ORBITAL STABILITY OF SOLITARY WAVE SOLUTIONS FOR AN INTERACTION EQUATION OF SHORT AND LONG DISPERSIVE WAVES.

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Abstract. We study the existence and orbital stability of solitary wave solutions for an interaction equation between a long internal wave and a short surface wave in a two layer fluid. If the short wave term is denoted by $u = u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ and the long wave term by $v = v(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the phenomena of interaction is described by the following equation (Funakoshi and Oikawa (J. Phys. Soc. Japan, **52** (1983), 1982-1995),

$$\begin{cases} iu_t + u_{xx} = \alpha vu, \\ v_t + \gamma\mathcal{H}v_{xx} = \beta(|u|^2)_x, \end{cases}$$

where $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$ and $\mathcal{H}$ is the Hilbert transform. Via the Implicit Function Theorem we show the existence de smooth real solutions ϕ and ψ of the system

$$\begin{cases} \phi'' - \sigma\phi = \alpha\,\psi\phi \\ \gamma\mathcal{H}\psi' - c\psi = \beta\phi^2, \end{cases}$$

where $\sigma, c > 0$ and γ in some neighbourhood of zero. Moreover, using perturbation theory of closed operators on Hilbert spaces we show that the functions $\Phi(\xi) = e^{ic\xi/2}\phi(\xi)$ and $\Psi(\xi) = \psi(\xi)$ are solutions orbitally stable in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$, at least when γ is negative near zero .

1. INTRODUCTION

In this work we continue our study on an interaction equation between a long internal wave and a short surface wave in a two layer fluid when the fluid depth of the lower layer

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is sufficiently large in comparison with the wavelength of the internal wave. The fluids are assumed with different densities, inviscid and incompressible, and their motions to be two-dimensional and irrotational. If the short wave term is denoted by $u = u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ and the long wave term by $v = v(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the phenomena of interaction is described by the following nonlinear coupled system (see Funakoshi and Oikawa [14]),

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_t + \gamma D v_x = \beta(|u|^2)_x, \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where α, β are positive constants, $\gamma \in \mathbb{R}$ and $D = \mathcal{H}\partial_x$, is a linear differential operator representing the dispersion of the internal wave. Here $\mathcal{H}$ denotes the Hilbert transform defined by

$$\mathcal{H}f(x) = p.v. \frac{1}{\pi}\int \frac{f(y)}{y-x}\,dy$$

therefore, D is the multiplier with Fourier operator defined as $\widehat{Dv}(\xi) = |\xi|\widehat{v}(\xi)$. Here the circumflex over a function denotes the function Fourier transform.

Several results for Eq. (1.1) have been obtained. Funakoshi and Oikawa ([14]) computed numerical solitary wave solutions. Angulo and Montenegro ([4]) proved the existence of solitary wave solutions via the concentration compactness method (Lions [18], [19]), as well as, the evenness and analyticity of these solutions. Bekiranov, Ogawa and Ponce ([6]) proved the local well-posedness theory in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$. More precisely, if $|\gamma| < 1$ and $s \geqq 0$ then for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ there exists $T > 0$ such that the initial value problem (1.1) admits a unique solution $(u(t), v(t)) \in C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^{s-\frac{1}{2}}(\mathbb{R}))$. Moreover, for $T > 0$ the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^{s-\frac{1}{2}}(\mathbb{R}))$. For the case $|\gamma| = 1$, they obtain the same results as above, but for $s > 0$. We note that as consequence of the relations (1.2) and (1.3) below, T can be chosen arbitrarily large if $\gamma \leqq 0$ and $s = 1$.

For any $s \geqq 0$, we have that the solution u preserves its $L^2(\mathbb{R})$-norm, *i.e.,* if

$$H(u) = \int_{\mathbb{R}} |u(x)|^2\,dx \tag{1.2}$$

then for any $0 < t < T$, $H(u(t)) = H(u_0)$. Moreover, we have the conservation laws of momentum and energy:

$$
\begin{aligned}
G(u,v) &\equiv Im \int_{\mathbb{R}} u(x)\overline{u_x(x)}\,dx + \frac{\alpha}{2\beta}\int_{\mathbb{R}} |v(x)|^2\,dx,\\
E(u,v) &\equiv \int_{\mathbb{R}} |u_x(x)|^2\,dx + \alpha\int_{\mathbb{R}} v(x)|u(x)|^2\,dx - \frac{\alpha\gamma}{2\beta}\int_{\mathbb{R}} |D^{1/2}v(x)|^2\,dx,
\end{aligned}
\tag{1.3}
$$

where $D^{1/2}$ is the multiplier with Fourier operator defined as $\widehat{D^{1/2}v}(\xi) = |\xi|^{\frac{1}{2}}\widehat{v}(\xi)$.

The purpose of this paper is to consider the existence and orbital stability of solitary wave solutions for Eq. (1.1) of the form

$$
\begin{cases} u(x,t) = e^{i\omega t}\varphi(x-ct),\\ v(x,t) = \psi(x-ct), \end{cases} \tag{1.4}
$$

where we have that, $\varphi : \mathbb{R} \to \mathbb{C}$, $\psi : \mathbb{R} \to \mathbb{R}$, are smooth functions such that for each $n \in \mathbb{N}$, $|\varphi^{(n)}(\xi)| \to 0$, and $\psi^{(n)}(\xi) \to 0$, as $|\xi| \to \infty$, $c > 0$ and $\omega \in \mathbb{R}$. Thus, substituting (u,v) as above in (1.1) we obtain the coupled system of equations

$$
\begin{cases} \varphi'' - \omega\varphi - ic\varphi' = \alpha\,\psi\varphi\\ \gamma\mathcal{H}\psi' - c\psi = \beta\,|\varphi|^2 \end{cases} \tag{1.5}
$$

with "$'$" $= \frac{d}{d\xi}$ and $\xi = x - ct$. Now, if we consider $\varphi(\xi) = e^{ic\xi/2}\phi(\xi)$, for ϕ real-valued, and replace it in (1.5) finally we obtain the pseudo-differential system

$$
\begin{cases} \phi'' - \sigma\phi = \alpha\,\psi\phi\\ \gamma\mathcal{H}\psi' - c\psi = \beta\phi^2, \end{cases} \tag{1.6}
$$

where $\sigma = \omega - \frac{c^2}{4}$.

We shall show the existence of smooth real solutions ϕ, ψ of (1.6) for $\sigma > 0$ and γ in some neighbourhood of zero via the Implicit Function Theorem. Moreover, adapting a method developed by Albert, Bona and Henry ([1]), Benjamin ([7]) and Weinstein ([23], [24]), together with some results from perturbation theory of closed operators on Hilbert spaces (Kato [16]), we shall prove that these solutions are orbitally stable in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ at least when γ is negative near zero.

In comparison with Eq. (1.1), we consider the following system

$$
\begin{cases} iu_t + u_{xx} = \alpha v u\\ v_t = \beta(|u|^2)_x, \end{cases} \tag{1.7}
$$

i.e., $\gamma = 0$ in (1.1). This model is the most typical in the theory of wave interaction and occurs when the fluid depth is sufficiently small in comparison with the wavelength of the internal wave. Eq. (1.7) also has been considered under various settings, see for example, Benney ([8],[9]), Bekiranov, Ogawa and Ponce ([5]), Grimshaw ([15]), Laurençot ([17]), Ma ([20]) and Tsutsumi and Hatano ([22]). In the particular case of the existence and stability theory of solitary wave solutions, the results are more definitive (see [17]), in the sense that solitary waves for (1.7) ($\gamma = 0$ and $\sigma > 0$ in (1.6)) are unique (up to translations) and may be computed explicitly as

$$\begin{cases} \phi_0(\xi) = \sqrt{\frac{2c\sigma}{\alpha\beta}}\,\mathrm{sech}(\sqrt{\sigma}\xi) \\ \psi_0(\xi) = -\frac{\beta}{c}\phi_0^2(\xi), \end{cases} \tag{1.8}$$

however, formulas of the form given in (1.8) have no known counterpart for equations of type (1.6) with $\gamma \neq 0$.

The plan of this paper is as follows. In section 2, we prove the existence and stability of solitary-wave solutions corresponding to values of γ negative near zero. In the Appendix, we briefly review some results about perturbation theory of closed linear operators on Hilbert spaces necessary in the development of our work.

Notations. Throughout this paper we will denote by $\widehat{f}$ the Fourier transform of f, defined as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$. $|f|_{L^p}$ denotes the $L^p(\mathbb{R})$ norm of f, $1 \leqq p \leqq \infty$. In particular, $|\cdot|_{L^2} = \|\cdot\|$ and $|\cdot|_{L^\infty} = |\cdot|_\infty$. We denote by $H^s(\mathbb{R})$ the Sobolev space of all f (tempered distributions) for which the norm $\|f\|_s^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi$ is finite. The product norm in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$ is denoted by $\|\cdot\|_{s\times r}$. $D^s = (-\partial_x^2)^{s/2}$ is the Riesz potential of order $-s$, and is defined by $\widehat{D^s f}(\xi) = |\xi|^s\widehat{f}(\xi)$. $B(X;Y)$ denote the space of all bounded linear operators from X into Y. If $X = Y$, $B(X;Y) = B(X)$. If S is any closed operator on $L^2(\mathbb{R})$, we denote its spectrum by $\Sigma(S)$.

2. EXISTENCE AND STABILITY OF SOLITARY WAVE SOLUTIONS

For any $(c,\omega) \in \mathbb{R}^+ \times \mathbb{R}$ define the functions $\Phi(\xi) = e^{ic\xi/2}\phi(\xi)$ and $\Psi(\xi) = \psi(\xi)$, where (ϕ,ψ) is a solution of (1.6). Then we said that the solitary-wave solution $(U_s(x,t), V_s(x,t)) = (e^{i\omega t}\Phi(x-ct), \Psi(x-ct))$ is orbitally stable in $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that when $(u_0,v_0) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ and satisfies both $\|u_0 - \Phi\|_1 < \delta$

and $\|v_0 - \Psi\|_{\frac{1}{2}} < \delta$, and (u,v) is the solution of (1.1) corresponding to (u_0, v_0), then $(u,v) \in C([0,\infty); H^1(\mathbb{R})) \times C([0,\infty); H^{\frac{1}{2}}(\mathbb{R}))$ and

$$\operatorname*{Inf}_{\substack{x_0\in\mathbb{R}\\ \theta\in[0,2\pi)}} \|e^{i\theta}u(\cdot + x_0, t) - \Phi\|_1 < \epsilon$$

$$\operatorname*{Inf}_{x_0\in\mathbb{R}} \|v(\cdot + x_0, t) - \Psi\|_{\frac{1}{2}} < \epsilon, \tag{2.1}$$

for all $t \in [0,\infty)$.

The main result to be proved in this section is that stable solitary-wave solutions of (1.1) exist when $\sigma > 0$ and γ is negative and near 0.

Theorem 2.1. *Let $\alpha, \beta > 0$, and $\sigma = \omega - \frac{c^2}{4} > 0$, for $c > 0$ and $\omega \in \mathbb{R}$ fixed numbers. Then there exists $\gamma_0 > 0$ such that for each $\gamma \in (-\gamma_0, 0)$, Eq. (1.6) has a solution $(\phi_\gamma, \psi_\gamma)$. Moreover, for $\Phi_\gamma(\xi) = e^{ic\xi/2}\phi_\gamma(\xi)$ and $\Psi_\gamma(\xi) = \psi_\gamma(\xi)$, we have that $(e^{i\omega t}\Phi(x-ct), \Psi(x-ct))$ is a stable solitary-wave solution of (1.1).*

The proof of Theorem 2.1 will proceed by adapting a method developed by Benjamin ([7]), Albert, Bona, and Henry ([1]) and Weinstein ([24]), together with some results on spectral theory of closed linear operators on Hilbert spaces. We begin our study proving the existence of solitary-wave solutions of (1.1). Since our argument is based on the Implicit Function Theorem, we need to establish several facts on the structure spectral of the self-adjoint operator on $L^2(\mathbb{R})$,

$$\mathcal{L}_0 = -\frac{d^2}{d\xi^2} + \sigma - \frac{3\alpha\beta}{c}{\phi_0}^2, \tag{2.2}$$

where ϕ_0 is defined in (1.8). In fact, since ϕ_0' has a single zero and $\phi_0(\xi) \to 0$ as $|\xi| \to \infty$, we obtain from the Sturm-Liouville theory (see, [13], [10]) and perturbation theory of closed linear operators on Hilbert spaces (see, Kato [16]) that $\mathcal{L}_0$ has one simple negative eigenvalue λ_0 with eigenfunction $\varphi_0 > 0$, a simple eigenvalue at zero with eigenfunction ϕ_0', the essential spectrum is the interval $[\sigma, \infty)$, and the remainder of the spectrum of $\mathcal{L}_0$ consist of isolated eigenvalues of finite multiplicity. Moreover, there exists $\eta_0 > 0$ such that for $\beta_0 \in \Sigma(\mathcal{L}_0) - \{\lambda_0, 0\}$, we have that $\beta_0 > \eta_0$.

For $r \geqq 0$, let $H^r_e(\mathbb{R})$ denote the closed subspace of all even functions in $H^r(\mathbb{R})$. In order to prove Theorem 2.1 we have first,

Lemma 2.2. *Let $\alpha, \beta > 0$, and $\sigma = \omega - \frac{c^2}{4} > 0$, for $c > 0$ and $\omega \in \mathbb{R}$ fixed numbers. Then there exists $\gamma_1 > 0$ such that for every $\gamma \in (-\gamma_1, \gamma_1)$, Eq. (1.6) has a solution $(\phi_\gamma, \psi_\gamma) \in H_e^2(\mathbb{R}) \times H_e^1(\mathbb{R})$, and the correspondence $\gamma \to (\phi_\gamma, \psi_\gamma)$ defines a continuous map from $(-\gamma_1, \gamma_1)$ to $H_e^2(\mathbb{R}) \times H_e^1(\mathbb{R})$. In particular, for $\gamma < 0$ and $\gamma \to 0$, $(\phi_\gamma(x), \psi_\gamma(x))$ converges to $(\phi_0(x), \psi_0(x))$, uniformly for $x \in \mathbb{R}$, where ϕ_0 and ψ_0 are defined as in (1.8).*

Proof. Without loss of generality take $\alpha = 1$ and $\beta = 1/2$. Let $X_e = H_e^2(\mathbb{R}) \times H_e^1(\mathbb{R})$ and define a map $G : \mathbb{R} \times X_e \to L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ by

$$G(\gamma, \phi, \psi) = (-\phi'' + \sigma\phi + \phi\psi, -\gamma D\psi + c\psi + \phi^2).$$

A calculation shows that the Fréchet derivative $G_{(\phi,\psi)} = \frac{\partial G(\gamma,\phi,\psi)}{\partial(\phi,\psi)}$ exists on $\mathbb{R} \times X_e$ and is defined as a map from $\mathbb{R} \times X_e$ to $B(X_e; L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R}))$ by

$$G_{(\phi,\psi)}(\gamma, \phi, \psi) = \begin{pmatrix} -\frac{d^2}{d\xi^2} + \sigma + \psi & \phi \\ \phi & -\gamma D + c \end{pmatrix}.$$

From (1.8) it follows that for $\Phi_0 = (\phi_0, \psi_0)$, $G(0, \Phi_0) = \vec{0}^t$ and that the operator $\mathcal{S}_0 = G_{(\phi,\psi)}(0, \Phi_0)$ has a one-dimensional nullspace $\mathcal{N}(\mathcal{S}_0)$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. In fact, it follows immediately from (1.8) that $\mathcal{S}_0({\Phi_0'}^t) = \vec{0}^t$, and hence $\Phi_0' \in \mathcal{N}(\mathcal{S}_0)$. Now, we consider $\Psi = (f, g)$ such that $\mathcal{S}_0(\Psi^t) = \vec{0}^t$, then f satisfies the differential equation

$$-f'' + \sigma f - \frac{3}{2c}{\phi_0}^2 f = 0.$$

Thus, $f \in \mathcal{N}(\mathcal{L}_0)$ and therefore $f = \eta_1 {\phi_0}'$ for some $\eta_1 \in \mathbb{R}$. Since $g = -\frac{1}{c}\phi_0 f$ it follows that $\Psi = \eta_1 \Phi_0'$ and therefore $\mathcal{N}(\mathcal{S}_0)$ is generated by Φ_0'. Since $\Phi_0' \notin X_e$, it follows that $S_0 : X_e \to L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ is invertible. Finally, since G and $G_{(\phi,\psi)}$ are continuous maps on their domains, we have from the Implicit Function Theorem that there exist a number $\gamma_1 > 0$ and a continuous map $\gamma \to (\phi_\gamma, \psi_\gamma)$ from $(-\gamma_1, \gamma_1)$ to X_e such that $G(\gamma, \phi_\gamma, \psi_\gamma) = 0$ for all $\gamma \in (-\gamma_1, \gamma_1)$. This shows the Lemma. ■

Remark 2.3. Notice that for γ near 0, ϕ_γ obtained in Lemma 2.2 is strictly positive. In fact, since $\psi(\xi) \to 0$ as $|\xi| \to \infty$ there exists $M > 0$ such that for each $x \geqq M$, $\sigma + \alpha\psi(x) > \frac{\sigma}{2}$, thus

$$-\phi_\gamma(x)\phi_\gamma'(x) \geqq \frac{\sigma}{2}\int_x^\infty \phi_\gamma^2(y)dy + \int_x^\infty (\phi_\gamma'(y))^2 dy > 0.$$

Therefore, $\phi_\gamma(x) \neq 0$ for each $x \geqq M$. Now, since ϕ_0 is strictly positive and $\phi_\gamma \to \phi_0$ as $\gamma \to 0$ uniformly in $[0, M]$, we have that for γ near zero $\phi_\gamma(x) > 0$ for $x \in [0, M]$. Thus from the continuity of ϕ_γ we obtain the affirmation.

The existence of the desired family of solitary-wave solutions of Eq. (1.6) established in Theorem 2.1 has now been demonstrated, and it remains to prove that these solitary waves are stable at least when γ is negative and near zero. In order to show this, we need to establish some results on the spectrum of a operator associated with the solitary wave $(\phi_\gamma, \psi_\gamma)$, it will necessary to use some results from perturbation theory of closed linear operator on Hilbert spaces (ch. IV-V of Kato [16]). The linear, self-adjoint, closed, and unbounded operator on $L^2(\mathbb{R})$ of our interest here is given by

$$\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma - 2\alpha\beta\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma, \tag{2.3}$$

where $(\phi_\gamma, \psi_\gamma)$ is a solution of (1.6) for $\gamma \leqq 0$ obtained in Lemma 2.2, the operator $\mathcal{K}_\gamma^{-1}$, defined as $\widehat{\mathcal{K}_\gamma^{-1} f}(\xi) = \frac{1}{-\gamma|\xi|+c}\widehat{f}(\xi)$, is the inverse operator of $\mathcal{K}_\gamma \in B(H^s(\mathbb{R}); H^{s-1}(\mathbb{R}))$, $s \geqq 1$, defined by $\mathcal{K}_\gamma = -\gamma D + c$, and $\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma$ is given by $[\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma](f) = \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f)$.

When $\gamma = 0$ in (2.3) we obtain from (1.8) the operator $\mathcal{L}_0$ defined in (2.2). The following Lemma shows that for γ negative and near zero, the spectrum of $\mathcal{L}_\gamma$ is similar to $\mathcal{L}_0$.

Lemma 2.4. *There exists a constant $\gamma_2 > 0$ such that if $\gamma \in (-\gamma_2, 0)$, then the self-adjoint operator $\mathcal{L}_\gamma$ on $L^2(\mathbb{R})$, defined in (2.3) with domain $H^2(\mathbb{R})$, has the following properties:*
(1) $\mathcal{L}_\gamma$ has simple negative eigenvalue λ_γ with eigenfunction φ_γ, such that $\int_{\mathbb{R}} \varphi_\gamma \phi_\gamma dx > 0$.
(2) $\mathcal{L}_\gamma$ has a simple eigenvalue at zero with eigenfunction ϕ'_γ, and
(3) There is $\eta_\gamma > 0$ such that for $\beta_\gamma \in \Sigma(\mathcal{L}_\gamma) - \{\lambda_\gamma, 0\}$, we have that $\beta_\gamma > \eta_\gamma$.

Moreover, the essential spectrum of $\mathcal{L}_\gamma$ is the interval $[\sigma, \infty)$, $\lambda_\gamma \to \lambda_0$ as $\gamma \to 0^-$, and $\varphi_\gamma \to \varphi_0$ as $\gamma \to 0^-$ in $L^2(\mathbb{R})$-norm, where φ_0 is the eigenfunction of the operator $\mathcal{L}_0$ associated to the eigenvalue λ_0.

Proof. Without loss of generality take $\alpha = 1$ and $\beta = 1/2$. First we state the following facts:

(a) $\lim_{\gamma\to 0^-} \phi_\gamma(x) = \phi_0(x)$, and $\lim_{\gamma\to 0^-} \psi_\gamma(x) = \psi_0(x)$, uniformly for $x \in \mathbb{R}$,

(b) $\lim_{\gamma\to 0^-} \widehat{\delta}(\mathcal{L}_\gamma, \mathcal{L}_0) = 0$,

where $\widehat{\delta}$ is a metric on $C(L^2(\mathbb{R}))$, the space of closed operators on $L^2(\mathbb{R})$ (see Appendix).

Part (a) follows from Lemma 2.2. To prove part (b), first write $\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \mathcal{M}_\gamma$ and $\mathcal{L}_0 = -\frac{d^2}{d\xi^2} + \mathcal{M}_0$, where $\mathcal{M}_\gamma = \sigma + \psi_\gamma - \phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma$ and $\mathcal{M}_0 = \sigma - \frac{3}{2c}{\phi_0}^2$. Next, note that $\mathcal{M}_\gamma - \mathcal{M}_0 \in B(L^2(\mathbb{R}))$, with operator norm tending to 0 as $\gamma \to 0^-$. In fact, first one has for $f \in L^2(\mathbb{R})$ that

$$\|(\mathcal{M}_\gamma - \mathcal{M}_0)f\| \leqq |\psi_\gamma - \psi_0|_\infty \|f\| + \|\frac{1}{c}\phi_0^2 f - \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f)\|. \tag{2.4}$$

Next, we estimate the second norm on the right hand side of (2.4). From the relations

$$\frac{1}{c}\phi_0^2 f - \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f) = \frac{1}{c}\phi_0^2 f - \phi_0 \mathcal{K}_\gamma^{-1}(\phi_0 f) - (\phi_\gamma - \phi_0)\mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma \mathcal{K}_\gamma^{-1}((\phi_\gamma - \phi_0)f)$$

and $\|\mathcal{K}_\gamma^{-1}(g)\| \leqq \frac{1}{c}\|g\|$ we have that,

$$\begin{aligned}\|\frac{1}{c}\phi_0^2 f - \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f)\| \leqq |\phi_0|_\infty \|[\frac{1}{c} - \mathcal{K}_\gamma^{-1}](\phi_0 f)\| + \frac{1}{c}|\phi_0|_\infty |\phi_\gamma - \phi_0|_\infty \|f\| \\ + \frac{1}{c}|\phi_\gamma|_\infty |\phi_\gamma - \phi_0|_\infty \|f\|.\end{aligned} \tag{2.5}$$

Now, for the kernel K_μ defined by

$$K_\mu(x) = \frac{1}{\pi}\int_0^\infty \frac{e^{-x\tau}}{\mu^2 + \tau^2}\, d\tau, \qquad \text{for } x > 0,$$

and $K_\mu(x) = K_\mu(-x)$ for $x < 0$, we have that $K'_\mu \in L^1(\mathbb{R})$. Moreover, if $\mu = \frac{-c}{\gamma}$ then $\widehat{K_\mu}(\xi) = \frac{1}{|\xi|+\mu}$. Thus, it follows from the Plancherel Theorem and Young's inequality that

$$\begin{aligned}\|[\frac{1}{c} - \mathcal{K}_\gamma^{-1}](\phi_0 f)\|^2 &= \frac{1}{c^2}\int_{\mathbb{R}} |\xi|^2 |\widehat{K_\mu}(\xi)\widehat{\phi_0 f}(\xi)|^2\, d\xi \\ &= \frac{1}{c^2}\|K'_\mu * (\phi_0 f)\|^2 \leqq \frac{1}{c^2}|\phi_0|_\infty^2 |K'_\mu|_{L^1}^2 \|f\|^2.\end{aligned} \tag{2.6}$$

Therefore, from the equality $|K'_\mu|_{L^1} = \frac{1}{\mu}$, item (a) above, and (2.5), (2.6), it follows from (2.4) that $\|\mathcal{M}_\gamma - \mathcal{M}_0\|_{B(L^2)} \to 0$ as $\gamma \to 0^-$. Hence, from Theorem A.1 in the Appendix we have that

$$\begin{aligned}\widehat{\delta}(\mathcal{L}_\gamma, \mathcal{L}_0) &= \widehat{\delta}\Big(-\frac{d^2}{d\xi^2} + \mathcal{M}_\gamma, -\frac{d^2}{d\xi^2} + (M_0 - M_\gamma) + \mathcal{M}_\gamma\Big) \\ &\leqq 2(1 + \|\mathcal{M}_\gamma\|_{B(L^2)}^2)\, \widehat{\delta}\Big(-\frac{d^2}{d\xi^2}, -\frac{d^2}{d\xi^2} + M_0 - M_\gamma\Big) \\ &\leqq 2(1 + \|\mathcal{M}_\gamma\|_{B(L^2)}^2)\, \|\mathcal{M}_\gamma - \mathcal{M}_0\|_{B(L^2)},\end{aligned}$$

and since $\|\mathcal{M}_\gamma\|_{B(L^2)}$ is uniformly bounded for γ, we have immediately the advertised result in (b).

We now turn to the proof of the Lemma. That the essential spectrum of $\mathcal{L}_\gamma$ is $[\sigma, \infty)$ it follows from the fact that the operator $\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma$ is relatively compact with respect to the operator $\mathcal{J} = -\frac{d^2}{d\xi^2} + \sigma + \psi_\gamma$, because $|\phi_\gamma(\xi)| \to 0$ as $|\xi| \to \infty$, $\mathcal{K}_\gamma^{-1}$ is a bounded operator, and the essential spectrum of $\mathcal{J}$ is $[\sigma, \infty)$ (see [2], [16]). Now, from Theorem A.1 in the Appendix and following similar arguments to those of the proof of Theorem 5 of Albert, Bona and Henry ([1]) we have that there exists a positive constant γ_2 such that for each $\gamma \in (-\gamma_2, 0)$, the spectrum of $\mathcal{L}_\gamma$ have the properties established in the Lemma. Finally, since $\int_{\mathbb{R}} \varphi_0 \phi_0 dx > 0$ we have from Lemma 2.2 and Theorem A.1 that for γ small $\int_{\mathbb{R}} \varphi_\gamma \phi_\gamma dx > 0$. ■

Remark 2.5. We note that another proof of the existence of unique negative eigenvalue (simple) and that 0 is a nondegenerate eigenvalue, can be given using the min-max principle. In fact, for $f \in D(\mathcal{L}_\gamma)$, we have that

$$
\begin{aligned}
< \mathcal{L}_\gamma f, f > = < \mathcal{L}_0 f, f > -\frac{\gamma}{c}\alpha^2 < \phi_0 f, D\mathcal{K}_\gamma^{-1}(\phi_0 f) > +\alpha \int_{\mathbb{R}} (\psi_\gamma - \psi_0) f^2 \, dx \\
+ \alpha^2 \int_{\mathbb{R}} \left[\phi_0 f \mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f \mathcal{K}_\gamma^{-1}(\phi_\gamma f) \right] dx \\
\geqq < \mathcal{L}_0 f, f > +\alpha \int_{\mathbb{R}} (\psi_\gamma - \psi_0) f^2 \, dx + \alpha^2 \int_{\mathbb{R}} \left[\phi_0 f \mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f \mathcal{K}_\gamma^{-1}(\phi_\gamma f) \right] dx
\end{aligned}
$$

where the last inequality is due to that $D\mathcal{K}_\gamma^{-1}$ is a positive operator and $\gamma < 0$. Thus, if $\|f\| = 1$ and $f \perp \varphi_0$, $f \perp \phi_0'$ then from the spectral structure of $\mathcal{L}_0$ and item (a) in the proof of Lemma 2.4 it follows that for γ near zero $< \mathcal{L}_\gamma f, f > > \eta_0/2$. Therefore, from min-max principle ([21]) we have the advertised result.

As the stability considered here is with respect to form, *i.e.,* up to translation in space and phase, it is propitious to introduce the following map on $H^1(\mathbb{R})$ (see [7], [11], [12] and [24]), namely, consider $(\phi_\gamma, \psi_\gamma)$ a solution of (1.6) obtained in Lemma 2.2 for $\gamma < 0$. If $(u_0, v_0) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ and (u, v) is the solution to (1.1) corresponding to these initial data (see [6]), we define for all $t \geqq 0$

$$
\rho_\sigma(u(\cdot, t), \phi_\gamma)^2 \equiv \operatorname*{Inf}_{\substack{x_0 \in \mathbb{R} \\ \theta \in [0, 2\pi)}} \{ \| e^{i\theta} (T_c u)'(\cdot + x_0, t) - \phi_\gamma'(\cdot) \|^2 + \sigma \| e^{i\theta} (T_c u)(\cdot + x_0, t) - \phi_\gamma(\cdot) \|^2 \}, \tag{2.7}
$$

where we denote by T_c the bounded linear operator from $L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}))$ to itself defined by $(T_c u)(x,t) = e^{-ic(x-ct)/2}u(x,t)$.

The following Lemma stating essential properties of the map ρ_σ (analog of Lemma 1 in [11]), will show that there exist maps $\theta = \theta(t)$ and $x_0 = x_0(t)$ minimizing the function

$$\Omega_t(x_0,\theta) = \|e^{i\theta}(T_c u)'(\cdot + x_0, t) - \phi'_\gamma(\cdot)\|^2 + \sigma\|e^{i\theta}(T_c)u(\cdot + x_0, t) - \phi_\gamma(\cdot)\|^2.$$

Lemma 2.6. *We consider $(u_0, v_0) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ such that $\|u_0\| = \|\phi_\gamma\|$ and (u,v) the solution to (1.1) corresponding to this initial data. Suppose that for some $t_0 \in [0,\infty)$ and some $(\widetilde{x_0}, \widetilde{\theta}) \in \mathbb{R} \times [0, 2\pi)$, it is the case that*

$$\Omega_{t_0}(\widetilde{x_0}, \widetilde{\theta}) < \sigma\|\phi_\gamma\|^2. \tag{2.8}$$

Then, it follows that

$$\mathit{Inf}\,\{\Omega_{t_0}(x_0,\theta)|\ (x_0,\theta) \in \mathbb{R} \times [0,2\pi)\} \tag{2.9}$$

is attained at least once in $\mathbb{R} \times [0, 2\pi)$.

Proof. It is immediate that $\Omega_{t_0}(x_0, \theta)$ is a continuous function of (x_0, θ) on $\mathbb{R} \times [0, 2\pi)$. Moreover, for any $\theta \in [0, 2\pi)$, we have

$$\lim_{|x_0|\to\infty} \Omega_{t_0}(x_0,\theta) = \|(T_c u)'(\cdot, t_0)\|^2 + \|\phi'_\gamma\|^2 + 2\sigma\|\phi_\gamma\|^2. \tag{2.10}$$

The hypothesis (2.8), continuity of Ω_{t_0}, and (2.10) imply the result. ■

Next, it is established that the infimum in (2.9) is attained at points (x_0, θ) at least for t_0 in some interval of the form $[0, T]$. To this end, it is sufficient to obtain condition (2.8) in such an interval. Let $\epsilon > 0$ be such that $\epsilon^2 < \sigma\|\phi_\gamma\|^2/2(1+\omega)$. The solitary-wave solution $(U_s(x,t), V_s(x,t)) = (e^{i\omega t}e^{ic(x-ct)/2}\phi_\gamma(x-ct), \psi_\gamma(x-ct))$ is globally defined. Hence from continuous dependence theory for (1.1) with $|\gamma| < 1$, established in Bekiranov, Ogawa and Ponce ([6]), we deduced that for $T > 0$ there exists a $\delta > 0$ such that if $\|(u_0, v_0) - (e^{icx/2}\phi_\gamma, \psi_\gamma)\|_{1\times\frac{1}{2}} < \delta$, then the solution (u, v) of (1.1) corresponding to (u_0, v_0) exists at least for $0 \leqq t \leqq T$. Moreover

$$\|(u(\cdot,t), v(\cdot,t)) - (U_s(\cdot,t), V_s(\cdot,t))\|_{1\times\frac{1}{2}} < \epsilon,$$

for all $t \in [0,T]$. Now, since

$$\|e^{-i\omega t}\frac{d}{dy}(T_c u)(\cdot + ct, t) - \frac{d}{dy}\phi_\gamma(\cdot)\|^2 \leqq 2\|u'(\cdot,t) - U_s'(\cdot,t)\|^2 + \frac{2c^2}{4}\|u(\cdot,t) - U_s(\cdot,t)\|^2$$
$$< 2(1+\frac{c^2}{4})\epsilon^2$$

it follows that $\Omega_t(ct, -\omega t) < 2(1+\omega)\epsilon^2$, and therefore we have (2.8) because of the value of ϵ just specified above. Thus, the infimum (2.9) is taken on at values $(x_0(t), \theta(t))$ throughout the time-interval $[0,T]$.

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1, (Stability). We consider the functions $\Phi_\gamma(\xi) = e^{ic\xi/2}\phi_\gamma(\xi)$ and $\Psi_\gamma(\xi) = \psi_\gamma(\xi)$, where $(\phi_\gamma, \psi_\gamma)$ is the solution of (1.6) obtained in Lemma 2.2 and γ is negative close to zero such that Remark 2.3, Lemma 2.4 are true. Moreover, we consider $(u_0, v_0) \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ such that $\|u_0\| = \|\phi_\gamma\|$ and (u,v) the solution to (1.1) corresponding to this initial data.

The proof of stability is based on the continouos functional L defined on $H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ by

$$L(u,v) = E(u,v) + c\,G(u,v) + \omega\,H(u,v)$$

where E, G, H are defined by (1.2) and (1.3). We observe that for (u,v) solution of (1.1), $L(u(t), v(t)) = L(u_0, v_0)$ at any time $t \in [0,\infty)$.

To prove (2.1), write a renormalized version of (u,v), namely,

$$\xi(x,t) = e^{i\theta}(T_c u)(x + x_0, t) - \phi_\gamma(x)$$
$$\eta(x,t) = v(x + x_0, t) - \psi_\gamma(x) \tag{2.11}$$

where $\theta = \theta(t)$ and $x_0 = x_0(t)$ are chosen such that the infimum (2.9) is take on at this values, at least for $t \in [0,T]$. Thus, if $p(x,t) = \mathrm{Re}(\xi(x,t))$ and $q(x,t) = \mathrm{Im}(\xi(x,t))$ the result of Lemma 2.6 together with (1.6) provide us with compatibility relations on p and q, namely

$$\int_{\mathbb{R}} q(x,t)\phi_\gamma(x)\psi_\gamma(x)dx = 0, \tag{2.12}$$

and

$$\int_{\mathbb{R}} p(x,t)(\phi_\gamma(x)\psi_\gamma(x))'dx = 0 \tag{2.13}$$

for all $t \in [0, T]$.

Now, using the representation (2.11) and (1.6) we have

$$\begin{aligned}\Delta L(t) &:= L(u(t), v(t)) - L(\Phi_\gamma, \Psi_\gamma) = L(\Phi_\gamma + e^{icx/2}\xi, \psi_\gamma + \eta) - L(\Phi_\gamma, \psi_\gamma)\\ &=< \mathcal{L}_\gamma p, p > + < \mathcal{L}_\gamma^+ q, q > + \frac{\alpha}{2\beta}\int_{\mathbb{R}} \left[\mathcal{K}_\gamma^{1/2}\eta + 2\beta\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p) + \beta\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\right]^2 dx\\ &- \frac{\alpha\beta}{2}\int_{\mathbb{R}} \left[|\mathcal{K}_\gamma^{-1/2}(p^2+q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\right] dx \qquad (2.14)\end{aligned}$$

where $\mathcal{L}_\gamma$ is defined as in (2.3), $\mathcal{L}_\gamma^+$ is given by

$$\mathcal{L}_\gamma^+ = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma, \qquad (2.15)$$

and $\mathcal{K}_\gamma^{1/2}$, $\mathcal{K}_\gamma^{-1/2}$ are the positive roots of $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^{-1}$ respectively. Now, we need to find a lower bound for $\Delta L(t)$. The first step will be to obtain a suitable lower bound of the last term on the right-hand side of (2.14). In fact, since $\mathcal{K}_\gamma^{-1/2}$ is a bounded operator on $L^2(\mathbb{R})$, ϕ_γ is uniformly bounded, and from the continuous embedding of $H^1(\mathbb{R})$ in $L^4(\mathbb{R})$ and in $L^\infty(\mathbb{R})$, we have that

$$-\frac{\alpha\beta}{2}\int_{\mathbb{R}} \left[|\mathcal{K}_\gamma^{-1/2}(p^2+q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\right] dx \geq -C_1\|\xi\|_1^3 - C_2\|\xi\|_1^4, \quad (2.16)$$

where C_1 and C_2 are positive constants.

The estimates for $< \mathcal{L}_\gamma p, p >$ and $< \mathcal{L}_\gamma^+ q, q >$ are obtained in the next two Lemmas.

Lemma 2.7. *Consider $\gamma < 0$ and near zero such that Lemma 2.4 is true. If $\mathcal{L}_\gamma$ is defined as in (2.3), then*

(*a*) $\inf\{< \mathcal{L}_\gamma f, f > \mid \|f\| = 1, \ < f, \phi_\gamma >= 0\} \equiv D_0 = 0$

(*b*) $\inf\{< \mathcal{L}_\gamma f, f > \mid \|f\| = 1, \ < f, \phi_\gamma >= 0, \ < f, (\phi_\gamma\psi_\gamma)' >= 0\} \equiv D > 0$,

Proof. Proof of part (a). From Lemma 2.4, $\mathcal{L}_\gamma$ has exactly one negative eigenvalue λ_γ (simple) with eigenfunction φ_γ, such that $\int_{\mathbb{R}} \varphi_\gamma\phi_\gamma > 0$ for γ near zero. Moreover, since zero is not an eigenvalue for $\mathcal{L}_0$ in $L_e^2(\mathbb{R})$, we have that $\mathcal{L}_\gamma$ when considered as an operator on $L_e^2(\mathbb{R})$, it does not have zero in its spectrum for values of γ sufficiently close to zero, because $\lim_{\gamma\to 0^-} \widehat{\delta}(\mathcal{L}_\gamma, \mathcal{L}_0) = 0$. Thus $\mathcal{L}_\gamma^{-1}$ is well-defined on $L_e^2(\mathbb{R})$, and from Theorem IV.

2.25 of Kato ([16]) and Lemma 2.2, it follows that the function $\chi_\gamma = \mathcal{L}_\gamma^{-1}(\phi_\gamma)$ depends continuously on $\gamma \leqq 0$ in the $L^2(\mathbb{R})$-norm. Now, since $\mathcal{L}_0(-\frac{d}{d\sigma}\phi_0) = \phi_0$ and the nullspace of $\mathcal{L}_0$ is spanned by ϕ_0', then $< \chi_0, \phi_0 >=< -\frac{d}{d\sigma}\phi_0, \phi_0 > < 0$. Therefore by continuity $< \chi_\gamma, \phi_\gamma > < 0$ for γ near zero. Thus, as consequence of the last reasoning and Theorem A.3 in the Appendix we have that $D_0 = 0$ for γ in some neighbourhood of zero.

Proof of part (b). Because of part (a), we know that $D \geq 0$. Suppose that $D = 0$ and let $\{f_j\}$ be a sequence of $H^1(\mathbb{R})$-functions with $\|f_j\| = 1$, $< f_j, \phi_\gamma >= 0$, $< f_j, (\phi_\gamma\psi_\gamma)' >= 0$, and $< \mathcal{L}_\gamma f_j, f_j >\to 0$ as $j \to \infty$. Then, for any $\delta > 0$, there is a natural number j_0 such that for $j > j_0$,

$$\begin{aligned} 0 < \sigma \leq \int_{\mathbb{R}} |f_j'|^2 dx + \sigma &\leq -\alpha \int_{\mathbb{R}} \psi_\gamma |f_j|^2 dx + 2\alpha\beta \int_{\mathbb{R}} \phi_\gamma f_j \mathcal{K}_\gamma^{-1}(\phi_\gamma f_j) dx + \delta \\ &\leq \alpha |\psi_\gamma|_\infty + \frac{2\alpha\beta}{c} |\phi_\gamma|_\infty^2 + \delta. \end{aligned} \tag{2.17}$$

Then (2.17) implies that there is a subsequence of $\{f_j\}$, which we denote again by $\{f_j\}$, and a function $f^* \in H^1(\mathbb{R})$ such that $f_j \rightharpoonup f^*$ weakly in $H^1(\mathbb{R})$ and uniformly for each compact subset of $\mathbb{R}$. So f^* satisfies the conditions $f^* \perp \phi_\gamma$ and $f^* \perp (\phi_\gamma\psi_\gamma)'$. Moreover, from the just mentioned properties of the sequence $\{f_j\}$, the decay of $\phi_\gamma(\xi)$ to zero as $|\xi| \to \infty$, and that $\mathcal{K}_\gamma^{-1}$ is a bounded operator, we have

$$\int_{\mathbb{R}} \phi_\gamma |f_j|^2 dx \to \int_{\mathbb{R}} \phi_\gamma |f^*|^2 dx$$
$$\int_{\mathbb{R}} \phi_\gamma f_j \mathcal{K}_\gamma^{-1}(\phi_\gamma f_j) dx \to \int_{\mathbb{R}} \phi_\gamma f^* \mathcal{K}_\gamma^{-1}(\phi_\gamma f^*) dx$$

as $j \to \infty$. Taking the limit in (2.17) as $j \to \infty$ yields

$$0 < \sigma \leq -\alpha \int_{\mathbb{R}} \psi_\gamma |f^*|^2 dx + 2\alpha\beta \int_{\mathbb{R}} \phi_\gamma f^* \mathcal{K}_\gamma^{-1}(\phi_\gamma f^*) dx + \delta.$$

Since $\delta > 0$ is arbitrary and $\psi_\gamma < 0$, it must be the case that $f^* \neq 0$. It is now show that the infimum is achieved. Indeed, weak convergence is lower semi-continuous, so

$$0 \leq < \mathcal{L}_\gamma f^*, f^* > \leq \liminf_{j\to\infty} < \mathcal{L}_\gamma f_j, f_j >= 0$$

Now, define $g^* = \frac{f^*}{\|f^*\|}$. Then, $\|g^*\| = 1$, $g^* \perp \phi_\gamma$, $g^* \perp (\phi_\gamma\psi_\gamma)'$, and $< \mathcal{L}_\gamma g^*, g^* >= 0$. A consequence of the last reasoning is that there exist non-trivial critical points $(g^*, \lambda, \theta, \eta)$

for the Lagrange problem,

$$\begin{cases} \mathcal{L}_\gamma f = \lambda f + \theta \phi_\gamma + \eta(\phi_\gamma \psi_\gamma)' \\ \|f\| = 1, \;\; < f, \phi_\gamma >= 0, \;\; < f, (\phi_\gamma \psi_\gamma)' >= 0. \end{cases} \tag{2.18}$$

Using (2.18) and $< \mathcal{L}_\gamma g^*, g^* >= 0$, it is easily seen that $\lambda = 0$. Taking the inner product of (2.18) with ϕ'_γ, we have from equality $\mathcal{L}_\gamma \phi'_\gamma = 0$ that

$$0 = \eta \int_{\mathbb{R}} \phi'_\gamma (\phi_\gamma \psi_\gamma)' dx, \tag{2.19}$$

but the integral in (2.19) converge to

$$\int_{\mathbb{R}} \phi'_0 (\phi_0 \psi_0)' dx = \frac{-3\beta}{c} \int_{\mathbb{R}} \phi_0^2 (\phi'_0)^2 dx < 0$$

as $\gamma \to 0$. Then, for γ negative and near zero we have from (2.19) that $\eta = 0$ and therefore

$$\mathcal{L}_\gamma g^* = \theta \phi_\gamma. \tag{2.20}$$

Now, taking inner product of (2.20) with $\chi_\gamma = \mathcal{L}_\gamma^{-1} \phi_\gamma$, we have

$$0 =< g^*, \phi_\gamma >=< g^*, \mathcal{L}_\gamma \chi_\gamma >= \theta < \phi_\gamma, \chi_\gamma >,$$

since $< \phi_\gamma, \chi_\gamma >\neq 0$, for γ near zero, it follows that $\theta = 0$ and $\mathcal{L}_\gamma g^* = 0$, therefore from Lemma 2.4 $g^* = \nu \phi'_\gamma$ for some $\nu \neq 0$, which is a contradiction since ϕ'_γ is not orthogonal to $(\phi_\gamma \psi_\gamma)'$ in $L^2(\mathbb{R})$, for γ near zero. Thus, the minimum is positive and the proof of the Lemma is completed. ■

Lemma 2.8. *Consider $\gamma < 0$ near zero such that Remark 2.3 is true. If $\mathcal{L}_\gamma^+$ is defined as in (2.15), then there is a positive value C_0 such that*

$$\inf\{< \mathcal{L}_\gamma^+ f, f > \mid \; \|f\| = 1, \;\; < f, \phi_\gamma \psi_\gamma >= 0\} \equiv C_0 > 0. \tag{2.21}$$

Proof. Because of Remark 2.3, ϕ_γ is strictly positive for γ in some neighbourhood of zero. Moreover, ϕ_γ satisfies that $\mathcal{L}_\gamma^+ \phi_\gamma = 0$, thus it follows that $\mathcal{L}_\gamma^+$ is a non-negative operator on $L^2(\mathbb{R})$. Therefore, the infimum on the left-hand side of (2.21) is non-negative.

Suppose that $C_0 = 0$. Following the proof of part (b) in Lemma 2.7, we have that the minimum is attained at an admissible function $g^* \neq 0$ and there is $(\lambda, \theta) \in \mathbb{R}^2$ such that

$$\mathcal{L}_\gamma^+ g^* = \lambda g^* + \theta \phi_\gamma \psi_\gamma. \tag{2.22}$$

Thus, as a consequence of the condition $< \mathcal{L}_\gamma^+ g^*, g^* >= 0$, we have $\lambda = 0$. Now, taking the inner product of (2.22) with ϕ_γ it is deduced that

$$0 =< \mathcal{L}_\gamma^+ \phi_\gamma, g^* >=< \phi_\gamma, \mathcal{L}_\gamma^+ g^* >= \theta \int_{\mathbb{R}} \phi_\gamma^2 \psi_\gamma dx$$

and therefore $\theta = 0$, because $\phi_\gamma^2 \psi_\gamma < 0$ for γ near zero. Then, since ϕ_γ is the ground state for $\mathcal{L}_\gamma^+$, it follows that $g^* = \nu \phi_\gamma$ for some $\nu \neq 0$, which is a contradiction since ϕ_γ is not orthogonal to $\phi_\gamma \psi_\gamma$ in $L^2(\mathbb{R})$. This completes the proof. ■

We now again turn to the proof of Theorem 2.1. Attention is now turned to estimating the terms $< \mathcal{L}_\gamma p, p >$ and $< \mathcal{L}_\gamma^+ q, q >$ in (2.14), where p and q satisfy the relations (2.12) and (2.13) respectively. Thus, from Lemma 2.8, there exists $C_0 > 0$ such that

$$< \mathcal{L}_\gamma^+ q, q > \geqq C_0 \|q\|^2. \tag{2.23}$$

Then from the particular form of the operator $\mathcal{L}_\gamma^+$ it follows that there is $C_1 > 0$ such that

$$< \mathcal{L}_\gamma^+ q, q > \geqq C_1 \|q\|_1^2. \tag{2.24}$$

Now, suppose without loss of generality that $\|\phi_\gamma\| = 1$. We write $p_\perp = p - p_{||}$, where $p_{||} =< p, \phi_\gamma > \phi_\gamma$. Then, from (2.13) and the positivity of operator $\mathcal{K}_\gamma^{-1}$ it follows that $< p_\perp, (\phi_\gamma \psi_\gamma)' >= 0$, and therefore from Lemma 2.7, it follows $< \mathcal{L}_\gamma p_\perp, p_\perp > \geqq D \|p_\perp\|^2$. Thus, from hypothesis $\|u_0\| = \|\phi_\gamma\|$, $< \mathcal{L}_\gamma \phi_\gamma, \phi_\gamma > < 0$, Cauchy-Schwartz inequality and the specific form of the operator $\mathcal{L}_\gamma$, we obtain

$$< \mathcal{L}_\gamma p, p > \geqq D_1 \|p\|_1^2 - D_2 \|\xi\|_1^3 - D_3 \|\xi\|_1^4, \tag{2.25}$$

with $D_i > 0$.

Finally, collecting the results in (2.16),(2.24),(2.25) and substituting them in (2.14), we obtain

$$\Delta L(t) \geqq d_1 \|\xi\|_1^2 - d_2 \|\xi\|_1^3 - d_3 \|\xi\|_1^4, \tag{2.26}$$

where $d_i > 0$. Therefore, from standard arguments (Bona [11], Weinstein [24]) for $\epsilon > 0$ sufficiently small there is a $\delta(\epsilon) >$ such that if $\|u_0 - \Phi_\gamma\|_1 < \delta(\epsilon)$ and $\|v_0 - \Psi_\gamma\|_{\frac{1}{2}} < \delta(\epsilon)$ then

$$\rho_\sigma(u(t), \phi_\gamma) = \|\xi(t)\|_1 < \epsilon \tag{2.27}$$

for $t \in [0, \infty)$.

Now, it follows from (2.14) and from the above study of ξ that

$$\epsilon \geqq \frac{\alpha}{2\beta} \int_{\mathbb{R}} \left[\mathcal{K}_\gamma^{1/2} \eta + 2\beta \mathcal{K}_\gamma^{-1/2} (\phi_\gamma p) + \beta \mathcal{K}_\gamma^{-1/2} (p^2 + q^2) \right]^2 dx.$$

Thus, from (2.27) and the equivalence of the norms $\|\mathcal{K}_\gamma^{1/2} \eta\|$ and $\|\eta\|_{\frac{1}{2}}$ we obtain (2.1). We have thus proved that $(\Phi_\gamma, \Psi_\gamma)$ is stable relative to small perturbation which preserve the $L^2(\mathbb{R})$ norm of Φ_γ.

Now we discuss the stability relative to general perturbation. First,we remark that from Lemma 2.2 we have that for

$$\begin{cases} f_\gamma(x) = (\frac{1}{c^3})^{1/2} \phi_\gamma(\frac{x}{c}) \\ g_\gamma(x) = \frac{1}{c^2} \psi_\gamma(\frac{x}{c}), \end{cases}$$

the correspondence $\gamma \to (f_\gamma, g_\gamma)$ defines a continuous map from $(-\gamma_1, 0)$ to $H_e^2(\mathbb{R}) \times H_e^1(\mathbb{R})$. Moreover, (f_γ, g_γ) satisfies the system

$$\begin{cases} f_\gamma'' - \frac{\sigma}{c^2} f_\gamma = \alpha f_\gamma g_\gamma \\ \gamma D g_\gamma - g_\gamma = \beta f_\gamma^2. \end{cases} \tag{2.28}$$

Let (u_0, v_0) be such that $\|u_0 - \Phi_\gamma\|_1 < \delta(\epsilon)$ and $\|v_0 - \Psi_\gamma\|_{\frac{1}{2}} < \delta(\epsilon)$, then we can find $c_1 > 0$ and $\omega_1 \in \mathbb{R}$ such that $\frac{\sigma_1}{c_1^2} = \frac{\sigma}{c^2}$ and $c_1\|f_\gamma\| = \|u_0\|$, where $\sigma_1 = \omega_1 - \frac{c_1^2}{4}$. Thus, we have that the functions

$$\begin{cases} \phi_{\gamma,1}(x) = (c_1^3)^{1/2} \, f_\gamma(c_1 x) \\ \psi_{\gamma,1}(x) = c_1^2 \, g_\gamma(c_1 x), \end{cases} \tag{2.29}$$

satisfy the system

$$\begin{cases} \phi_{\gamma,1}'' - \sigma_1 \phi_{\gamma,1} = \alpha \phi_{\gamma,1} \psi_{\gamma,1} \\ \gamma D \psi_{\gamma,1} - c_1 \psi_{\gamma,1} = \beta \phi_{\gamma,1}^2 \end{cases}$$

and $\|\phi_{\gamma,1}\| = \|u_0\|$. Moreover, from the choice of c_1 it follows that $\|\phi_\gamma - \phi_{\gamma,1}\|_1 \leqq \epsilon/2$ (see [3] pg. 17 for a similar situation). Therefore, applying the preceding theory to the case of $\Phi_{\gamma,1} = e^{ic_1\xi/2} \phi_{\gamma,1}(\xi)$ and $\Psi_{\gamma,1}(\xi) = \psi_{\gamma,1}(\xi)$ we have (2.1) for $(\Phi, \Psi) = (\Phi_\gamma, \Psi_\gamma)$. Theorem 2.1 is now established ■

APPENDIX

In this Appendix, we state some facts from perturbation theory of closed linear operators on Hilbert spaces that we have used along this work (see Kato [16] for details).

We consider $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ the Hilbert space with norm defined by $|||(f,g)||| = (\|f\|^2 + \|g\|^2)^{1/2}$, and for any closed operator T on $L^2(\mathbb{R})$ with domain $D(T)$, its graph, $\mathcal{G}(T) = \{(f,g) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) | f \in D(T),\ T(f) = g\}$. Then a metric $\widehat{\delta}$ on $C(L^2(\mathbb{R}))$, the space of closed operators on $L^2(\mathbb{R})$, may be defined as follows: for any $S, T \in C(L^2(\mathbb{R}))$,

$$\widehat{\delta}(S,T) = \|P_S - P_T\|_{B(L^2 \times L^2)},$$

where P_S and P_T are the orthogonal projections on $\mathcal{G}(S)$ and $\mathcal{G}(T)$, and $\|\cdot\|_{B(L^2 \times L^2)}$ denotes the operator norm on the space of bounded operators on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Theorem A.1. *Let $S, T \in C(L^2(\mathbb{R}))$, and suppose A is a bounded operator on $L^2(\mathbb{R})$ with operator norm $\|A\|_{B(L^2)}$. Then*

a) $\widehat{\delta}(T + A, T) \leqq \|A\|_{B(L^2)}$

b) $\widehat{\delta}(S + A, T + A) \leqq 2(1 + \|A\|^2_{B(L^2)})\, \widehat{\delta}(S,T)$.

Theorem A.2. *Let $T \in C(L^2(\mathbb{R}))$ and let $\mathcal{U}$ denote an open subset of the complex plane whose boundary is a smooth contour Γ. Suppose that $\Sigma(T) \cap \Gamma = \emptyset$ and $\Sigma(T) \cap \mathcal{U}$ consists of a finite number of eigenvalue of T, each with finite (algebraic) multiplicity. Then there exists $\delta > 0$ such that if $S \in C(L^2(\mathbb{R}))$ and $\widehat{\delta}(S,T) < \delta$, then $\Sigma(S) \cap \bar{\mathcal{U}}$ consists of a finite number of eigenvalue of finite multiplicity, the sum of their multiplicities being equal to the sum of the multiplicities of the eigenvalues of T in $\mathcal{U}$.*

In particular, suppose $\Sigma(T) \cap \mathcal{U}$ consists of a single, simple eigenvalue λ_0 with eigenfunction f_0. If $\{S_n\}$ is a sequence in $C(L^2(\mathbb{R}))$ such that $\widehat{\delta}(S_n, T) \to 0$ as $n \to \infty$, then for n large, $\Sigma(S_n) \cap \mathcal{U}$ consists of a single simple eigenvalue λ_n, and $\lambda_n \to \lambda_0$ as $n \to \infty$. Moreover, there is an eigenfunction f_n associated to λ_n such that $f_n \to f_0$ as $n \to \infty$ in $L^2(\mathbb{R})$-norm.

Finally, the next result is discussed in Weinstein [23].

Theorem A.3. *Let S be a self-adjoint operator on $L^2(\mathbb{R})$ having exactly one negative eigenvalue λ with corresponding ground-state eigenfunction f_λ and let $g \in \mathcal{N}^{\perp}(S)$. Assume $< g, f_\lambda > \neq 0$ and that*

$$-\infty < \eta \equiv \underset{\substack{\|f\|=1 \\ <f,g>=0}}{Min} < Sf, f > .$$

If $< S^{-1}g, g > \leqq 0$, *then it must be the case that* $\eta \geqq 0$.

REFERENCES

1. J. P . Albert, J .L . Bona and D. Henry, Sufficient conditions for stability of solitary-wave solutions of model equations for long waves, *Physica D* **24** (1987), 343-366.
2. J. P. Albert, J. L. Bona and J.-C. Saut, Model equations for waves in stratified fluids, *Proc. Soc. Lond.* **A 453** (1997), 1233–1260.
3. J. Angulo, J. L. Bona, F. Linares, and M. Scialom, On the structure of singularities in solutions of the nonlinear Schrödinger equation for the critical case, $p = 4/n$, *in* " Nonlinear Theory of Generalized Functions, " Vol. 401, Research Notes in Mathematics, Chapman& Hall/CRC, 1999.
4. J. Angulo and J. F. Montenegro, Existence and evenness of solitary wave solutions for an equation of short and long dispersive waves, submitted.
5. D. Bekiranov, T. Ogawa and G. Ponce, On the well-posedness of Benney's interaction equation of short and long waves, *Advances Diff. Equations, in press.*
6. D. Bekiranov, T. Ogawa and G. Ponce, Interaction equation for short and long dispersive waves, *J. Funct. Anal.* **158** (1998), 357-388.
7. T. B. Benjamin, The Stability of solitary waves, *Proc. Roy. Soc. London A* **338** (1972), 153-183.
8. D. J. Benney, Significant interactions between small and large scale surface waves, *Stud. Appl. Math.* **55** (1976), 93-106.
9. D. J. Benney, A general theory for interactions equation between short and long waves, *Stud. Appl. Math.* **56** (1977), 81-94.
10. F. A. Berezin and M. A. Shubin, " The Schrödinger equation ", Kluwer Academic, 1991.
11. J. L. Bona, On the stability theory of solitary waves, *Proc. Roy. Soc. London A* **344** (1975), 363–374.
12. T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.* **85** (1982), 549–561.
13. E. A. Coddington and N. Levison, " Theory of Ordinary Differential Equations ", McGraw-Hill, New York, 1955.
14. M. Funakoshi and M. Oikawa, The resonant interaction between a long internal gravity wave and a surface gravity wave packet, *J. Phys. Soc. Japan* **52** (1983), 1982–1995.
15. R.H.J. Grimshaw, The modulation of an internal gravity-wave packet and the resonance with the mean motion, *Stud. Appl. Math.* **56** (1977), 241-266.
16. T. Kato, " Perturbation Theory for Linear Operators ", 2nd ed., Springer - Verlag, New York, 1976.
17. Ph. Laurençot, On a nonlinear Schrödinger equation arising in the theory of water waves, *Nonlinear Anal. TMA* **24** (1995), 509-527.
18. P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, *Ann. Inst. H. Poincaré, Anal. Non linéare* **1** (1984), 109–145.
19. P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. H. Poincaré, Anal. Non linéare* **4** (1984), 223–283.
20. Y-C. Ma, The complete solution of the long-wave-short-wave resonance equations, *Stud. Appl. Math.* **59** (1978), 201-221.
21. M. Reed and B. Simon, " Methods of Modern Mathematical Physics: Analysis of Operator, " Vol. IV, Academic Press, New York, 1978.
22. M. Tsutsumi and S. Hatano, Well-posedness of the Cauchy problem for Benney's first equations of long wave short wave interactions, *Funkcialaj Ekvacioj* **37** (1994), 289-316.
23. M. I. Weinstein, Modulation stability of ground states of nonlinear Schrödinger equations, *SIAM J. Math. Anal.* **16** (1985), 472–491.
24. M. I. Weinstein, Liapunov stability of ground states of nonlinear dispersive evolutions equations, *Comm. Pure Appl. Math* **39** (1986), 51–68.