An Operational Characterization of (lazy) Strong Normalization

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Outline

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- The parametric λ-calculus.
- Lazy evaluation.
- The $\lambda\Phi$-calculus.
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- A new characterization of (lazy)Strong Normalization.

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- Potential valuability: all non potentially valuable terms can be consistently equated.
- We explore the relation between (lazy)-potential valuability and (lazy)-β-strong normalization.
- We ask for two call-by-value λ-calculi, such that the set of potentially valuable terms in them coincide with the set of (lazy)-β-strongly normalizing terms.

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- The classical evaluation rule is

$$(\lambda \mathbf{x}.\mathbf{M})\mathbf{N} \rightarrow_{\beta} \mathbf{M}[\mathbf{N}/\mathbf{x}]$$

where $(\lambda x.M)N$ is named β-redex.

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- A term M is strongly β-normalizing if both M has β-normal form and every reduction sequence starting from M eventually stops.

The Parametric Lambda-Calculus

Let $\Delta \subseteq \Lambda$.

- The Δ-reduction ($\rightarrow_\Delta$) is the contextual closure of:

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- Standardization holds under one more condition.

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- The lazy β-normal form of a term, if there exists, may not be unique. In fact, $(\lambda xy.x)(II) \to^*_{\beta\ell} \lambda y.II$ and $(\lambda xy.x)(II) \to^*_{\beta\ell} \lambda y.I$ where both $\lambda y.II$ and $\lambda y.I$ are lazy β-normal forms.

Δ-Solvability and Potential Valuability

- A term M is Δ-solvable if and only if there is a sequence $\vec{N}$ of Δ-values such that:

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- A term M is potentially Δ-valuable iff there is a substitution $\mathbf{s}$ replacing variables by closed values, such that $\mathbf{s}(M)$ is Δ-valuable

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- $\Phi = \mathrm{Var} \cup (\Upsilon)^0$ where $\Upsilon = \cup_i \Upsilon_i$ and Υ_i, Φ_i are defined as follows:

$$\Upsilon_0 = \mathrm{Var} \qquad \Phi_i = \mathrm{Var} \cup (\Upsilon_i)^0$$

$$\begin{aligned}\Upsilon_{i+1} = \mathrm{Var} \cup & \ \{xM_1 \ldots M_n \mid M_k \in \Upsilon_i (1 \leq k \leq n)\} \\ \cup & \ \{\lambda\vec{x}.M \mid M \in \Upsilon_i\} \\ \cup & \left\{ (\lambda x.P)QM_1\ldots M_n \;\middle|\; \begin{array}{l} Q \in \Upsilon_i - (\Lambda^0 \cup \mathrm{Var}), \\ M_k \in \Upsilon_i (1 \leq k \leq n) \\ P[Q/x]M_1 \ldots M_n \to^*_{\Phi_i} R \in \Upsilon_i \end{array} \right\}\end{aligned}$$

The Result 1

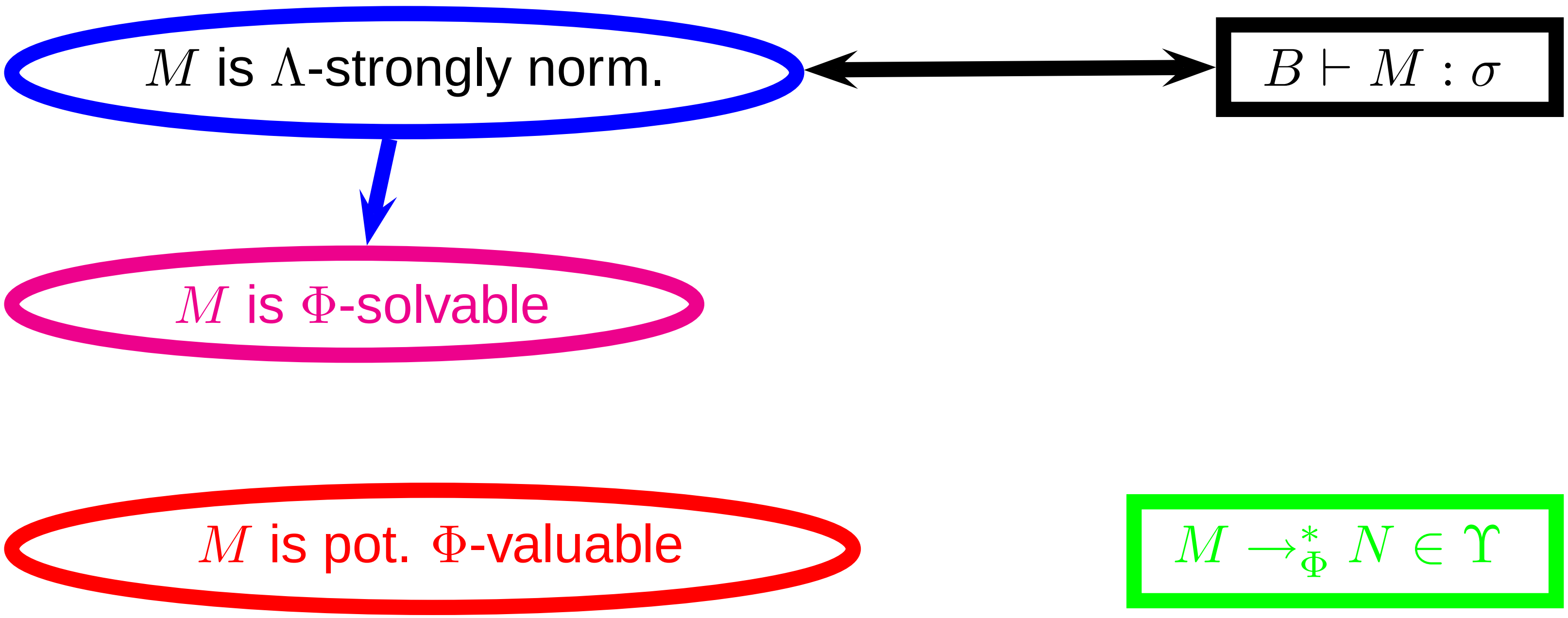
M is Λ-strongly norm. $\longleftrightarrow$ $B \vdash M : \sigma$

M is Φ-solvable

M is pot. Φ-valuable

$M \to_{\Phi}^{*} N \in \Upsilon$

The Result 1



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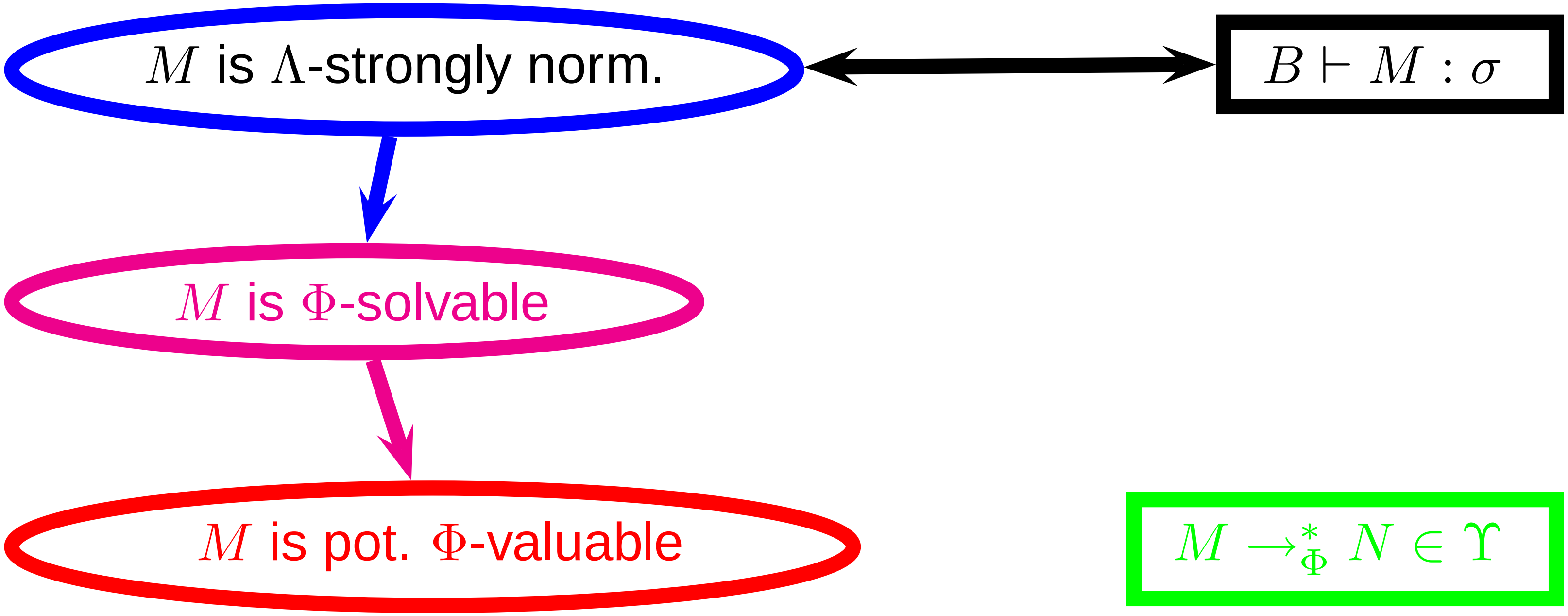

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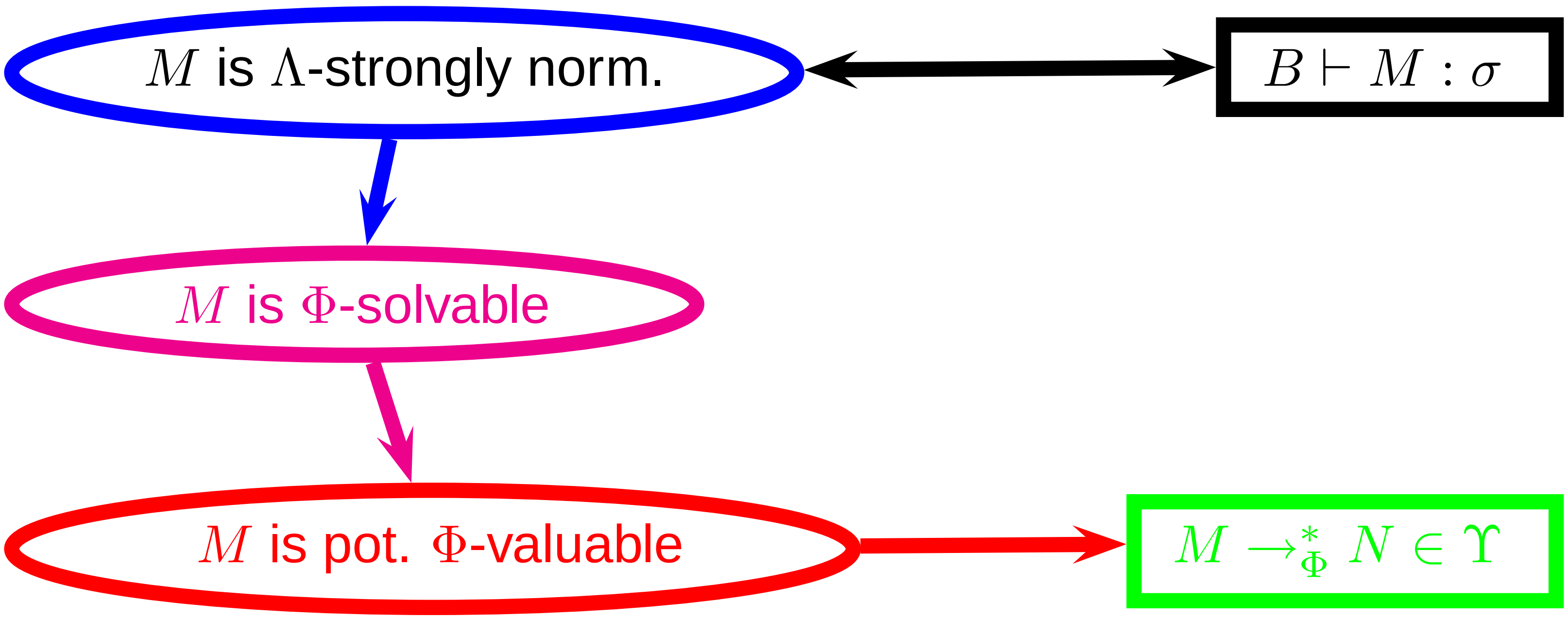

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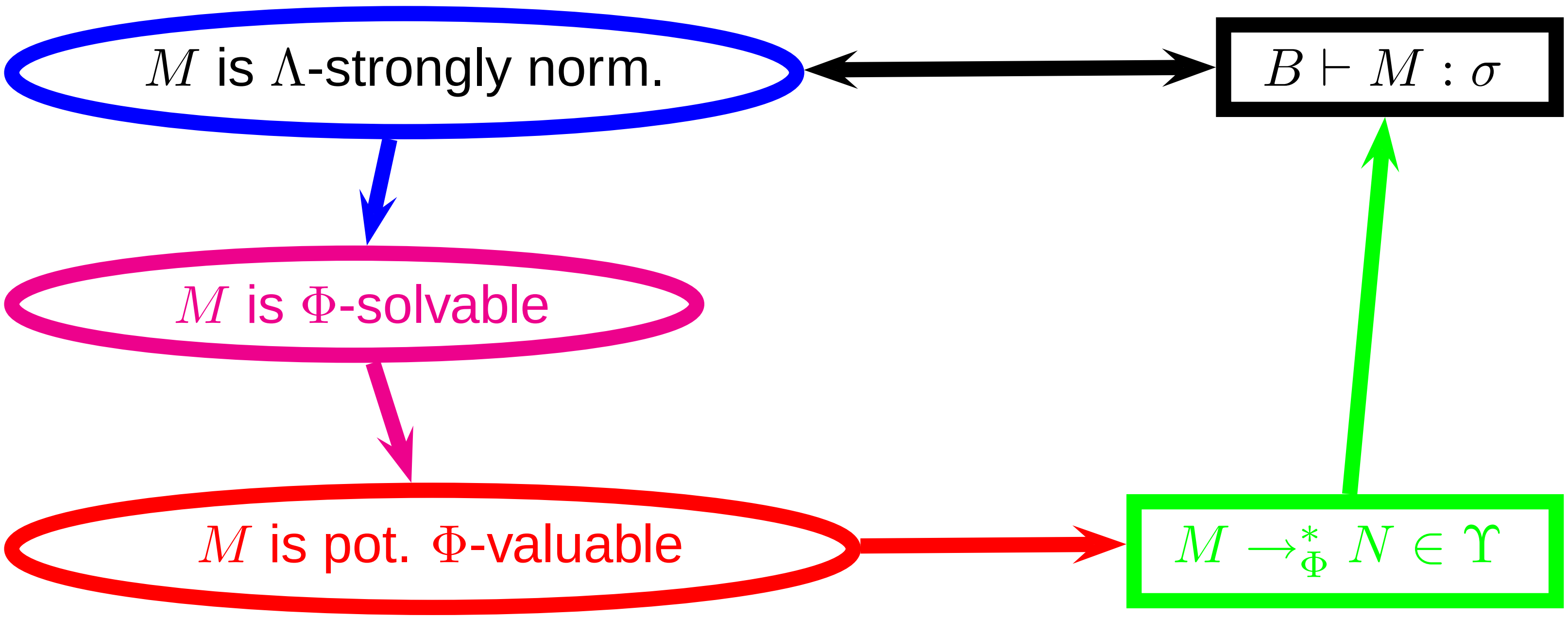

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- Intersection Types: $\sigma ::= a \mid \sigma \to \sigma \mid \sigma \cap \sigma$.
- The type assignment system is the following:

$$\frac{}{B[\sigma/x] \vdash_\nu x : \sigma}\,(var) \qquad \frac{B[\sigma/x] \vdash_\nu M : \tau}{B \vdash_\nu \lambda x.M : \sigma \to \tau}\,(\to I)$$

$$\frac{B \vdash_\nu M : \sigma \to \tau \quad B \vdash_\nu N : \sigma}{B \vdash_\nu MN : \tau}\,(\to E) \qquad \frac{B \vdash_\nu M : \sigma \quad B \vdash_\nu M : \tau}{B \vdash_\nu M : \sigma \cap \tau}\,(\cap I)$$

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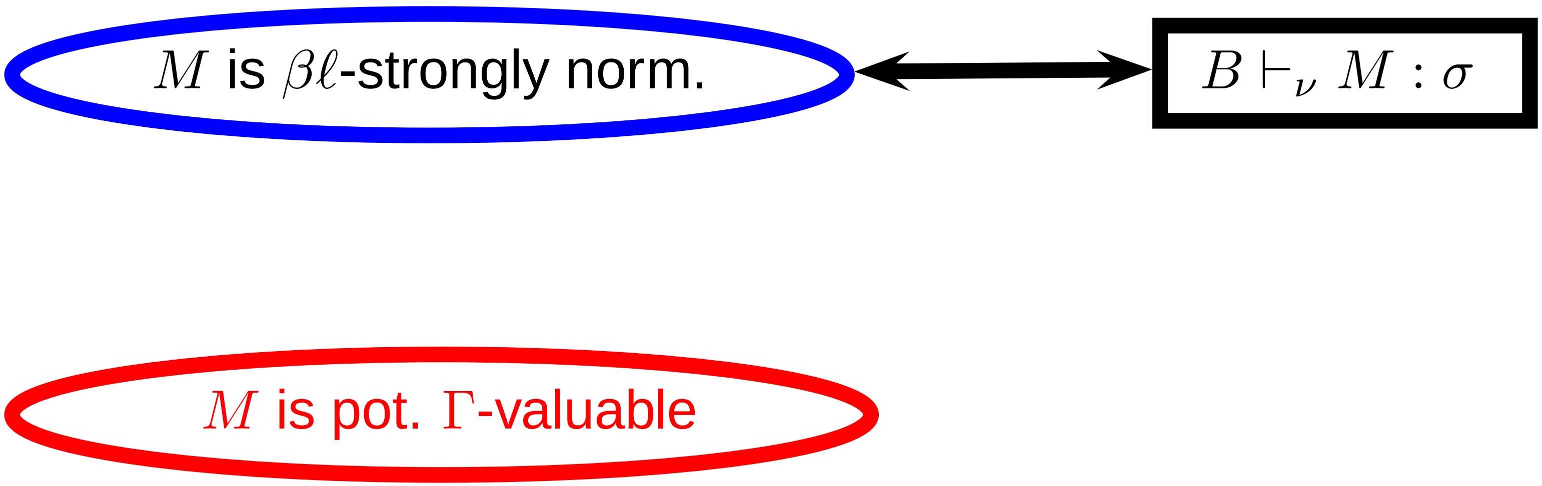

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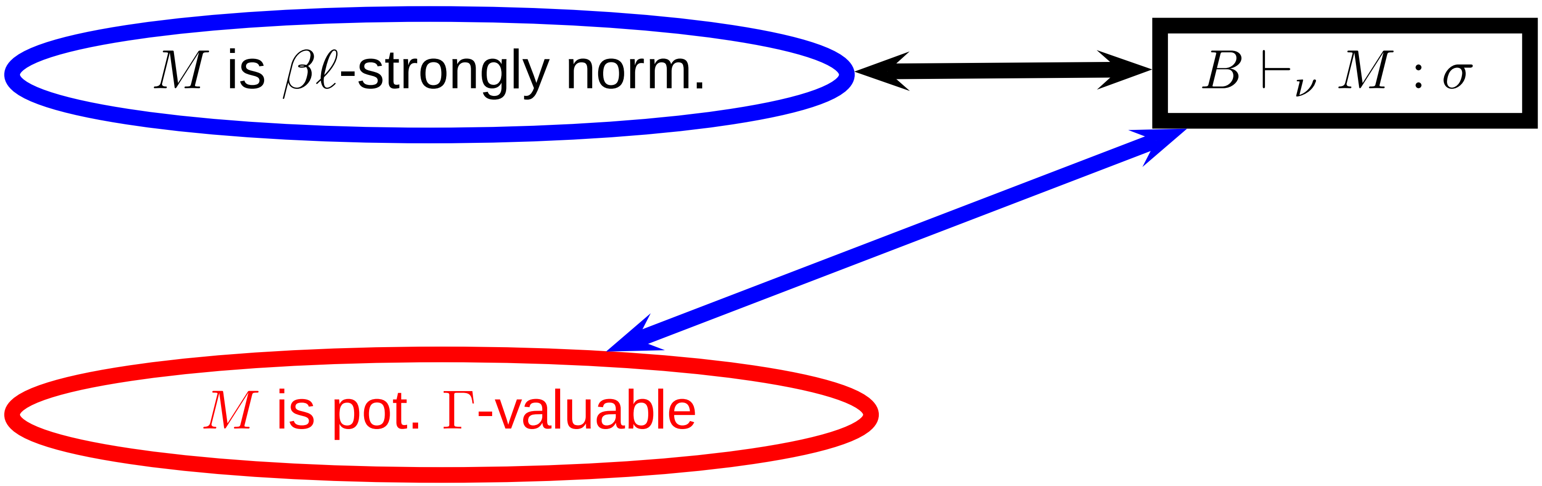

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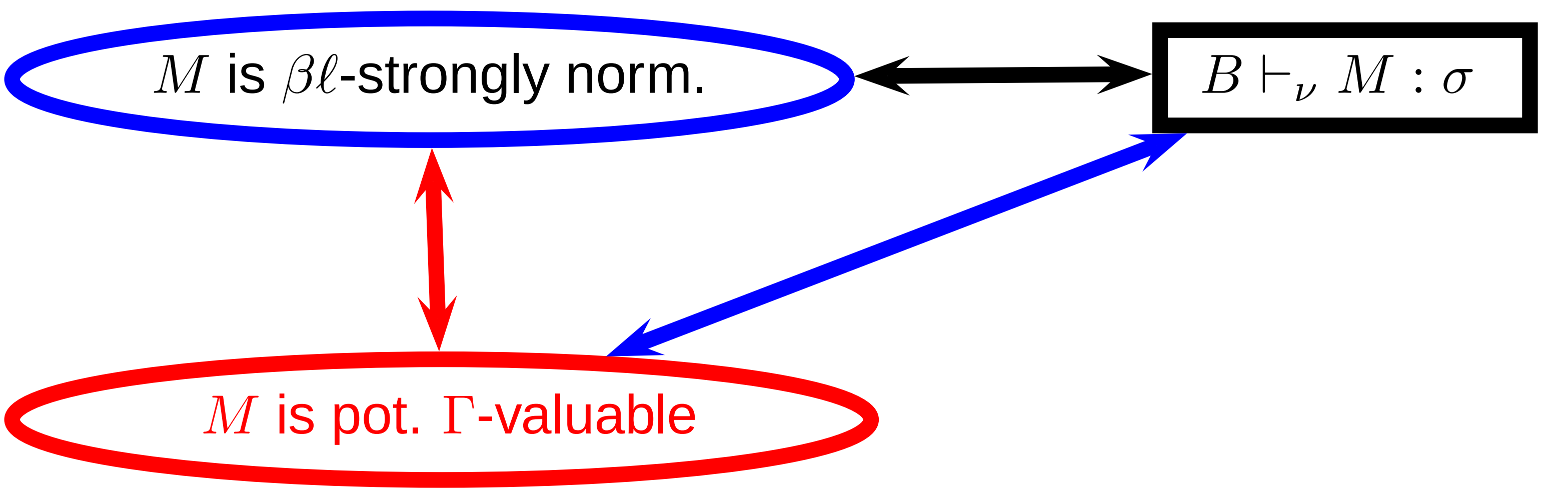

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- We conjecture that the answer to this question is negative.