text{Milnor's question} \bigcap \text{General Burnside Problem}$$

$$= \text{the first Grigorchuk group}, \ldots$$

Automorphisms of regular rooted trees

Regular rooted trees

Seriously: the regular rooted tree $\mathcal{T}_d$



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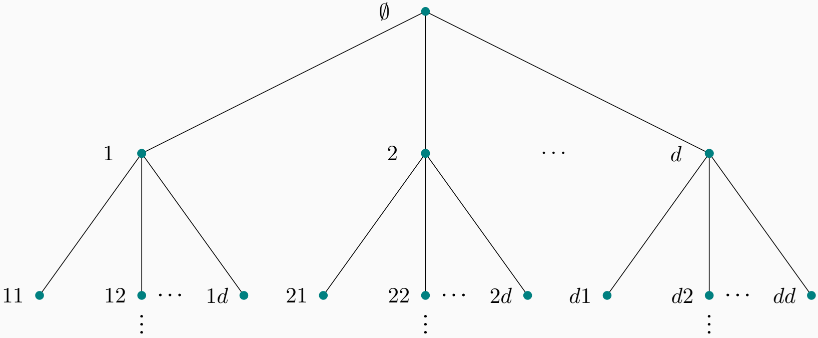

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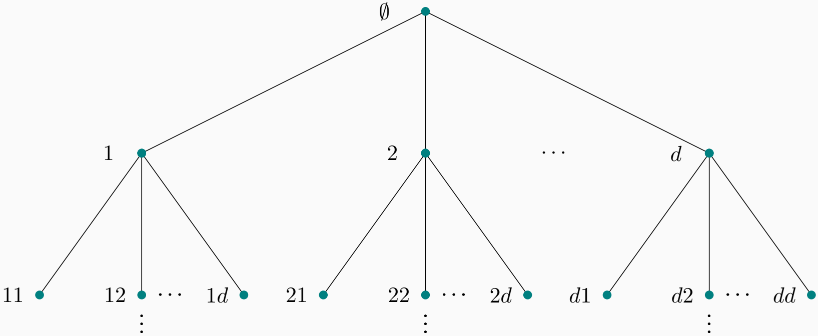
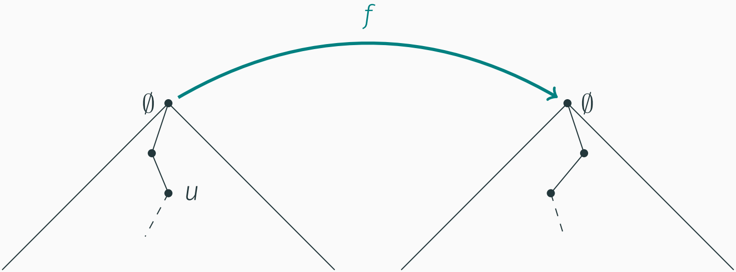


- X^n denotes the nth level of the tree, and X^* denotes all the vertices of the tree.

Automorphisms of $\mathcal{T}_d$

Bijections of the vertices that preserve incidence.



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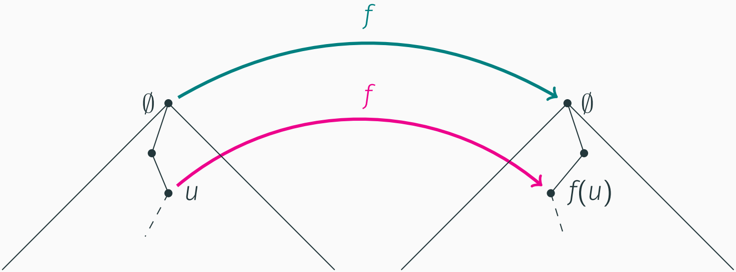

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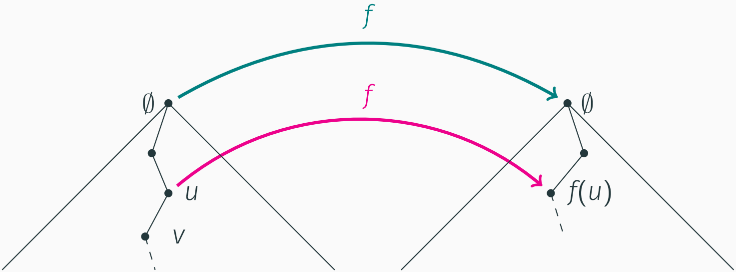

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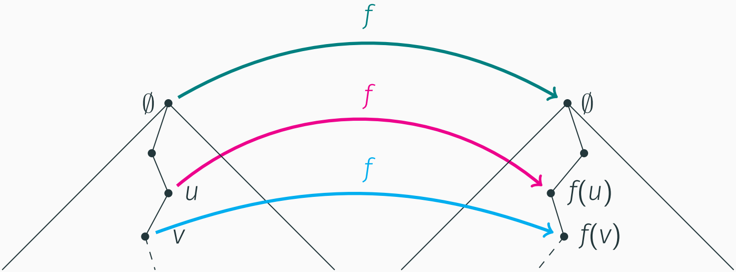

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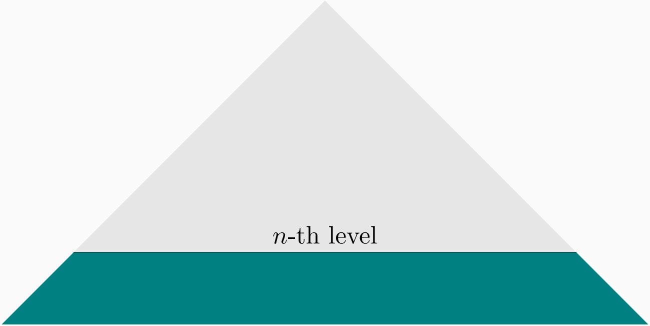
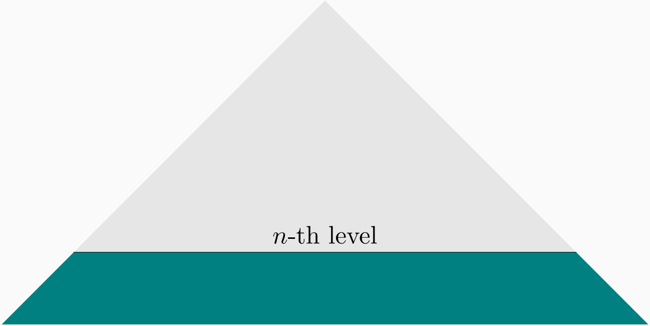
Sometimes we write $\mathcal{T}$ for $\mathcal{T}_d$, and, consequently, Aut $\mathcal{T}$ for Aut $\mathcal{T}_d$.



- The nth level stabilizer $\mathrm{st}(n)$ fixes all vertices up to level n.

A subgroup of $\operatorname{Aut}\mathcal{T}$: the stabilizer



- The nth level stabilizer $\mathsf{st}(n)$ fixes all vertices up to level n.
- If $H \leq \operatorname{Aut}\mathcal{T}$, we define $\mathsf{st}_H(n) = H \cap \mathsf{st}(n)$.

The stabilizer

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$$\operatorname{Aut}\mathcal{T} \supseteq \operatorname{st}(1) \supseteq \operatorname{st}(2) \supseteq \cdots \supseteq \operatorname{st}(n) \supseteq \dots$$

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  where $\bigcap_{n\in\mathbb{N}} \mathrm{st}(n) = 1$.
- Hence $\operatorname{Aut}\mathcal{T}$ is a residually finite group (i.e. a group in which the intersection of all its normal subgroups of finite index is trivial).

Describing elements of $\operatorname{Aut}\mathcal{T}$

An automorphism $f \in \operatorname{Aut}\mathcal{T}_d$ can be represented by writing in each vertex v a permutation $\sigma_v \in \operatorname{Sym}(d)$ which represents the action of f on the descendants of v.

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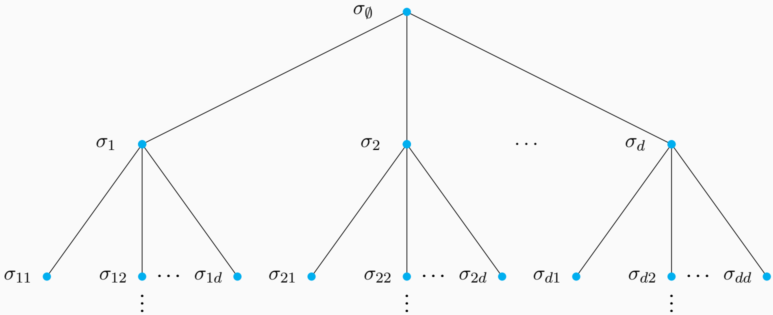

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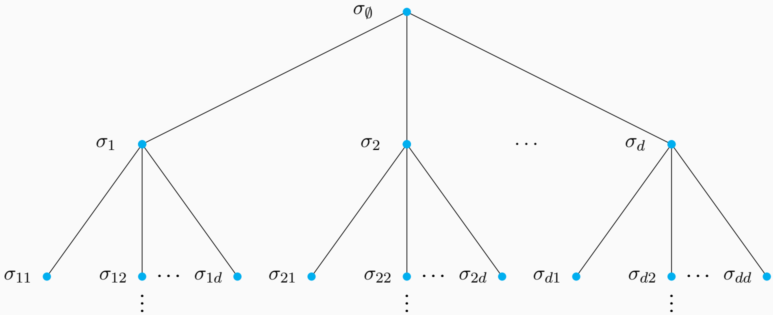

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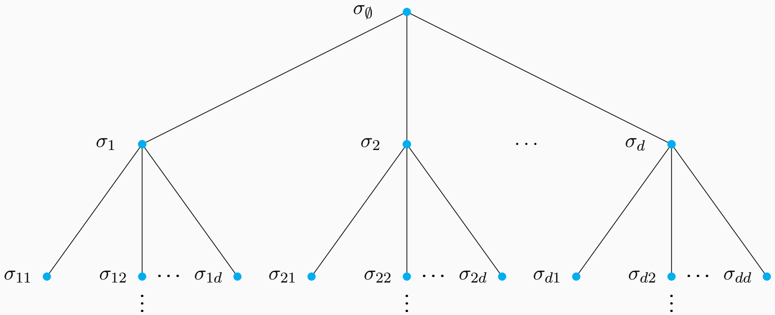
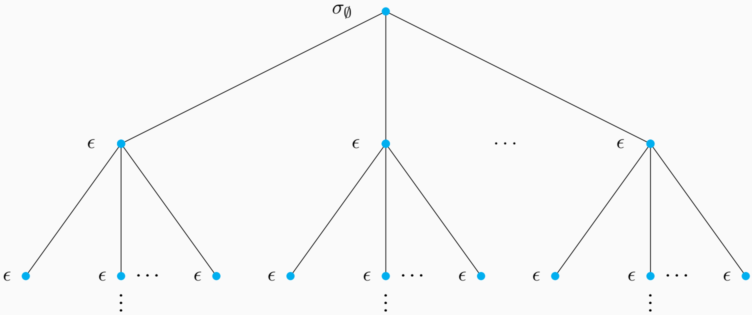


We say that $\sigma_v \in \mathsf{Sym}(d)$ is the *label* of f at the vertex v. The set of all labels is the *portrait* of f.

Describing elements of Aut $\mathcal{T}$

The simplest type are rooted automorphisms: given $\sigma \in \mathsf{Sym}(d)$, they simply permute the d subtrees hanging from the root according to σ.



We denote with ϵ the identity element of $\mathsf{Sym}(d)$.

Example of a rooted automorphism

Let $\mathcal{T}_3$ be the ternary tree, and a the rooted automorphism corresponding to the cycle $\sigma = (1\ 2\ 3)$.

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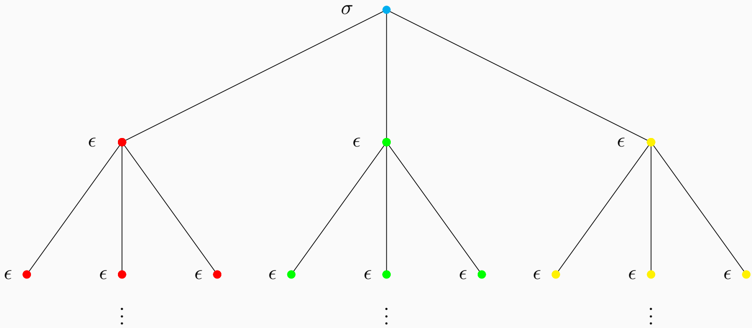

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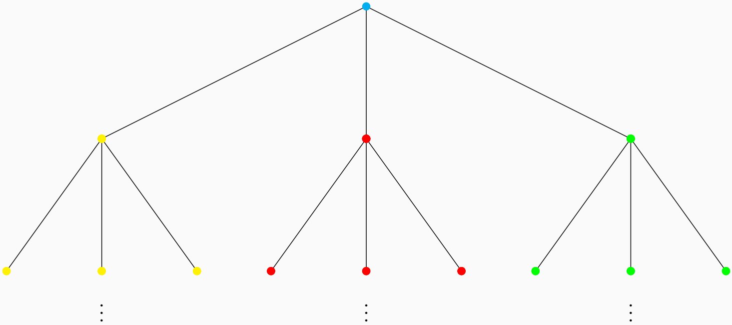

Note: sometimes we will identify a with σ.

Some facts about $\operatorname{Aut} \mathcal{T}$: I

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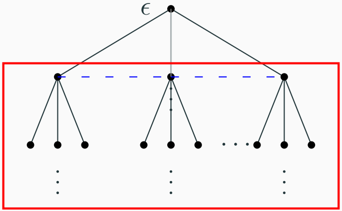

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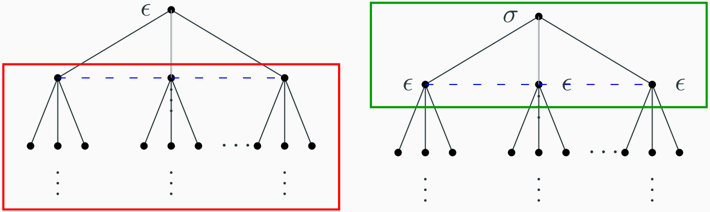

We define the isomorphism

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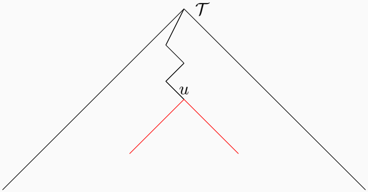

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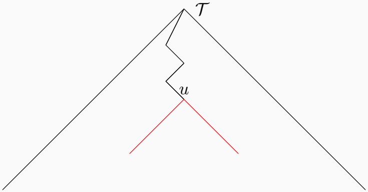

Digression: this implies that Aut $\mathcal{T}$ contains products Aut $\mathcal{T} \times \cdots \times$ Aut $\mathcal{T}$.

- Any $g \in \mathrm{Aut}\,\mathcal{T}_d$ can be seen as

$$g = h\sigma, \quad \sigma \in \mathrm{Sym}(d), \quad h \in \mathrm{st}(1) \cong \mathrm{Aut}\,\mathcal{T}_d \times \overset{d}{\ldots} \times \mathrm{Aut}\,\mathcal{T}_d$$

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In other words, every $f \in \mathrm{Aut}\,\mathcal{T}_d$ can be written as

$$f = (f_1, \ldots, f_d)a,$$

where $f_i \in \mathrm{Aut}\,\mathcal{T}_d$ and a is rooted corresponding to some permutation $\sigma \in \mathrm{Sym}(d)$.

Let $f \in \operatorname{Aut} \mathcal{T}_d$ with $f = (f_1, f_2, \ldots, f_d)a$, where $f_i \in \operatorname{Aut} \mathcal{T}_d$ and a is rooted corresponding to σ. If $f_1 = f_2 = \cdots = f_d = 1$, then f is rooted.

Example

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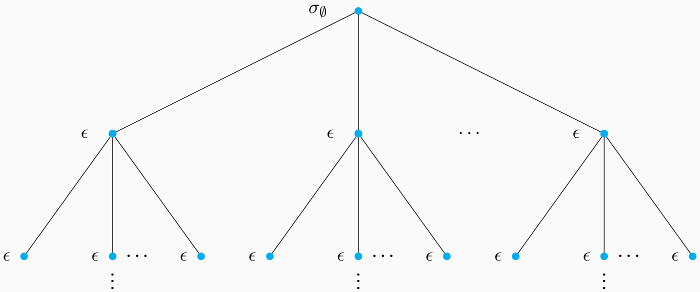

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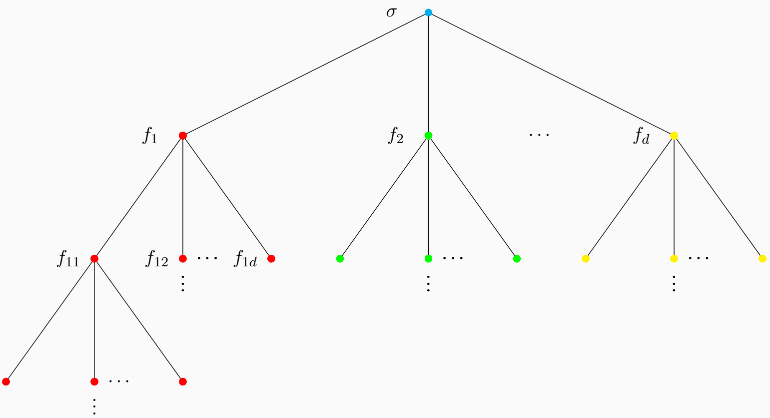

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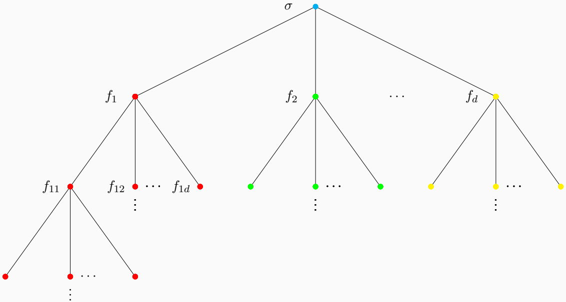

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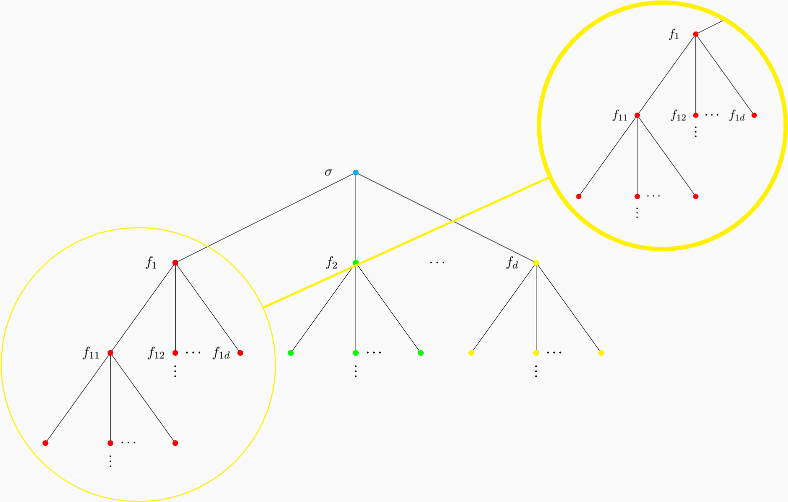

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If $\mathcal{T}_2$ is the binary tree and a is rooted corresponding to $(1\ 2)$, let

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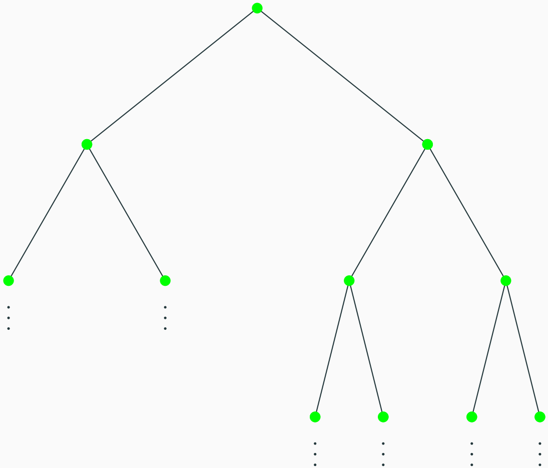

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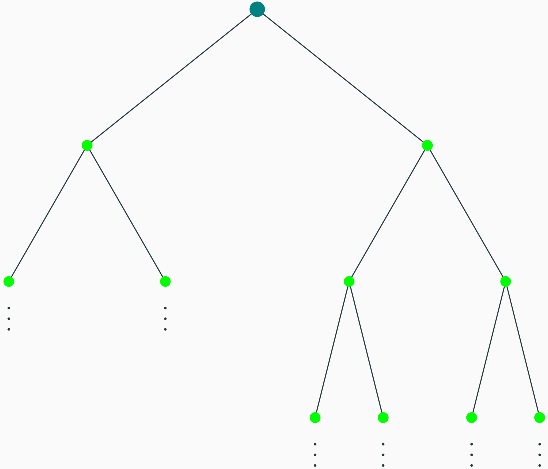

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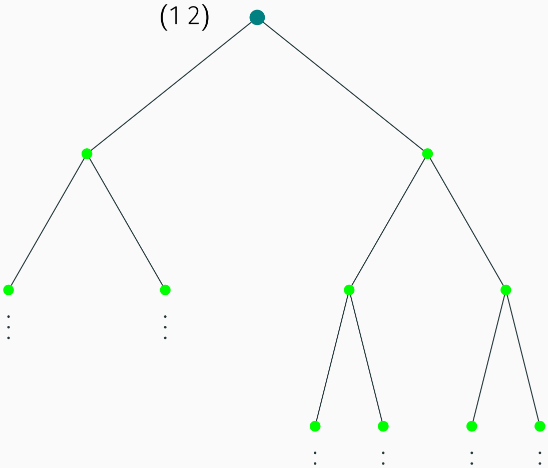

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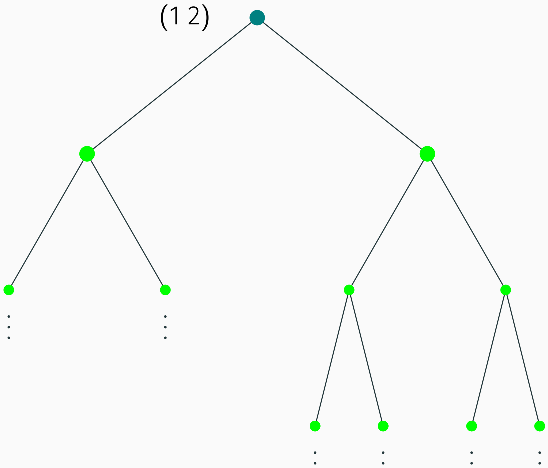

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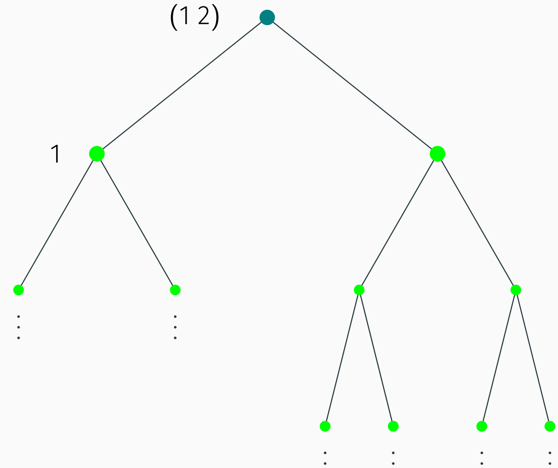

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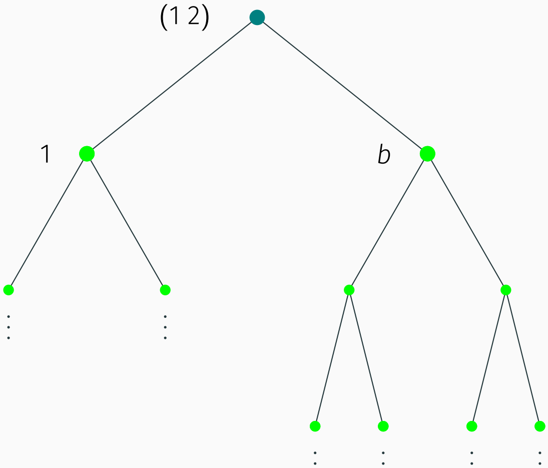

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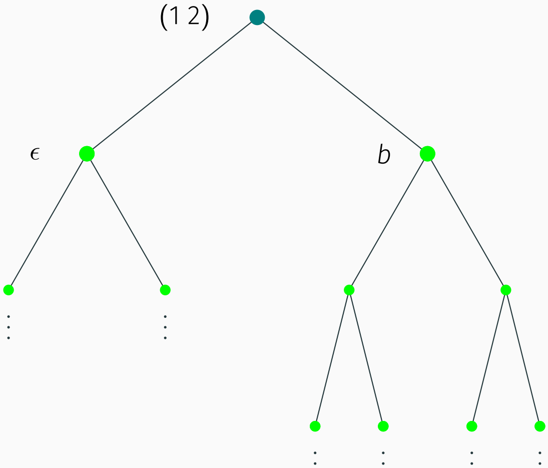

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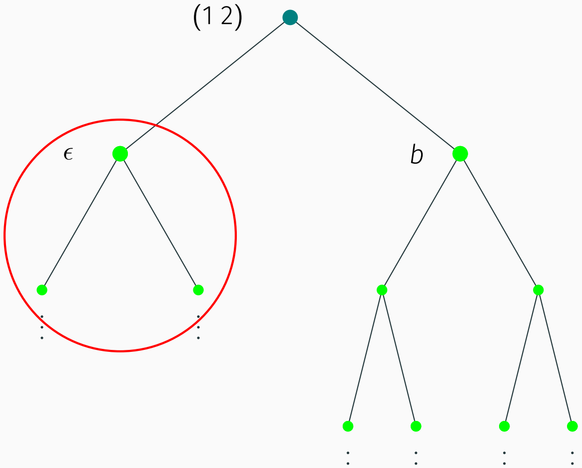

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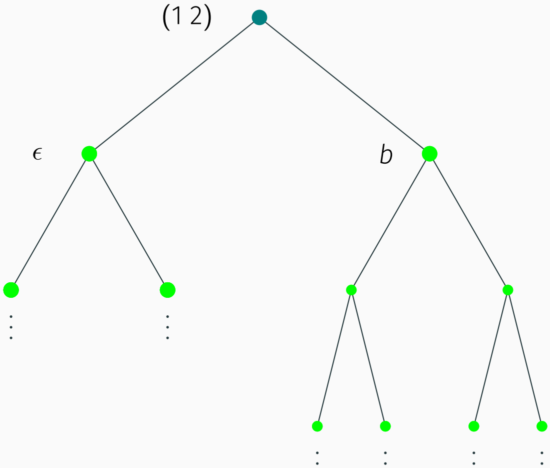

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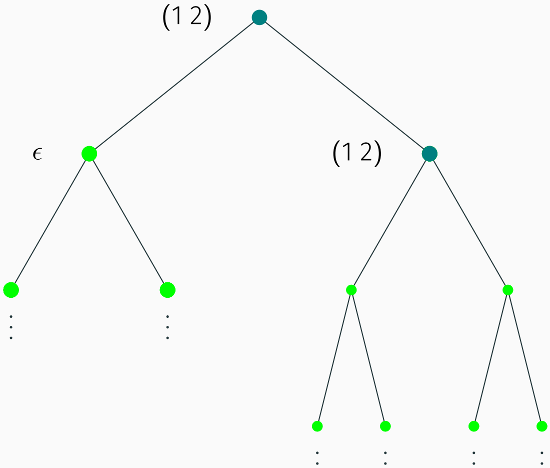

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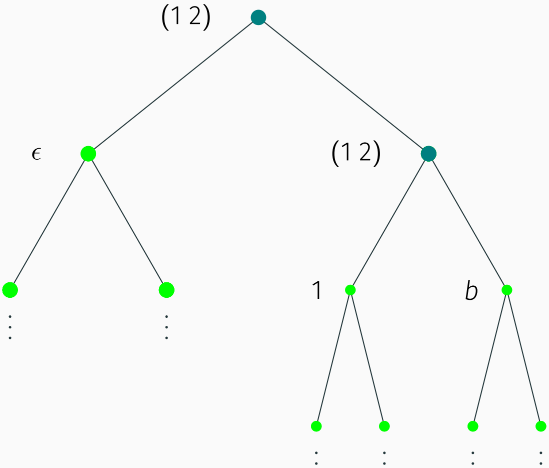

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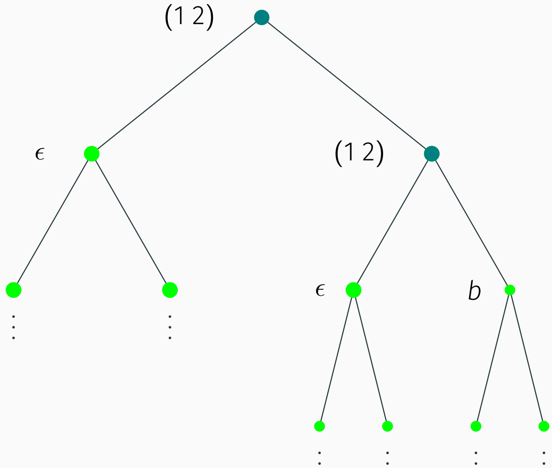

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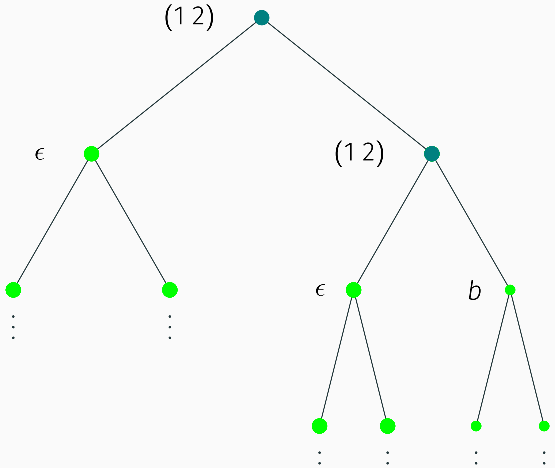

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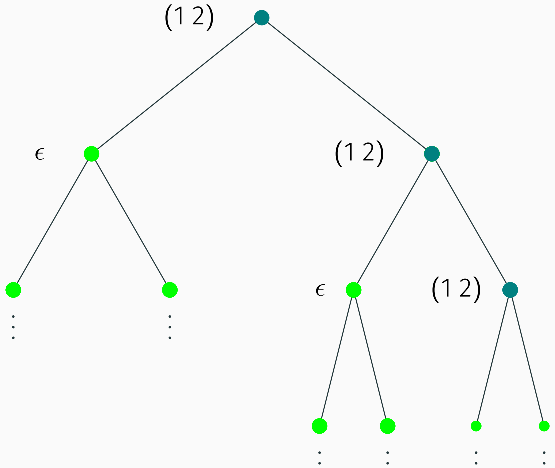

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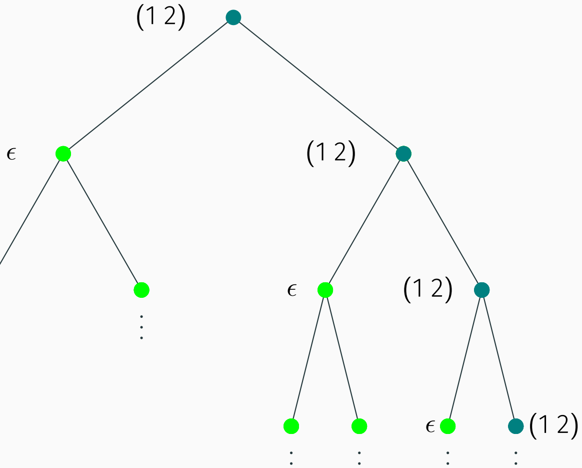

Exercise

If $\mathcal{T}_7$ is the 7-adic tree and a is rooted corresponding to $(1\,2\,3\,4\,5\,6\,7)$, let

$$b = (a, a^{-1}, a^2, 1, 1, 1, b)a.$$

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Self-similar groups

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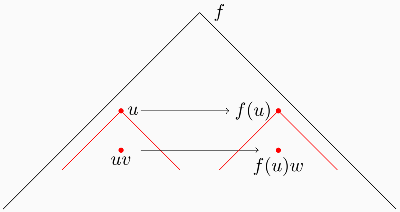

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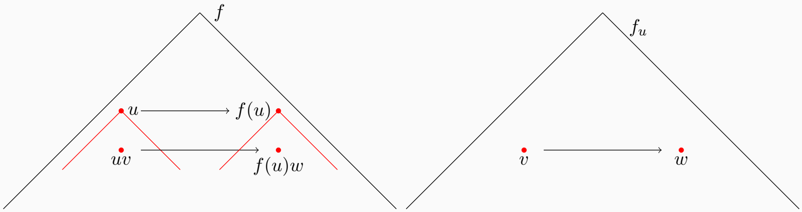

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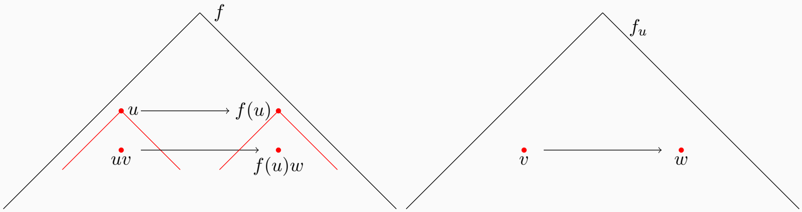

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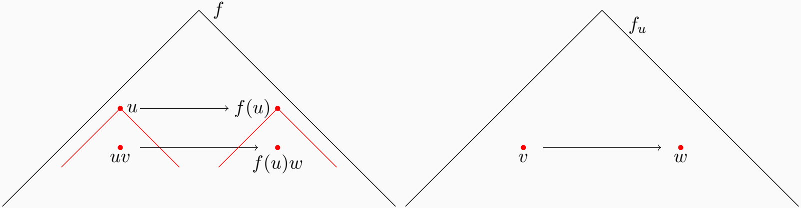

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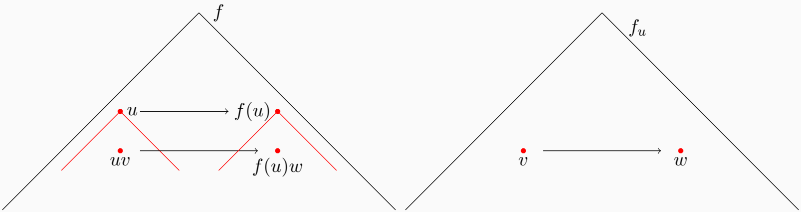

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- $a(21322) = a(2)a_2(1322) = 31322$
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Let $G \leq \operatorname{Aut} \mathcal{T}$.

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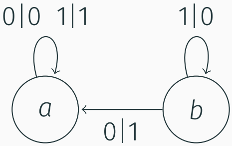
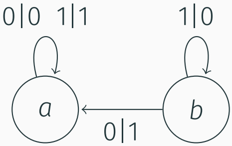
- An automata group is a finitely generated self-similar and finite-state group (i.e. $\{g_u \mid u \in X^*\}$ is finite).



Suppose we have $w = 01$. Denote blue: output, purple: states.

Automata groups vs self-similar groups



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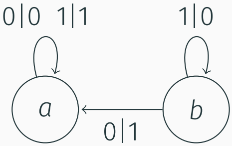

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The set of states of the automaton corresponds to a generating set of the self-similar group.


Take a break

Branch groups

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$$\mathrm{st}(n) \simeq \operatorname{Aut}\mathcal{T} \times \overset{d^n}{\cdots} \times \operatorname{Aut}\mathcal{T},$$

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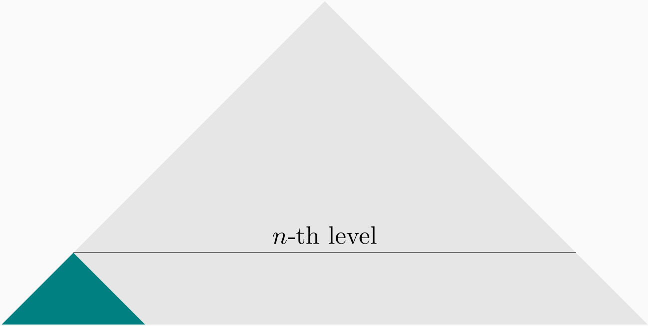
$$\psi_n : \mathrm{st}_G(n) \longrightarrow \psi_n(\mathrm{st}_G(n)),$$

  where $\psi_n(\mathrm{st}_G(n))$ need not be a direct product.
- The question is: given $G \leq \mathrm{Aut}\,\mathcal{T}$, can we find for every $n \in \mathbb{N}$ a subgroup (eventually of finite index) of $\mathrm{st}_G(n)$ which is a direct product?

The *rigid stabilizer* of the vertex u is

$$\mathrm{rst}_G(u) = \{g \in G : g \text{ fixes all vertices outside } \mathcal{T}_u\}$$



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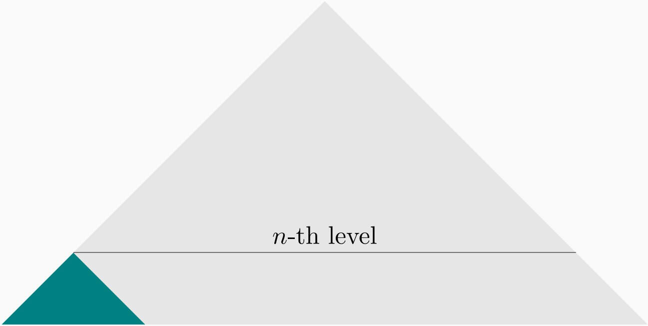

The *rigid stabilizer* of the nth level is $\mathrm{rst}_G(n) = \prod_{u \in X^n} \mathrm{rst}_G(u)$.

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- Good news: in some cases, there exist "nice" rigid stabilizers.
- Informally speaking: the subgroup $\psi_n(\mathrm{rst}_G(n))$ is the largest subgroup of $\psi_n(\mathrm{st}_G(n))$ which is a "geometric" direct product.

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- The first Grigorchuk group is a branch group.

Regular branch groups

Let G be a self-similar group. We say that G is a *regular branch* if there exists a subgroup K of $\mathrm{st}_G(1)$ of finite index such that

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More precisely we have this situation:

$$\begin{array}{ccc}
G & & G \times \overset{d}{\ldots} \times G \\
\Big| & & \Big| \\
\mathrm{st}_G(1) & \xrightarrow{\psi} & \psi(\mathrm{st}_G(1)) \\
\Big| & & \Big| \\
K & \xrightarrow{\psi} & \psi(K) \\
\Big| & & \Big| \\
L & \xrightarrow{\psi} & K \times \overset{d}{\ldots} \times K
\end{array}$$

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- Regular branch $\longrightarrow$ branch.

Examples of (weakly) branch groups

Now we will present the following groups of automorphisms of rooted trees together with their main properties:

- The Grigorchuk groups
- The GGS-groups
- The Basilica group
- The Hanoi Tower group

The first Grigorchuk group (finally!)

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (1, 1)(12) \quad b = (a, c) \quad c = (a, d) \quad d = (1, b)$$

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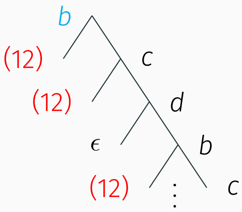

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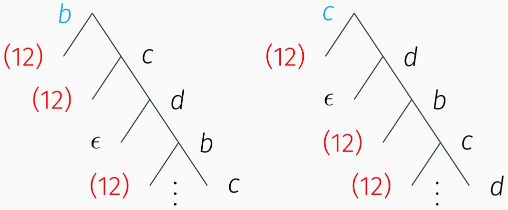

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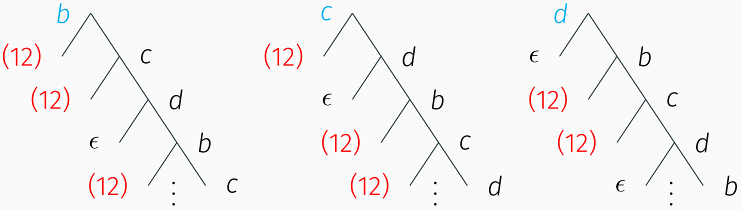
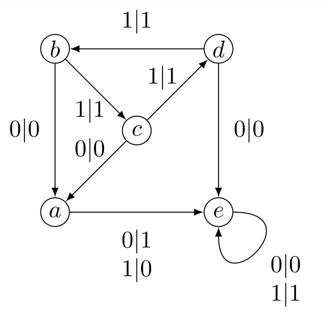


Can you guess what is the Grigorchuk automaton?

The Grigorchuk automaton



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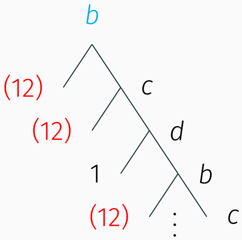

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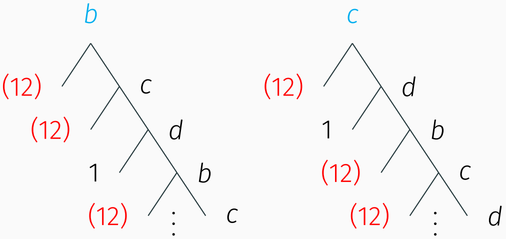

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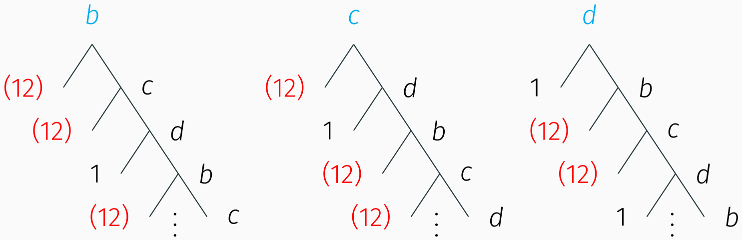

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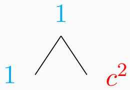

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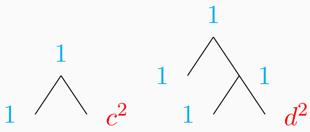

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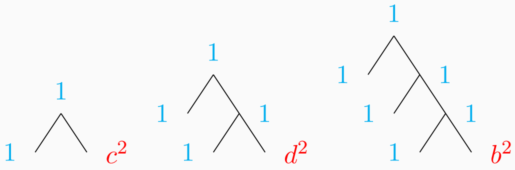

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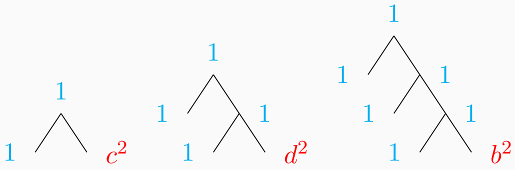

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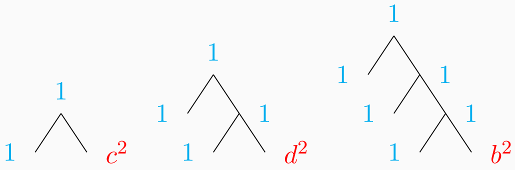

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- As a consequence, $c^2 = d^2 = 1$.

Summarizing some properties of Γ

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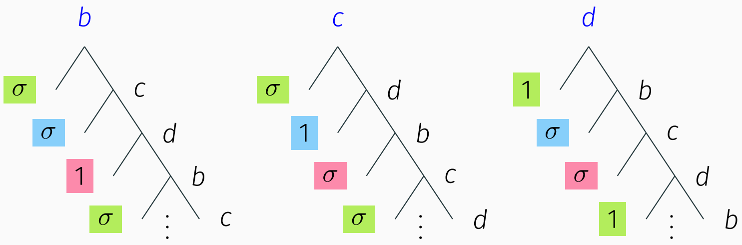
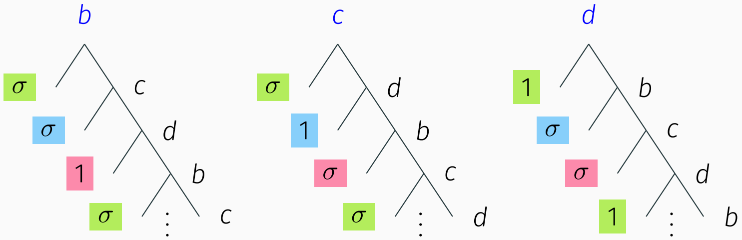
- Many other exotic properties ....



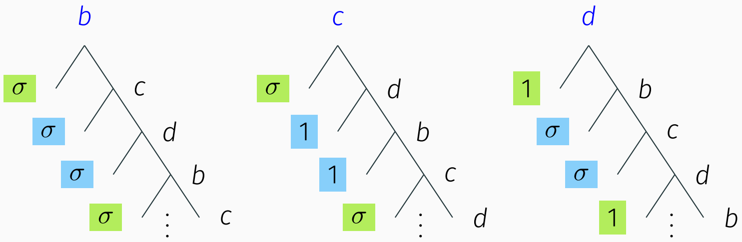
where $\sigma = (1\,2)$.



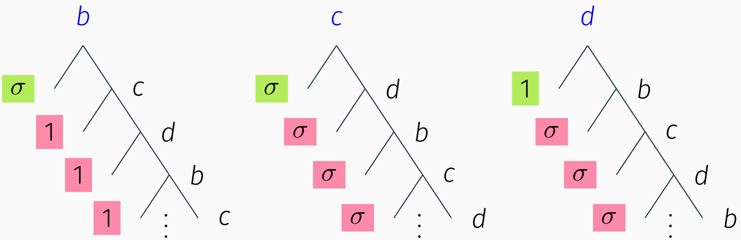
where $\sigma = (1\ 2)$.
Let **0, 1, 2** be the three non-trivial homomorphisms from $C_2 \times C_2 = \{1, b, c, d\}$ to $C_2 = \{1, \sigma\}$ such that:

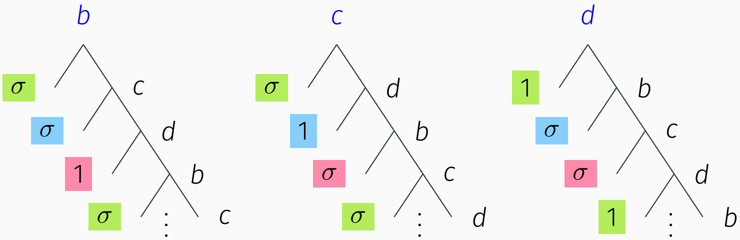
0 : $b \mapsto \sigma$	**1** : $b \mapsto \sigma$	**2** : $b \mapsto 1$
$c \mapsto \sigma$	$c \mapsto 1$	$c \mapsto \sigma$
$d \mapsto 1$	$d \mapsto \sigma$	$d \mapsto \sigma.$

Example 1: 0, 1, 1, 0...



Example 2: 0, 2, 2, 2...




b
σ
c
σ
d
1
b
σ
c
c
σ
d
1
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1
b
σ
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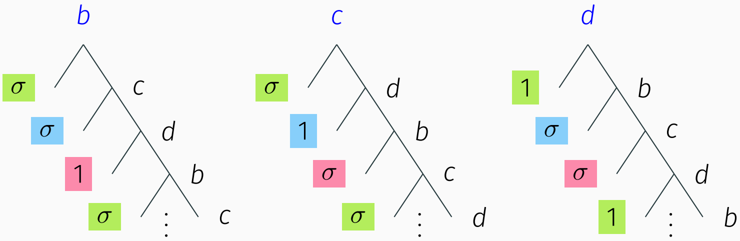
d
1
b

Grigorchuk groups



- Let $\Omega = \{0, 1, 2\}^\infty$ be the space of infinite sequences over letters $\{0, 1, 2\}$.

Given $\omega \in \Omega$ the Grigorchuk group is $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$.

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- Otherwise, G_ω is of intermediate growth.
- The group G_ω is periodic if and only if ω contains all three letters 0, 1, 2 infinitely often.

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A GGS-group is torsion if and only if $\sum_{i=1}^{p-1} e_i \equiv 0 \mod p$.

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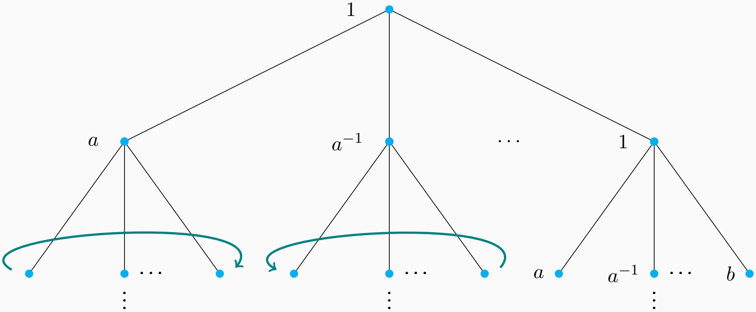

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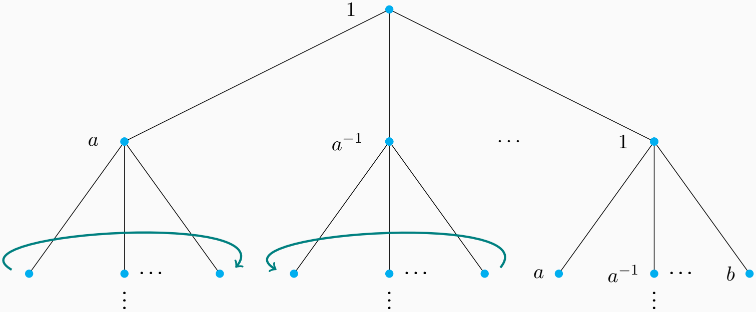

- Can you prove that this group is infinite and generated by elements of order p? And that is a p-group?.

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- In both cases G is not a counterexample to the GBP.
- Then if you want a group generated by elements of order 2, you must add generators: $\Gamma = \langle a, b, c, d \rangle$.

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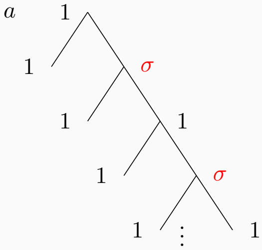

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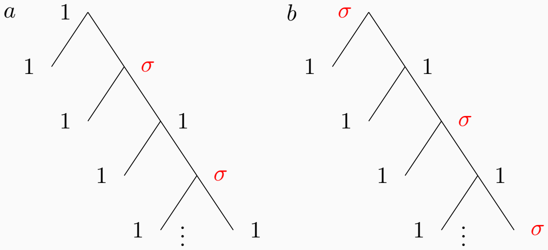

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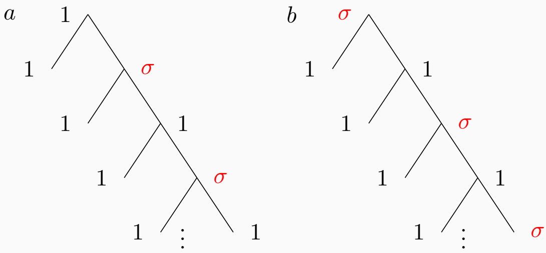

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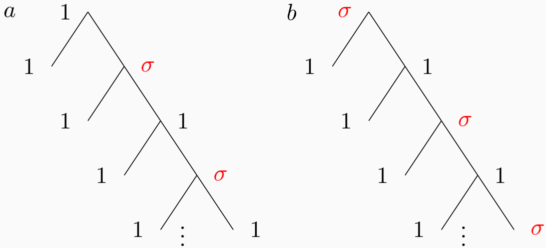

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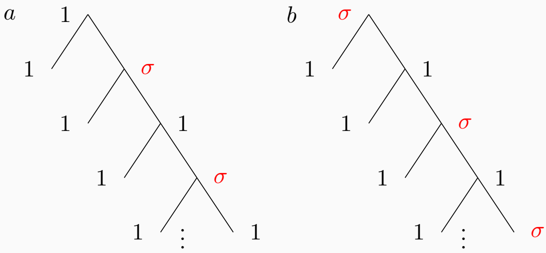

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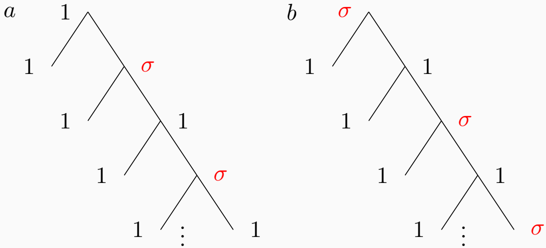

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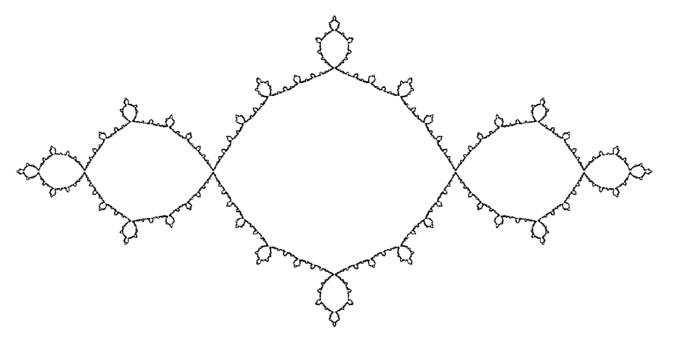

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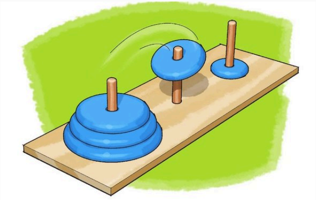
Some properties of the Basilica group

- It is torsion-free (Can you prove that a and b have infinite order?)
- It is weakly regular branch over its derived subgroup B'.
- It has exponential word growth.

The Hanoi Tower group

The tower of Hanoi was invented by a French mathematician Édouard Lucas in the 19th century.



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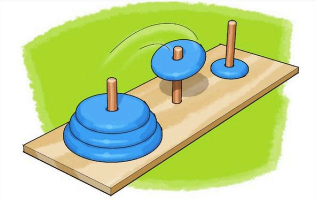

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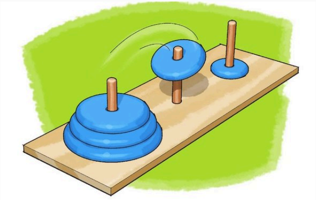

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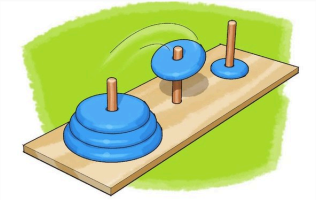

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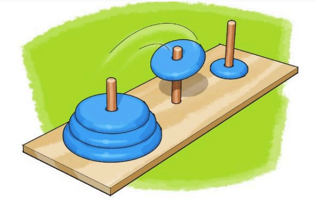

- The goal: to move the entire stack to another peg.
- The rules:
  - One disk can be moved at a time;
  - Each move consists of taking the upper disk from one of the stacks and placing it on top of another or on an empty peg;
  - No disk may be placed on top of a smaller disk.

- Let 3 be the number of pegs, then consider $X = \{1, 2, 3\}$. A word in X is a configuration of the disks and the length of the word is the number of disks.

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- Example:

$$23112.$$

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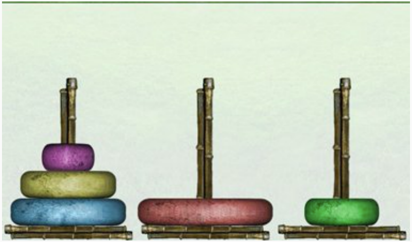
- Example:

$$231112.$$

- The length of the word above is 6 $\longrightarrow$ 6 disks.
- This means that the smaller disk is in the 2nd position, the second smaller disk is in the 3rd position, the third smaller disk is in the 1st position, and so on.

- Other example: can you guess how to write the configuration below?



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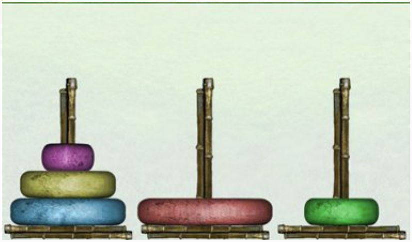

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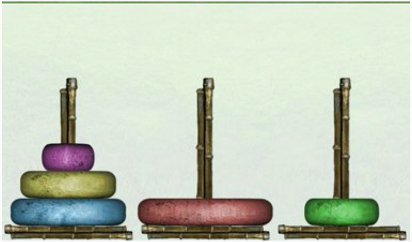

- The configuration is:

1

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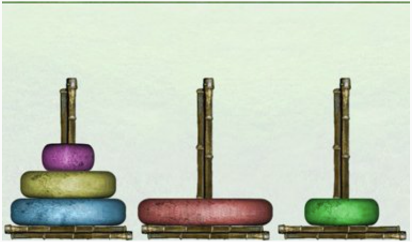
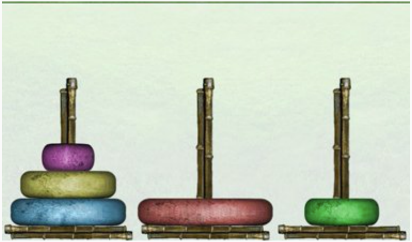


- The configuration is:

13

- Other example: can you guess how to write the configuration below?

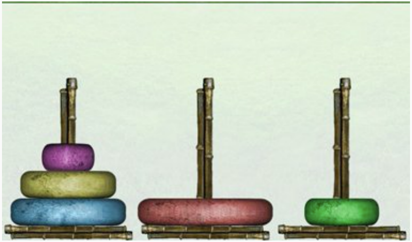


- The configuration is:

131

The Hanoi towers game II

- Other example: can you guess how to write the configuration below?



- The configuration is:

1311

The Hanoi towers game II

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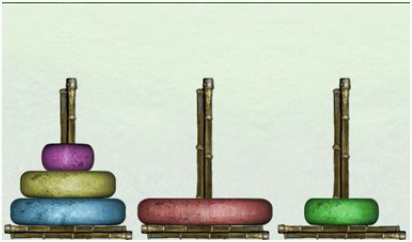

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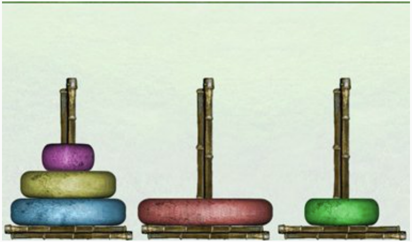
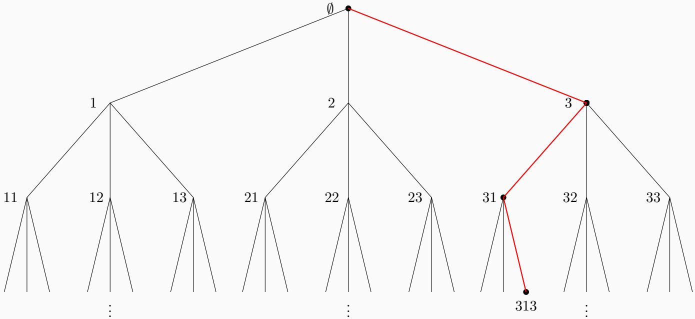


- The configuration is:

$$13112.$$

- Goal: to send $11\ldots1$ to $33\ldots3$.

- Configurations (sequences of length n of $1, 2, 3$) can be seen as vertices on the n-th level in a rooted ternary tree.



- Any move takes one vertex on the n-th level on the tree to another vertex on the n-th level. Then each move can be thought of as an automorphism of the rooted ternary tree.

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Conclusions

To conclude ...

Automata theory plays an important role not only in Computer Science but also in group theory.

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Automata theory plays an important role not only in Computer Science but also in group theory.

Some questions:

- What is your favourite automata group...? :)
- Nice topic: study algorithmic problems in branch groups.
- Do there exist finitely presented branch groups?

References

[1] G. Baumslag *Topics in combinatorial group theory* - Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel (1993).

[2] L. Bartholdi, R.I. Grigorchuk, Z. Sunik *Branch groups* - Handbook of Algebra, Volume 3, North-Holland (2003), 989-1112.

[3] R.I. Grigorchuk *Just infinite branch groups* - New Horizons in pro-p Groups, Progress in Mathematics, Volume 184 (2000), 121-179.

[4] P. de la Harpe *Topics in Geometric Group Theory* - Chicago Lectures in Mathematics (2000).

[5] D. Holt, S. Rees,and C. E. Röver *Groups, Languages, and Automata* - London Mathematical Society Student Texts 88 (2017).

[6] V. Nekrashevych *Self-similar groups* - Mathematical Surveys and Monographs, 117, American Mathematical Society (2005).

Obrigada.
Grazie :)