Part II

Approximated Theorem Proving

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1. Motivation

- Suppose that we have a huge database and we want to check whether a formula follows from it.
- We may not have enough resources to check the whole database.
- But we may be able to find the answer even without checking all formulas:
  - Let B=$\{\alpha, \beta, \gamma, \delta, p, p \rightarrow q, \varphi, \psi\}$
  - We want to know whether $B \models q$.

1.1 Example

Triangles are polygons.

Rectangles are polygons.

Rectangles have four right angles.

Cows eat grass.

Animals that eat grass do not have canine teeth.

Carnivorous animals are mammals.

Mammals have canine teeth or molar teeth.

Animals that eat grass are mammals.

Mammals are vertebrate.

Vertebrates are animals.

Brazil is in South America.

Volcanic soil is fertile.

1.2 Structure of the talk:

- Approximate Entailment (S_3).
- Extended S_3.
- Incremental Proof Method: Tableaux for S_3.
- Dynamic properties.
- Static expressivity.
- Recovering control.
- Conclusions and Future Work.

2. Approximate Entailment

(Schaerf and Cadoli, 1995)

- **Goal:** Approximated theorem proving in the propositional clausal fragment.
- **Main idea:** Consider only some propositional letters.
  - L: propositional letters of the language.
  - Context set $S \subseteq L$.
  - Define two approximate entailments:
    - $\models^1_S$: complete but not sound (wrt CL).
    - $\models^3_S$: sound and incomplete (wrt CL).

2.1 Approximate Entailment – S_3

- $S = L$: classical entailment.
- Every theorem of S_3 is a classical theorem.
- Approximations behave classically for $p \in S$:
  - $v(p) = 1$ iff $v(\neg p) = 0$.
- If $p \notin S$, there are 3 possibilities:
  - $v(p) = 1$ and $v(\neg p) = 0$
  - $v(p) = 0$ and $v(\neg p) = 1$
  - $v(p) = 1$ and $v(\neg p) = 1$

2.2 Example – Cadoli and Schaerf

$$
\begin{aligned}
B = \quad & \{\neg\texttt{cow} \vee \texttt{grass-eater}, \neg\texttt{dog} \vee \texttt{carnivore},\\
& \neg\texttt{grass-eater} \vee \neg\texttt{canine-teeth},\\
& \neg\texttt{grass-eater} \vee \texttt{mammal}, \neg\texttt{carnivore} \vee \texttt{mammal},\\
& \neg\texttt{mammal} \vee \texttt{canine-teeth} \vee \texttt{molar-teeth},\\
& \neg\texttt{mammal} \vee \texttt{vertebrate},\\
& \neg\texttt{vertebrate} \vee \texttt{animal}\}.
\end{aligned}
$$

Let $\alpha = \neg\texttt{cow} \vee \texttt{molar-teeth}$.

$B \models \alpha$?

For $S = \{\texttt{grass-eater}, \texttt{mammal}, \texttt{canine-teeth}\}$,

$B \models_S^3 \alpha$, hence $B \models \alpha$.

2.3 Problems with S_3

- Only defined for clauses.
- No axiomatisation.
- No indications on how to get S.
- Our work:
  - Extended S_3 to full propositional logic.
  - Tableaux for extended S_3.
  - Heuristics for expanding S.
  - Axiomatisation.

3. KE-Tableaux for S_3 (KES_3)

- Based on KE-Tableaux [D'Agostino 94].
- Formulas are T- and F-marked.
- One basic rule $(T\ \neg)$ is restricted:

$$\frac{T\ \neg\alpha}{F\ \alpha} \quad \text{provided } \alpha \in S$$

- Other tableau rules are as in classical KE

3.1 Classical KE-Rules

$$\frac{T\ \alpha \to \beta \quad T\ \alpha}{T\ \beta}\ (T \to_1) \qquad \frac{T\ \alpha \to \beta \quad F\ \beta}{F\ \alpha}\ (T \to_2) \qquad \frac{F\ \alpha \to \beta}{T\ \alpha \quad F\ \beta}\ (F \to)$$

$$\frac{F\ \alpha \wedge \beta \quad T\ \alpha}{F\ \beta}\ (F\wedge_1) \qquad \frac{F\ \alpha \wedge \beta \quad T\ \beta}{F\ \alpha}\ (F\wedge_2) \qquad \frac{T\ \alpha \wedge \beta}{T\ \alpha \quad T\ \beta}\ (T\wedge)$$

$$\frac{T\ \alpha \vee \beta \quad F\ \alpha}{T\ \beta}\ (T\vee_1) \qquad \frac{T\ \alpha \vee \beta \quad F\ \beta}{T\ \alpha}\ (T\vee_2) \qquad \frac{F\ \alpha \vee \beta}{F\ \alpha \quad F\ \beta}\ (F\vee)$$

$$\frac{T\ \neg\alpha}{F\ \alpha}\ (T\neg) \qquad \frac{F\ \neg\alpha}{T\ \alpha}\ (F\neg) \qquad \frac{}{T\ \alpha \quad F\ \alpha}\ (\mathrm{PB})$$

3.2 Example

- $S = \varnothing$.
- $\neg\alpha \vee \beta \vdash \alpha \rightarrow \beta$?

1.	$T\ \neg\alpha \vee \beta$		Initial Configuration
2.	$F\ \alpha \rightarrow \beta$		

3.3 Example (cont)

- $S = \varnothing$.
- $\neg\alpha \vee \beta \vdash \alpha \to \beta$?

1.	$T\ \neg\alpha \vee \beta$	Next Rule: $(F\ \to)$
2.	$F\ \alpha \to \beta$	$\dfrac{F\ \varphi \to \psi}{\begin{array}{c} T\ \varphi \\ F\ \psi \end{array}}$

3.4 Example (cont)

- $S = \varnothing$.
- $\neg\alpha \vee \beta \vdash \alpha \rightarrow \beta$?

		Next Rule: $(T\ \vee)$
1.	$T\ \neg\alpha \vee \beta$	$T\ \varphi \vee \psi$
2.	$F\ \alpha \rightarrow \beta$	$F\ \psi$
3.	$T\ \alpha$	
4.	$F\ \beta$	$T\ \varphi$

$$\frac{T\ \varphi \vee \psi \qquad F\ \psi}{T\ \varphi}$$

3.5 Example (cont)

- $S = \varnothing$.

1.	$T\ \neg\alpha \vee \beta$	Cannot apply: $(T\ \neg)$
2.	$F\ \alpha \rightarrow \beta$	$\dfrac{T\ \neg\varphi}{F\ \varphi}$
3.	$T\ \alpha$	
4.	$F\ \beta$	only applicable if $\varphi \in S$
5.	$T\ \neg\alpha$	
	?	

- Rule $(T\ \neg)$ is blocked because $\alpha \notin S = \varnothing$.
- The tableau does not close: in $S_3(\varnothing)$, $\neg\alpha \vee \beta \nvdash \alpha \rightarrow \beta$.
- Indeed, in $S_3(\varnothing)$, $\neg\alpha \vee \beta \not\models_S^{3.2} \alpha \rightarrow \beta$.

4. An Incremental Method to Compute S

- The idea:

  When the KES_3 tableau is blocked, expand S so as to unblock it.

- By expanding form S to $S' \supset S$, we are changing the logic.
- No need to redo the tableau. Expansion can resume from where it stopped.
- The bigger S, the closer to classical logic.
- When S contains all propositions in the sequent, we are in classical logic.

4.1 Example (cont)

- $S = \{\alpha\}$.

1.	$T\ \neg\alpha \vee \beta$	**Can apply:** $(T\ \neg)$
2.	$F\ \alpha \to \beta$	$\dfrac{T\ \neg\varphi}{F\ \varphi}$
3.	$T\ \alpha$	
4.	$F\ \beta$	for now $\varphi \in S$
5.	$T\ \neg\alpha$	

4.2 Example (final)

- $S = \{\alpha\}$.

		Branch Closing Condition
1.	$T\ \neg\alpha \vee \beta$	$T\ \varphi$
2.	$F\ \alpha \to \beta$	$F\ \varphi$
3.	$\boxed{T\ \alpha}$	$\times$
4.	$F\ \beta$	
5.	$T\ \neg\alpha$	
6.	$\boxed{F\ \alpha}$	
	$\times$	

- The tableau is closed: $\neg\alpha \vee \beta \vdash \alpha \to \beta$ is derivable in $S_3(\{\alpha\})$.
- Indeed, in $S_3(\{\alpha\})$, $\neg\alpha \vee \beta \models_S^{3.2} \alpha \to \beta$.

5. Static Properties

- Given S, what is $Th(S_3(S))$? (expressivity)
- For a fixed S, KE$S_3(S)$ proves more than CS$S_3(S)$.

$$
\begin{array}{c}
T\ \neg l \vee \alpha \\
T\ l \vee \beta \\
F\ \alpha \vee \beta \\
\hline
F\ \alpha \\
F\ \beta \\
T\ \neg l \\
F\ l\ (l \in S) \\
T\ l \\
\times
\end{array}
\qquad
\begin{array}{c}
T\ l \to \alpha \\
T\ l \vee \beta \\
F\ \alpha \vee \beta \\
\hline
F\ \alpha \\
F\ \beta \\
F\ l \\
T\ l \\
\times
\end{array}
$$

6. Dynamic Properties

- How do we expand S to $S' \supset S$? (control)
- $\mathrm{KE}S_3$ can linearly simulate the dynamics of $\mathrm{CS}S_3$, generating the same S.
- For the same ΔS, $\Delta T(\mathrm{CS}S_3) \subseteq \Delta T(\mathrm{KE}S_3)$.

6.1 Control

- In CSS_3, S controls the atoms over which resolution can be applied.
- In KES_3, S controls the formulas over which $(T\neg)$ can be applied.
- If we eliminate $\neg$-formulas we reduce the control of S on KES_3.
- How can we recover control?

6.2 Recovering control

- We can restrict the application of Modus Ponens:

$$\frac{\begin{array}{c}T\ \alpha \to \beta\\ T\ \alpha\end{array}}{T\ \beta \text{ if } \alpha \in S^T_\to} \qquad \frac{T\ \neg\alpha}{F\ \alpha \text{ if } \alpha \in S^T_\neg}$$

- With this, the two systems have the same expressivity and control.
- But KES_3 allows for fine-tuning (e.g. different weights for the rules).

Conclusions and Future Work

We have:

- Extended S_3 to full propositional logic.
- Given a proof method for the extended system.
- Compared the new system to the original one.
- Shown how to add control.

Future work includes:

- Studying the computational complexity of the proof method.
- Applications.