A combinatorial argument for termination properties

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Talk's Plan

The λ-calculus

Proposed by Church in 1932. [Church32]

Terms $\quad t := x \mid (t)\, t \mid \lambda x.t$

Computations (reductions) are made by a unique rule:

$$(\lambda x.t)\, s \longrightarrow t\{x := s\} \qquad (\beta)$$

Some renaming may be necessary:

$$\lambda x.t \longrightarrow \lambda y.t\{x := y\} \qquad (\alpha)$$

Foundation for the Lisp and functional programming languages in general.

Typing Systems

Simply typed λ-calculus proposed by Church.[Church40]

Classify objects (terms) in the formal system.

$\lambda_{x:int}.x : int \rightarrow int \quad \lambda_{x:bol}.x : bool \rightarrow bool$ (*à la Church*)

$\lambda_{x}.x : int \rightarrow int \quad \lambda_{x}.x : bool \rightarrow bool$ (*à la Curry*)

STLC is related to IPL: Curry-Howard(-de Bruijn) Isomorphism.

If $\Gamma \vdash t : \tau$ then $\langle \Gamma \vdash \tau \rangle$ is called a typing of t.

Simple types system for λ

Types $\sigma, \tau \in \mathcal{S} ::= \mathcal{A} \,|\, \mathcal{S} \to \mathcal{S}$

Contexts $\Gamma ::= \{x{:}\tau \,|\, x{\in}\mathcal{X},\ \tau{\in}\mathcal{S}\}$ s.t. $\mathtt{dom}(\Gamma) = \{x \,|\, x{:}\tau \in \Gamma\}$ is finite.

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ ax \qquad\qquad \frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x.t : \sigma \to \tau}\ \to_i$$

$$\frac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash s : \sigma}{\Gamma \vdash t\, s : \tau}\ \to_e$$

Typing Systems Properties

- Subject Reduction (SR)

  If $\Gamma \vdash t : \tau$ and $t \to_\beta s$, then $\Gamma \vdash s : \tau$

- Subject Expansion (SE)

  If $\Gamma \vdash s : \tau$ and $t \to_\beta s$, then $\Gamma \vdash t : \tau$

- Strong, Weak or Head Normalisation (SN, WN and HN) for typable terms.

- Type Inference ($t : ?$)

- Principal Typing (PT)

- Inhabitation Problem ($? : \langle \Gamma \vdash \tau \rangle$)

SR and SN/WN

$$\frac{\Gamma \vdash \lambda x.t : \sigma \to \tau \qquad \Gamma \vdash s : \sigma}{\Gamma \vdash (\lambda x.t)\, s : \tau} \quad \Rightarrow \quad \frac{x{:}\sigma; \Gamma \vdash t : \tau \qquad \Gamma \vdash s : \sigma}{\Gamma \vdash t\{x := s\} : \tau}$$

$$\frac{\dfrac{\Phi :: \dfrac{[\sigma]^x \; \nabla}{\tau}}{\sigma \to \tau} \qquad \Psi :: \dfrac{\nabla}{\sigma}}{\tau} \quad \Rightarrow \quad \Phi' :: \frac{\Psi :: \dfrac{\nabla}{[\sigma]}}{\tau}$$

Intersection type discipline

- Introduced by Coppo and Dezani-Ciancaglini. [CDC78, CDC80]
- Characterisation of the SN terms of the λ-calculus. [Pottinger80]
- It incorporates type polymorphism in a finitary way:

$$\lambda_x.x : (int \rightarrow int) \wedge (bool \rightarrow bool)$$

- PT has been verified in IT systems. [Bakel95, SM96a, KW04]
- Execution time of (head) normalising λ-terms is related with the size of derivations in a nonidempotent IT system [**?**]

$$\sigma \wedge \sigma \neq \sigma$$

Nonidempotent (restricted) intersection types

Definition (Nonidempotent intersection types and contexts)

1. The **nonidempotent intersection types** are defined by:

$$\tau, \sigma \in \mathcal{T} ::= \mathcal{A} \,|\, \mathcal{U} \to \mathcal{T} \qquad\qquad u \in \mathcal{U} ::= [\sigma_i]_{i \in I},\ I \text{ finite}$$

   $+$ is defined to be the multiset union.

2. **Contexts**: $\Gamma = \{x{:}u \,|\, x \in \mathcal{X},\ u \in \mathcal{U}\ \&\ u \neq [\,]\}$ s.t. $\mathtt{dom}(\Gamma) = \{x \,|\, x{:}u \in \Gamma\}$ is finite.

$$\Gamma(x) = \begin{cases} u, & \text{if } x{:}u \in \Gamma \\ [\,], & \text{otherwise} \end{cases}$$

The NIT system for λ

$$\frac{}{x : [\tau] \vdash x : \tau}\ ax \qquad \frac{\Gamma \vdash t : \tau}{\Gamma \setminus\!\!\setminus x \vdash \lambda x.t : \Gamma(x) \to \tau}\ \to_i$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \to \tau \qquad (\Gamma_i \vdash s : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Gamma_i \vdash t\, s : \tau}\ \to_e$$

Note that:

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash \lambda x.t : [\,] \to \tau}, \text{if } x \notin \mathtt{dom}(\Gamma)$$

and, for any $s \in \Lambda$,

$$\frac{\Gamma \vdash t : [\,] \to \tau}{\Gamma \vdash t\, s : \tau}, \text{if } m = 0$$

The NIT system for λ

Definition

Let $\Phi :: \Gamma \vdash t : \tau$. The $\mathrm{sz}(\Phi)$ is defined to be the number of typing rules applied in the typing derivation Φ.

Example

$$\Phi :: \dfrac{\dfrac{\Phi_t :: \dfrac{\nabla}{x{:}[\sigma_1, \sigma_2]; \Gamma \vdash t : \tau}}{\Gamma \vdash \lambda x.t : [\sigma_1, \sigma_2] \rightarrow \tau} \qquad \Psi_1 :: \dfrac{\nabla}{\Delta_1 \vdash s : \sigma_1} \quad \Psi_2 :: \dfrac{\nabla}{\Delta_2 \vdash s : \sigma_2}}{\Gamma +_{i \in \{1,2\}} \Delta_i \vdash (\lambda x.t)\, s : \tau}$$

$$\mathrm{sz}(\Phi) = \mathrm{sz}(\Phi_M) + 1 + \mathrm{sz}(\Psi_1) + \mathrm{sz}(\Psi_2) + 1$$

Theorem (Subject Reduction)

If $\Phi :: \Gamma \vdash (\lambda x.t)\, s : \tau$ *then* $\Phi' :: \Gamma \vdash t\{x/s\} : \tau$ *where* $\mathtt{sz}(\Phi') < \mathtt{sz}(\Phi)$.

Example

For $\tau = \alpha \to \alpha$, $\Delta = \{x : [\tau, \tau], y : [\alpha]\}$ and

$$\Phi :: \dfrac{\dfrac{\dfrac{}{x : [\tau] \vdash x : \tau} \quad \dfrac{\dfrac{}{x : [\tau] \vdash x : \tau} \quad \dfrac{}{y : [\alpha] \vdash y : \alpha}}{x : [\tau]; y : [\alpha] \vdash xy : \alpha}}{\Delta \vdash x^2 y : \alpha}}{\dfrac{x : [\tau, \tau] \vdash \lambda y.x^2 y : \tau}{\vdash \lambda xy.x^2 y : [\tau, \tau] \to \tau}}$$

one has

$$\dfrac{\Phi :: \dfrac{\nabla}{\vdash \lambda xy.x^2 y : [\tau, \tau] \to \tau} \qquad \Psi_1 :: \dfrac{\nabla}{\Gamma_1 \vdash s : \tau} \quad \Psi_2 :: \dfrac{\nabla}{\Gamma_2 \vdash s : \tau}}{\Gamma_1 + \Gamma_2 \vdash (\lambda xy.x^2 y)\, s : \tau}$$

$$
\Phi' :: \dfrac{\dfrac{\Psi_2 :: \dfrac{\nabla}{\Gamma_2 \vdash s : \tau} \qquad \dfrac{\Psi_1 :: \dfrac{\nabla}{\Gamma_1 \vdash s : \tau} \qquad \dfrac{}{y : [\alpha] \vdash y : \alpha}}{y : [\alpha] + \Gamma_2 \vdash sy : \alpha}}{y : [\alpha] + \Gamma_1 + \Gamma_2 \vdash s^2 y : \alpha}}{\Gamma_1 + \Gamma_2 \vdash \lambda y . s^2 y : \tau}
$$

Terminating reduction strategy

Theorem

Let $\Phi :: \Gamma \vdash t : \tau$.

1. *If* $t \to_\beta t'$ *for any untyped occurrence of* t *in* Φ *then* $\Phi' :: \Gamma \vdash t' : \tau$ *s.t.* $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi')$.
2. *If* $t \to_\beta t'$ *for any typed occurrence of* t *in* Φ *then* $\Phi' :: \Gamma \vdash t' : \tau$ *s.t.* $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.

Corollary

If t *is typable then any sequence reducing all and only redexes in typed occurrences terminates.*

Lemma

If $\Phi :: \Gamma \vdash t : \tau$ and t has no redex in a typed occurrence in Φ then t is a head-normal form.

Lemma

If $\Phi :: \Gamma \vdash t : \tau$ s.t. t has no typed redex occurrence in Φ and $[\,]$ has only negative occurrences in $\langle \Gamma \vdash \tau \rangle$ then t is a β-normal form.

Theorem (Subject Expansion)
If $\Gamma \vdash t\{x/s\} : \tau$ *then* $\Gamma \vdash (\lambda x.t)\, s : \tau$.

Theorem
If $\Gamma \vdash t' : \tau$ *and* $t \to_\beta t'$ *then* $\Gamma \vdash t : \tau$.

Lemma

Every head-normal form is typable.

Theorem

If t is head-normalisable then t is typable.

Lemma

Every β-normal form t is typable with some typing $\langle \Gamma \vdash \tau \rangle$ having no positive occurrence of $[\,]$.

Theorem

If t is weakly normalising then t is typable with some typing $\langle \Gamma \vdash \tau \rangle$, with no positive occurrence of $[\,]$.

The Klop calculus

Terms $t, p \quad ::= \quad x \mid t\ p \mid [t, p] \mid \lambda x.t$

Equation

$$[t, p]\ s \quad =_\sigma \quad [t\ s, p]$$

Rules

$$(\lambda x.t)\ p \quad \to_\beta \quad t\{x/p\} \quad , x \in \mathtt{fv}(t)$$

$$(\lambda x.t)\ p \quad \to_{mem} \quad [t, p] \quad , x \notin \mathtt{fv}(t)$$

The NIT system for the λK

$$\frac{}{x : [\tau] \vdash x : \tau}\ ax \qquad \frac{\Gamma \vdash t : \tau}{\Gamma \setminus\!\!\setminus x \vdash \lambda x.t : \Gamma(x) \rightarrow \tau}\ \rightarrow_i$$

$$\frac{\Gamma \vdash t : \tau \qquad \Delta \vdash s : \sigma}{\Gamma + \Delta \vdash [t, s] : \tau}\ mem$$

$$\frac{\Gamma \vdash t : [\,] \rightarrow \tau \qquad \Delta \vdash s : \sigma}{\Gamma + \Delta \vdash t\, s : \tau}\ \rightarrow_e^{\omega}$$

$$\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \rightarrow \tau \qquad (\Gamma_i \vdash s : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Gamma_i \vdash t\, s : \tau} * \ \rightarrow_e$$

$*$ where $I \neq \emptyset$.

Conclusions

- Nonidempontent intersection types system can be seem as a refinement of IT system, with more information about resources.
- The feature described above allowed us to give a simple measure to have characterisations of termination properties.
- The present technique, and the new reduction strategy derived from it, can be applied to other calculi.

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