A Nonstandard Standardisation Theorem

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Overview

1. Introduce a calculus of explicit substitutions called the Linear Substitution Calculus $\lambda^{\sim}_{\mathtt{lsub}}$
2. Introduce the notion of standardisation
3. Say a thing or two about standardisation for $\lambda^{\sim}_{\mathtt{lsub}}$

Approach:

- Informal, mostly via examples
- Intersperse the use of slides and the whiteboard

Lambda Calculus and Explicit Substitutions

Standardisation in the λ calculus

Standardisation for $\lambda^{\sim}_{\mathtt{lsub}}$

Review of the Lambda Calculus

$$t \;::=\; x \;\mid\; t\,t \;\mid\; \lambda x.t$$

$$(\lambda x.s)t \mapsto_\beta s\{x := t\}$$

Explicit Substitutions

$$t \ ::= \ x \ \mid \ t\,t \ \mid \ \lambda x.t \ \mid \ t[x/t]$$

$$(\lambda x.t)u \mapsto_{beta} t[x/u]$$

- We add rules describing behaviour of $t[x/t]$
- Typical examples

$$\begin{array}{lll l} (tu)[x/v] & \mapsto_{app} & t[x/v]u[x/v] & \\ (\lambda y.t)[x/u] & \mapsto_{abs} & \lambda y.t[x/u] & y \notin \mathtt{fv}(u) \\ x[x/u] & \mapsto_{var} & u & \end{array}$$

Problem with Traditional Presentations of ES

- Structure of reduction space is not amenable to algebraic treatment
- In particular, no obvious theory of residuals
- For example, the beta redex is lost in this step (non-orthogonality)

$$((\lambda y.t)u)[x/v] \mapsto_{app} (\lambda y.t)[x/v]u[x/v]$$

Recently – ES that act at a distance

- $\lambda^{\sim}_{\texttt{lsub}}$ or the Linear Substitution Calculus
- Arises from work of Milner on the one hand, and that of Accattoli and Kesner on the other
- Has two parts: rewrite rules + equations
- Rewrite rules:

$$\begin{array}{lll} (\lambda x.t)\mathtt{L}u & \mapsto_{\mathtt{db}} & t[x/u]\mathtt{L} \\ C[\![x]\!][x/u] & \mapsto_{\mathtt{ls}} & C[\![u]\!][x/u] \\ t[x/u] & \mapsto_{\mathtt{gc}} & t \qquad\qquad \text{if } x \notin \mathtt{fv}(t) \end{array}$$

  - $\mathtt{L} = [x_1/t_1] \ldots [x_k/t_k]$ (k may be 0)
  - C context (term with a hole); in $C[\![u]\!]$ the free variables of u are not captured by C

$\lambda^{\sim}_{\texttt{lsub}}$

- Rewrite rules

$$
\begin{array}{llll}
(\lambda x.t)\mathtt{L}u & \mapsto_{\texttt{db}} & t[x/u]\mathtt{L} & \\
C[\![x]\!][x/u] & \mapsto_{\texttt{ls}} & C[\![u]\!][x/u] & \\
t[x/u] & \mapsto_{\texttt{gc}} & t & \text{if } x \notin \texttt{fv}(t)
\end{array}
$$

- Equations (generate what we call graphical equivalence $\sim$)

$$
\begin{array}{llll}
t[x/u][y/v] & \approx_{\texttt{CS}} & t[y/v][x/u] & x \notin \texttt{fv}(v)\ \&\ y \notin \texttt{fv}(u) \\
(\lambda y.t)[x/u] & \approx_{\sigma_1} & \lambda y.t[x/u] & y \notin \texttt{fv}(u) \\
(tv)[x/u] & \approx_{\sigma_2} & t[x/u]v & x \notin \texttt{fv}(v)
\end{array}
$$

- Sample reduction (on the board): $(\lambda x.x[y/u]v)(\lambda z.z)$

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Introduction

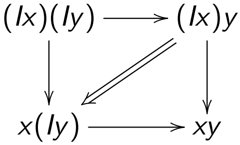
- Sorting a list of numbers.

$$
\begin{array}{rl}
 & \texttt{[3,4,1,2]} \\
\Rightarrow & \texttt{[3,1,4,2]} \\
\Rightarrow & \texttt{[3,1,2,4]} \\
\Rightarrow & \texttt{[1,3,2,4]} \\
\Rightarrow & \texttt{[1,2,3,4]}
\end{array}
$$

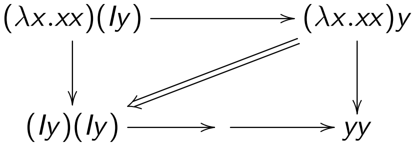
- We would like to do a similar thing with derivations: sort the redexes in a derivation

Sorting Redexes in Derivations

- left-to-right order



- Gets a little tricky due to duplication (below) and erasure



- These can be made into "square" diagrams using a notion of simultaneous rewrite step (not developed in this talk)

Residuals in λ-calculus

- Needed to formalise notion of sorting
- The idea: follow a redex along a derivation by coloring it or labeling it
- Example of labeling for λ-calculus:
  - Labeled terms

$$t \ ::= \ x \ | \ tt \ | \ \lambda x.t \ | \ (\lambda x^{\alpha}.s)t$$

  - Labeled β

$$(\lambda x^{\alpha}.s)t \mapsto_{\beta} s\{x := t\}$$

- Example of the residual relation A/B (on the board): the residuals of redex A after performing B

Residuals in $\lambda^{\sim}_{\texttt{lsub}}$ (1/2)

- Labeled terms

$$t ::= x \mid x^\alpha \mid tt \mid \lambda x.t \mid \lambda x^\alpha.t \mid t[x/t] \mid t[x^\alpha/t]$$

- Labeled rewriting

$$\begin{array}{llll} (\lambda x^\alpha.t)\texttt{L}u & \stackrel{\alpha}{\mapsto}_{\texttt{dB}} & t[x/u]\texttt{L} & \\ C[\![x^\alpha]\!][x/u] & \stackrel{\alpha}{\mapsto}_{\texttt{ls}} & C[\![u]\!][x/u] & \\ t[x^\alpha/u] & \stackrel{\alpha}{\mapsto}_{\texttt{gc}} & t & x \notin \texttt{fv}(t) \end{array}$$

- Anchor of a labeled redex is the variable containing the label
- Note: there is an additional well-labeled condition required which is omitted here (eg. $\lambda x.x^\alpha$ is not well-labeled)
- What about the graphical equivalence? We can do the same (next slide)

Residuals in $\lambda^{\sim}_{\mathtt{lsub}}$ (2/2)

- Labeled rewriting (same as above)

$$\begin{array}{llll}
(\lambda x^{\alpha}.t)\mathtt{L}u & \overset{\alpha}{\mapsto}_{\mathtt{dB}} & t[x/u]\mathtt{L} & \\
C[\![x^{\alpha}]\!][x/u] & \overset{\alpha}{\mapsto}_{\mathtt{ls}} & C[\![u]\!][x/u] & \\
t[x^{\alpha}/u] & \overset{\alpha}{\mapsto}_{\mathtt{gc}} & t & x \notin \mathtt{fv}(t)
\end{array}$$

- Labeled equivalence ((α) means α may or may not appear)

$$\begin{array}{llll}
t[x^{(\alpha)}/u][y^{(\beta)}/v] & \approx_{\mathrm{CS}} & t[y^{(\beta)}/v][x^{(\alpha)}/u] & x \notin \mathtt{fv}(v) \ \&\ y \notin \mathtt{fv}(u) \\
(\lambda y^{(\beta)}.t)[x^{(\alpha)}/u] & \approx_{\sigma_1} & \lambda y^{(\beta)}.t[x^{(\alpha)}/u] & y \notin \mathtt{fv}(u) \\
(tv)[x^{(\alpha)}/u] & \approx_{\sigma_2} & t[x^{(\alpha)}/u]v & x \notin \mathtt{fv}(v)
\end{array}$$

- Note: it can be shown that $s \sim t$ determines a bijective relation between the redexes of s and t
- Examples (on the board)

Standardisation via Inversion (for total orders)

- $\prec$-inversion diagram ($\prec$ total ordering on redexes)

$$\begin{array}{ccc} s & \xrightarrow[B]{} & t \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A/B} \\ s' & \xrightarrow[B/A]{} \!\!\!\!\!\!\rightarrow & t' \end{array} \qquad A \prec B$$

- $\prec$-inversion step $\Rightarrow_{\prec}$ in a derivation:

$$\sigma_1; B; A/B; \sigma_2 \qquad \Rightarrow_{\prec} \qquad \sigma_1; A; B/A; \sigma_2$$

- Definition: A derivation in which no $\Rightarrow_{\prec}$ steps are applicable is said to be $\prec$-standard

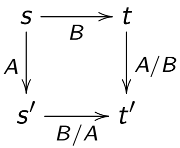
Theorem

If $\sigma : t \twoheadrightarrow_{\beta} u$ then there exists a unique $\prec_{left}$-standard β-derivation $\rho : t \twoheadrightarrow_{\beta} u$ s.t. $\sigma \Rightarrow^{*} \rho$.

Proof: $\Rightarrow_{\prec}$ SN+CR (Klop)

Standardisation via Inversion (for partial orders)

- $\prec$-inversion diagram ($\prec$ partial ordering on redexes)
  - Same as previous slide
- $\prec$-square diagram ($\prec$ partial ordering on redexes)



$A \not\prec B$ and $B \not\prec A$

- $\prec$-square step $\Diamond_{\prec}$ (symmetric)
- $\prec$-inversion step $\Rightarrow_{\prec}^{\Diamond}$ in a derivation: apply $\Rightarrow_{\prec}$ modulo $\Diamond_{\prec}$
- Examples (on the board)

Standardisation via Inversion (for partial orders)

- Definition: A derivation in which no $\Rightarrow_{\prec}^{\Diamond}$ steps are applicable is said to be $\prec$-standard

Theorem

If $\sigma : t \twoheadrightarrow_{\beta} u$ then there exists a unique $\prec_{left}$-standard β-derivation $\rho : t \twoheadrightarrow_{\beta} u$ s.t. $\sigma \Rightarrow^{*} \rho$. Note: uniqueness here means modulo $\Diamond$

Proof1: Repeatedly extract external redex in ρ (Huet,Lévy,Melliès)
Proof2: $\Rightarrow_{\prec}^{\Diamond}$ SN+CR (TERESE)

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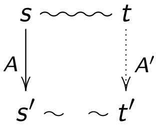
Standardisation for $\lambda^{\sim}_{\mathtt{lsub}}$

The requirement for the order on $\lambda^{\sim}_{\mathtt{lsub}}$ redexes

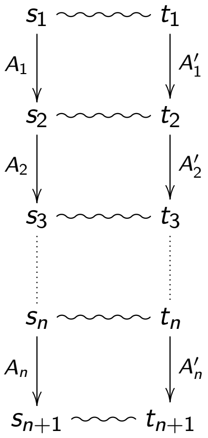
It must preserve the graphical equivalence

$A_1; \ldots; A_n$ standard iff $A'_1; \ldots; A'_n$ standard

$\sim$ is a strong bisimulation between $\lambda_{\mathtt{lsub}}$ and itself that reduces the "same" redexes



Thus standardisation should be "preserved" via the equations



An example

$$
\begin{array}{ccc}
t[x^{\alpha}/u][y^{\beta}/v] & \rightsquigarrow & t[y^{\beta}/v][x^{\alpha}/u] \\
\downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A'} \\
t[y^{\beta}/v] & \rightsquigarrow & t[y^{\beta}/v] \\
\downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B'} \\
t & \rightsquigarrow & t
\end{array}
$$

- Note $t[x^{\alpha}/u][y^{\beta}/v] \sim_{\mathtt{cs}} t[y^{\beta}/v][x^{\alpha}/u]$, assuming $y \notin \mathtt{fv}(u)$
- $A; B$ standard iff $A'; B'$ standard
- The left-to-right order does not make sense due to the graphical equivalence

Action Principle as Guideline

For devising appropriate partial order on redexes in $\lambda^{\sim}_{\mathtt{lsub}}$

$$\begin{array}{ccc} C[\![x]\!][x/s] & \longrightarrow & C[\![x]\!][x/s'] \\ \downarrow & & \downarrow \\ C[\![s]\!][x/s] & \twoheadrightarrow & C[\![s']\!][x/s'] \end{array}$$

Standard should be down-below since the `ls`-redex *acts on* (*i.e. nests*) the redexes in s

Action Principle as Guideline

$$\begin{array}{ccc} t[x/s] & \longrightarrow & t[x/s'] \\ \downarrow & \swarrow & \\ t & & \end{array}$$

Standard should be down since the erasing redex *acts on* the redexes in s

Action Principle as Guideline

$$\begin{array}{ccc} x[x/y][y/z] & \longrightarrow & x[x/z][y/z] \\ \downarrow & & \downarrow \\ y[x/y][y/z] & \twoheadrightarrow & z[x/z][y/z] \end{array}$$

`ls`-redex on x must nest the `ls`-redex on y

- Note that duplicated `ls`-redex on y is not syntactically contained in the acting `ls`-redex on x
- The same diagram applies to terms like $(x[x/y]yz)[y/z]$, where $[x/y]$ and $[y/z]$ are no longer next to each other.

Action Principle as Guideline

$$\begin{array}{ccc} x[x'/y][y/z] & \longrightarrow & x[x'/z][y/z] \\ \downarrow & \swarrow & \\ x[y/z] & & \end{array}$$

This is the version at a distance of the erasing diagram, requiring the same notion of nesting at a distance.

Definition of the partial "box" order

- *A immediately boxes B*, noted $A \prec_{\mathtt{B}}^{1} B$ if the anchor of B (*i.e.* the variable possibly carrying a label) is in the box of A
  - *i.e.* if the pattern of A is any of $(\lambda x.t)\mathtt{L}u$, $C[\![x]\!][x/u]$ or $t[x/u]$, then the anchor of B appears in u.
- *A boxes B*, noted $A \prec_{\mathtt{B}} B$ if $A(\prec_{\mathtt{B}}^{1})^{+}B$
- *A and B are disjoint*, noted $A \parallel B$, if $A \not\preceq_{\mathtt{B}} B$ and $B \not\preceq_{\mathtt{B}} A$.
- Key property: box order is stable by the equivalence $\sim$

Some Results

Theorem (Existence of Standard Derivations for $\lambda^{\sim}_{\mathtt{lsub}}$)

If $t \twoheadrightarrow_{\lambda^{\sim}_{\mathtt{lsub}}} u$ then there is a $\prec_{\mathrm{B}}$-standard $\lambda^{\sim}_{\mathtt{lsub}}$-derivation from t to u.

Proof uses axiomatics of Melliès

Theorem (Uniqueness Modulo for $\lambda^{\sim}_{\mathtt{lsub}}$)

If $t \twoheadrightarrow_{\lambda^{\sim}_{\mathtt{lsub}}} u$ then there exists a $\prec_{\mathrm{B}}$-standard $\lambda^{\sim}_{\mathtt{lsub}}$-derivation from t to u that is unique modulo $\Diamond$.

Proof uses

1. Existence of Standard Derivations for $\lambda^{\sim}_{\mathtt{lsub}}$;
2. Uniqueness of standardisation for $\lambda_{\mathtt{lsub}}$ w.r.t. the left-to-right order; and
3. A simple argument showing that $\prec_{\mathrm{L}}$-inversions of a $\prec_{\mathrm{B}}$-standard derivation swaps only disjoint (w.r.t. $\prec_{\mathrm{B}}$) redexes

Conclusions

- Quick overview of $\lambda^{\sim}_{\texttt{lsub}}$
- Quick overview of standardisation
- Standardisation for $\lambda^{\sim}_{\texttt{lsub}}$
- General context of this work: $\lambda^{\sim}_{\texttt{lsub}}$ as a vehicle to study the metatheory of the λ-calculus

Further reading: Standardisation (Ch.8:TERESE), This work (POPL 2014)