Interval Valued R-Implications and Automorphisms

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Introduction

- Triangular norms (t-norms) model conjunction in a fuzzy semantics
- There are several classes of t-norms.
- Automorphisms act on t-norms generating in most of the cases a new t-norm.
- When we see t-norms as a semi-group operation the automorphism yields an isomorphism between t-norms.
- Fuzzy implications play an important role in fuzzy logic.
- In the literature, several fuzzy implication properties have already been considered and their interrelationship with the other kinds of connectives are generally presented.
- In this paper, we are interested in fuzzy implications associated to fuzzy connectives named R-implications, which are generated by t-norms.

Interval Valued Fuzzy Logics

- Whenever intervals are considered as a particular type of fuzzy set, or interval membership degrees are used in the modeling of the uncertainty in the belief of specialists, it seems natural and interesting to deal with the interval fuzzy approach, i.e. where the membership degrees are intervals instead of punctual values.
- There are several works in the literature on interval valued fuzzy logics. Here we will consider the approach were the interval fuzzy connectives are seen as interval representations of the punctual fuzzy connectives guarantying aspects of optimality and correctness.

Contributions

- The aim of this work is to introduce the concept of interval-valued R-implication and to show that the action of the interval-valued automorphisms preserve the interval-valued R-implications.
- We use an interval constructor for obtaining an interval-valued R-implication from an R-implication canonically, such that the resulting interval implication is the best interval representation of the R-implication.
- We prove that there is a commutativity between the process for obtaining R-implications from t-norms and the process for obtaining interval-valued R-implications from interval-valued t-norms and those canonical interval constructions.
- We also show that the use of automorphism over R-implications, and of interval-valued automorphisms over interval-valued R-implications also commutes when the interval constructor is applied.

Notation

- $U = [0, 1] \subseteq \mathbb{R}$ and $\mathbb{U} = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$.
- The interval set has two projections $l, r : \mathbb{U} \rightarrow U$ defined by $l([a, b]) = a$ and $r([a, b]) = b$, respectively.
- For $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.
- Orders on $\mathbb{U}$:
  1. *Product*: $X \leq Y$ iff $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$;
  2. *Inclusion*: $X \subseteq Y$ iff $\underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

Interval Representations

- $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an *interval representation* of $f : U^n \longrightarrow U$ if, for each $\overrightarrow{X} \in \mathbb{U}^n$ and $\overrightarrow{x} \in \overrightarrow{X}$, $f(\overrightarrow{x}) \in F(\overrightarrow{X})$.
- This notion meet with the notion of correctness of interval computations.
- Let $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ and $G : \mathbb{U}^n \longrightarrow \mathbb{U}$ be two interval representations of the function $f : U \longrightarrow U$. F is a *better interval representation* of f than G, denoted by $G \sqsubseteq F$, if, for each $\overrightarrow{X} \in \mathbb{U}^n$, the inclusion $F(\overrightarrow{X}) \subseteq G(\overrightarrow{X})$ holds.
- For each function $f : U^n \longrightarrow U$, the interval function $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$ defined by

$$\widehat{f}(\overrightarrow{X}) = \left[\inf\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}, \sup\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}\right] \qquad (1)$$

  is *the best interval representation of f* in the sense that for any other interval representation F of f, $F \sqsubseteq \widehat{f}$.
- This notion meet with the notion of optimality of interval computations.

T-norms and Automorphism

- An **t-norm** is a increasing monotonic function $T : U \times U \longrightarrow U$ which is commutative, associative and has 1 as neutral element.
- Typical examples of t-norms are $T_G(x, y) = \min(x, y)$ and $T_L(x, y) = \max(x + y - 1, 0)$.
- A t-norm T is said to be left-continuous whenever $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$.
- An automorphism is a increasing monotonic function $\rho : U \longrightarrow U$ which is bijective.
- Automorphism acts on t-norms to obtain new similar t-norms (in general with the same characteristics than the original).
- $T^{\rho}(x, y) = \rho^{-1}(T(\rho(x), \rho(y)))$.
- Let $\rho(x) = x^2$ then $T_L^{\rho}(x, y) = \max(\sqrt{x^2 + y^2 - 1}, 0)$

Interval T-norms and Interval Automorphism

- An **interval t-norm** is a function $\mathbb{T} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ which is increasing monotonic (w.r.t. the two interval orders), commutative, associative and has $[1, 1]$ as neutral element.
- Proposition: A function $\mathbb{T} : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an interval t-norm iff there exist t-norms T_1 and T_2 such that $T_1 \leq T_2$ and $\mathbb{T} = I[T_1, T_2]$, where

$$I[T_1, T_2](X, Y) = [T_1(\underline{X}, \underline{Y}), T_2(\overline{X}, \overline{Y})]. \tag{2}$$

- Corollary: T is a t-norm iff $\widehat{T}$ is an interval t-norm. In fact, $\widehat{T} = I[T, T]$
- $\widehat{T_G}(X, Y) = \inf(X, Y) = [\min(\underline{XY}), \min(\overline{XY})]$ and $\widehat{T_L}(X, Y) = [\max(\sqrt{\underline{X} + \underline{Y} - 1}, 0), \max(\sqrt{\overline{X} + \overline{Y} - 1}, 0)]$.
- An interval automorphism is a increasing monotonic function $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ (w.r.t. the product order) which is bijective.
- Interval automorphism acts on interval t-norms to obtain new similar interval t-norms.
- $\mathbb{T}^{\varrho}(X, Y) = \varrho^{-1}(\mathbb{T}(\varrho(X), \varrho(Y)))$.

Fuzzy Implication

- Several definitions for fuzzy implication together with related properties have been given
- The unique consensus is that has the same behavior of the classical implication for the crisp case.
- Thus, a binary function $I : U^2 \longrightarrow U$ is a *fuzzy implication* if it satisfies the minimal boundary conditions:

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

Properties of Fuzzy Implications

- Other properties:

$$
\begin{aligned}
\text{I1} &: \text{If } y \le z \text{ then } I(x,y) \le I(x,z);\\
\text{I2} &: I(x, I(y,z)) = I(y, I(x,z));\\
\text{I3} &: I(x,y) = 1 \text{ iff } x \le y;\\
\text{I4} &: \lim_{n\to\infty} I(x, y_n) = I(x, \lim_{n\to\infty} y_n);\\
\text{I5} &: \text{If } x \le z \text{ then } I(x,y) \ge I(z,y);\\
\text{I6} &: I(x,1) = 1;\\
\text{I7} &: I(0,x) = 1;\\
\text{I8} &: I(1,x) = x;\\
\text{I9} &: I(x,y) \ge y;\\
\text{I10} &: I(x,x) = 1;
\end{aligned}
$$

- Proposition: Let I be a fuzzy implication satisfying **I1**, **I2** and **I3**. Then I also satisfies **I5** – **I10**.

R-implications

- Let T be a t-norm. Then the equation

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}, \forall x,y \in [0,1] \quad (3)$$

  defines a fuzzy implication, called *R-implication* or *residuum* of T.
- R-implication is well-defined only if the t-norm is left-continuous.
- A t-norm T is left-continuous iff it satisfies the residuation condition:

$$T(x,z) \leq y \text{ if and only if } I_T(x,y) \geq z. \quad (4)$$

- Proposition: Let $I : U^2 \rightarrow U$ be a fuzzy implication. Then, I is an R-implication with a left-continuous underlying t-norm iff I satisfies the properties **I1** to **I4**.

Interval-valued Fuzzy Implications

- A function $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an *interval fuzzy implication* if the following conditions hold:
$$\begin{aligned} \mathbb{I}([1,1],[1,1]) &= \mathbb{I}([0,0],[0,0]) = \mathbb{I}([0,0],[1,1]) = [1,1]; \\ \mathbb{I}([1,1],[0,0]) &= [0,0]. \end{aligned}$$
- Extra Properties:

$$\begin{aligned}
\mathbb{I}1 &: \text{If } Y \leq Z \text{ then } \mathbb{I}(X,Y) \leq \mathbb{I}(X,Z), \\
\mathbb{I}2 &: \mathbb{I}(X,\mathbb{I}(Y,Z)) = \mathbb{I}(Y,\mathbb{I}(X,Z)), \\
\mathbb{I}3 &: \mathbb{I}(X,Y) = [1,1] \text{ iff } \overline{X} \leq \underline{Y}, \\
\mathbb{I}4a &: \mathbb{I}_Y(X) = \mathbb{I}(X,Y) \text{ is Moore-continuous}, \\
\mathbb{I}4b &: \mathbb{I}_Y(X) = \mathbb{I}(X,Y) \text{ is Scott-continuous}, \\
\mathbb{I}5 &: \text{If } X \leq Z \text{ then } \mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y), \\
\mathbb{I}6 &: \mathbb{I}([0,0],X) = [1,1], \\
\mathbb{I}7 &: \mathbb{I}(X,[1,1]) = [1,1], \\
\mathbb{I}8 &: \mathbb{I}([1,1],X) = X, \\
\mathbb{I}9 &: \underline{\mathbb{I}(X,Y)} \geq Y, \\
\mathbb{I}10 &: \overline{\mathbb{I}(X,X)} = 1
\end{aligned}$$

Relating Fuzzy implication with Interval fuzzy implications

- Proposition: If I is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication.
- Proposition: Let I be a fuzzy implication. Then, for each $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then it holds that $\widehat{I}(X_1, Y_1) \subseteq \widehat{I}(X_2, Y_2)$.
- Theorem: Let I be a fuzzy implication. If I satisfies a property **Ik**, for $k = 1, \ldots, 10$, then $\widehat{I}$ satisfies the property $\mathbb{I}$**k**.
- Corollary Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying **I1**, **I2** and **I3**. Then $\widehat{I}$ satisfies $\mathbb{I}$**1**–$\mathbb{I}$**3** and $\mathbb{I}$**5**–$\mathbb{I}$**10**.
- Proposition: Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying the properties **I1**, **I2** and **I3**. Then a characterization of $\widehat{I}$ can be obtained as

$$\widehat{I}(X, Y) = [I(\overline{X}, \underline{Y}), I(\underline{X}, \overline{Y})]. \tag{5}$$

Interval-valued R-implications

- An interval fuzzy implication $\mathbb{I}$ is an *interval R-implication* if there is an interval t-norm $\mathbb{T}$ such that $\mathbb{I} = \mathbb{I}_{\mathbb{T}}$, where

$$\mathbb{I}_{\mathbb{T}}(X, Y) = \sup\{Z \in \mathbb{U} | \mathbb{T}(X, Z) \leq Y\}. \tag{6}$$

- Observe that, in Equation (6), the supremum is determined considering the product order, and, therefore, it results from the supremum considering the usual order on the real numbers (the interval endpoints).
- Proposition: Let $\mathbb{I}$ be an interval fuzzy implication. If $\mathbb{I}$ is an interval R-implication then $\mathbb{I}$ satisfies $\mathbb{I}\mathbf{1}$, $\mathbb{I}\mathbf{2}$ and $\mathbb{I}\mathbf{3}$.
- Theorem: Let T be a left continuous t-norm. Then it holds that

$$\widehat{I_T} = \mathbb{I}_{\widehat{T}}. \tag{7}$$

- Corollary: If I is an R-implication then $\widehat{I}$ is an interval R-implication.

Relating R-implication with Interval R-implications

$$\begin{tikzcd} T \arrow[r] \arrow[d] & I_T \arrow[d] \\ \widehat{T} \arrow[r] & \mathbb{I}_{\widehat{T}} = \widehat{I_T} \end{tikzcd}$$

Interval-valued Automorphisms

- A mapping $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an *interval automorphism* if it is bijective and monotonic with respect to the product order (i.e., $X \leq Y$ implies that $\varrho(X) \leq \varrho(Y)$).
- Theorem: $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval automorphism iff there exists an automorphism $\rho : U \longrightarrow U$ such that

$$\varrho(X) = [\rho(\underline{X}), \rho(\overline{X})]. \tag{8}$$

- This theorem implies in that each interval automorphism ϱ has the form: $\varrho = \widehat{\rho}$ for some automorphism ρ.

Interval Automorphism Acting on Interval Fuzzy Implications

- Automorphism also acts on fuzzy implication to obtain new fuzzy implication.
- $I^{\rho}(x, y) = \rho^{-1}(I(\rho(x), \rho(y)))$.
- Proposition: Let I be an R-implication and ρ be an automorphism. Then I^{ρ} is also an R-implication.
- Proposition: Let $\mathbb{I}$ be an interval fuzzy implication and ϱ be an interval automorphism. Then $\mathbb{I}^{\varrho}(X, Y) = \varrho^{-1}(\mathbb{I}(\varrho(X), \varrho(Y)))$, is also an interval fuzzy implication..
- Let I be an implication and ρ be an automorphism. Then it holds that

$$\widehat{I^{\rho}} = \widehat{I}^{\widehat{\rho}}. \tag{9}$$

Interval Automorphism Acting on Interval R-implication

- Theorem: Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{I}_{\mathbb{T}} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval R-implication. Then the mapping $\mathbb{I}_{\mathbb{T}}^{\varrho} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an interval R-implication, defined by

$$\mathbb{I}_{\mathbb{T}}^{\varrho}(X, Y) = \mathbb{I}_{\mathbb{T}^{\varrho}}(X, Y). \tag{10}$$

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I^{\rho} \arrow[d] \\
\widehat{I} \arrow[r] & \widehat{I^{\rho}} = \widehat{I}^{\rho}
\end{tikzcd}$$

Final remarks

- Although the methodology used here is analogous to the applied by the authors for interval-valued QL-implications and S-implications, we observe that, whereas S-implications, for example, are obtained directly from t-conorms and fuzzy negations, R-implications are obtained as limits (supremum) of applications of t-norms, which led us to a different and more elegant approach in this presentation, and in the proofs of propositions and theorems.