

Sequent Calculus for 'Generally'

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Overview

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Logics of ‘Generally’

Examples:

1. “Birds *generally* fly”
2. “Metals *rarely* are liquid under normal conditions”
3. “A reply to a message will be received *almost always*”

Motivation:

- ‘generally’, ‘rarely’, ‘many’,
- ‘most’, ‘several’, ‘almost always’, etc.

appear often in

ordinary language

some branches of science.

Logics of ‘Generally’: Syntax

- $L^\nabla \supseteq L$
- FOL + generalized quantifier ∇
- Generalized Formula: $\nabla v\varphi$

Expressive power:

- “Birds *generally* fly”: $\nabla xF(x)$
- “Metals *rarely* are liquid under normal conditions”: $\neg\nabla xL(x)$

Logics of ‘Generally’: Semantics

- Modulated Structure: $\mathcal{A}^{\mathcal{K}} = \langle \mathcal{A}, \mathcal{K} \rangle$ (complex \mathcal{K})
- Satisfaction (extension in \mathcal{K})

$\mathcal{A}^{\mathcal{K}} \models \nabla v \varphi[s]$ iff $\{b \in A : \mathcal{A}^{\mathcal{K}} \models \varphi[s(v \mapsto b)]\}$ is in complex \mathcal{K} .

- Model: $\mathcal{A}^{\mathcal{K}} \models \varphi$ & $\mathcal{A}^{\mathcal{K}} \models \Gamma$ (as usual).

Logics for ‘generally’: Semantics

Consequence relation:

Many boys love sports $M \in \mathcal{K}$

Sports lovers watch SporTv channel $M \subseteq W$

Many boys watch SporTv channel $W \in \mathcal{K}$

$\Gamma \models_c F$

Class of complexes \mathcal{C} :

iff

$\mathcal{A}^{\mathcal{K}} \models \Gamma \Rightarrow \mathcal{A}^{\mathcal{K}} \models F$ for $\mathcal{K} \in \mathcal{C}$

Axiomatic Systems

- Basic Logic: (complexes without restriction)

$$[\nabla\alpha] : \nabla x A(x) \leftrightarrow \nabla y A(y) \text{ (y is a new var)}$$

$$[\leftrightarrow \nabla] : \forall x(A \leftrightarrow B) \rightarrow (\nabla x A \rightarrow \nabla x B)$$

- Specific Logic: intersection-closure

$$[\nabla\wedge] : \nabla x A \wedge \nabla x B \rightarrow \nabla x(A \wedge B)$$

Sequent Calculus for ‘Generally’

- Marked formulas: $\langle A[x/-] \rangle$
 - ‘_’ represents a generic object;
 - ‘ \langle ’ and ‘ \rangle ’ emphasise that $A(x)$ is the scope of a generalized quantifier.
- Sequent: $A_1, \dots, A_n \Rightarrow B_1, \dots, B_m$
 $A_1, \dots, A_n, B_1, \dots, B_m$: unmarked or marked formulas.
- Meaning: $T(A_1) \wedge \dots \wedge T(A_n) \rightarrow T(B_1) \vee \dots \vee T(B_m)$
 - T : unmarking marked formulas;
 - $T(\langle A \rangle) = \nabla z A[-/z]$ (z is the first variable such that $z \notin \text{occ}[A]$).

Sequent Calculus for ‘Generally’

Sequent Calculus for Basic Logic: $SC(\mathcal{B})$

- Sequent Calculus for Classical Logic (SC) + $\{(\nabla a), (\nabla c), (\Downarrow)\}$.

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta}{\nabla v A[-/v], \Gamma \Rightarrow \Delta} (\nabla a) \qquad \frac{\Gamma \Rightarrow \Delta, \langle A \rangle}{\Gamma \Rightarrow \Delta, \nabla v A[-/v]} (\nabla c)$$

(with $v \notin \text{occ}[A]$)

$$\frac{A, \Gamma \Rightarrow \Delta, B \qquad B, \Gamma \Rightarrow \Delta, A}{\langle A[v/-] \rangle, \Gamma \Rightarrow \Delta, \langle B[v/-] \rangle} (\Downarrow)$$

(with $v \notin \text{free}(\Gamma \cup \Delta)$)

Sequent Calculus for ‘Generally’

Sequent Calculus for Basic Logic: $SC(\mathcal{B})$

- $SC(\mathcal{B})$ is equivalent to the basic logic for ‘generally’:

Proposition 1: $\vdash_{SC(\mathcal{B})} \Gamma \Rightarrow \Delta$

iff

$$\vdash_{SC} T(\Gamma), B_1, \dots, B_n \Rightarrow T(\Delta)$$

(with $\{B_1, \dots, B_n\} \subseteq [\nabla\alpha] \cup [\leftrightarrow \nabla]$)

Sequent Calculus for ‘Generally’

Sequent Calculi for Specific Logics

Correspondence: axiom/schemas \iff sequent rules

$[\nabla \wedge]$: $\nabla x A \wedge \nabla x B \rightarrow \nabla x(A \wedge B)$ can be formulated as:

$$\frac{\Gamma \Rightarrow \Delta, \nabla v A \quad \Gamma \Rightarrow \Delta, \nabla v B}{\Gamma \Rightarrow \Delta, \nabla v(A \wedge B)} (\nabla \wedge).$$

$(\nabla \wedge)$ can be reformulated as:

$$\frac{\Gamma \Rightarrow \Delta, \langle A \rangle \quad \Gamma \Rightarrow \Delta, \langle B \rangle}{\Gamma \Rightarrow \Delta, \langle A \wedge B \rangle} (\wedge^*c).$$

Sequent Calculus for ‘Generally’

Sequent Calculi for Specific Logics: $SC(\Omega) = SC(\mathcal{B}) \cup \Omega^*$

$$\frac{\langle A \rightarrow A \rangle, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (\top^* a)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle \perp \rangle}{\Gamma \Rightarrow \Delta} (\perp^* c)$$

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta}{\langle A \wedge B \rangle, \Gamma \Rightarrow \Delta} (\wedge^* a)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle A \rangle \quad \Gamma \Rightarrow \Delta, \langle B \rangle}{\Gamma \Rightarrow \Delta, \langle A \wedge B \rangle} (\wedge^* c)$$

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta \quad \langle B \rangle, \Gamma \Rightarrow \Delta}{\langle A \vee B \rangle, \Gamma \Rightarrow \Delta} (\vee^* a)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle A \rangle \quad \Gamma \Rightarrow \Delta, \langle B \rangle}{\Gamma \Rightarrow \Delta, \langle A \vee B \rangle} (\vee^* c)$$

$$\frac{\Gamma \Rightarrow \Delta, \langle A \rangle}{\langle \neg A \rangle, \Gamma \Rightarrow \Delta} (\neg^* a)$$

$$\frac{\langle A \rangle, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \langle \neg A \rangle} (\neg^* c)$$

Cut Elimination

- Cut Elimination for Sequent Calculus for Basic Logic:
 - The cut formula is a generalized formula:

$$\frac{\frac{\Sigma_1}{\Gamma \Rightarrow \Theta, \langle A \rangle} (\nabla c) \quad \frac{\Sigma_2}{\langle A \rangle, \Delta \Rightarrow \Lambda} (\nabla a)}{\Gamma, \Delta \Rightarrow \Theta, \Lambda} (cut)$$

- The cut formula is a marked formula:

$$\frac{\frac{\Sigma_1}{A, \Gamma \Rightarrow \Theta, B} \quad \frac{\Sigma_2}{B, \Gamma \Rightarrow \Theta, A} (\Downarrow) \quad \frac{\Sigma_3}{C, \Delta \Rightarrow \Lambda, D} \quad \frac{\Sigma_4}{D, \Delta \Rightarrow \Lambda, C} (\Downarrow)}{\langle A[x/-] \rangle, \Gamma, \Delta \Rightarrow \Theta, \Lambda, \langle D[y/-] \rangle} (cut)$$

where $\langle B[x/-] \rangle = \langle C[y/-] \rangle$, $x \notin free(\Gamma \cup \Theta)$ and $y \notin free(\Delta \cup \Lambda)$.

Cut Elimination

- Cut Elimination for Sequent Calculus for Basic Logic:

Example:

$$\frac{\frac{A \rightarrow B \Rightarrow \neg(A \wedge \neg B) \quad \neg(A \wedge \neg B) \Rightarrow A \rightarrow B \quad (\Downarrow)}{\langle A \rightarrow B \rangle \Rightarrow \langle \neg(A \wedge \neg B) \rangle} \quad \frac{\neg(A \wedge \neg B) \Rightarrow \neg A \vee B \quad \neg A \vee B \Rightarrow \neg(A \wedge \neg B) \quad (\Downarrow)}{\langle \neg(A \wedge \neg B) \rangle \Rightarrow \langle \neg A \vee B \rangle} \quad (\text{cut})}{\langle A \rightarrow B \rangle \Rightarrow \langle \neg A \vee B \rangle} \quad (\text{cut})$$

↓

$$\frac{\frac{A \rightarrow B \Rightarrow \neg(A \wedge \neg B) \quad \neg(A \wedge \neg B) \Rightarrow \neg A \vee B \quad (\text{cut})}{A \rightarrow B \Rightarrow \neg A \vee B} \quad \frac{\neg A \vee B \Rightarrow \neg(A \wedge \neg B) \quad \neg(A \wedge \neg B) \Rightarrow A \rightarrow B \quad (\text{cut})}{\neg A \vee B \Rightarrow A \rightarrow B} \quad (\Downarrow)}{\langle A \rightarrow B \rangle \Rightarrow \langle \neg A \vee B \rangle} \quad (\text{cut})$$

↓ (Classical Cut Elimination)

Cut Elimination

- Cut Elimination for Sequent Calculi for Specific Logics:

$$\frac{\frac{\Gamma \Rightarrow \Theta, \langle A \rangle \quad \Gamma \Rightarrow \Theta, \langle B \rangle}{\Gamma \Rightarrow \Theta, \langle A \wedge B \rangle} (\wedge^*c) \quad \frac{\langle A \rangle, \Delta \Rightarrow \Lambda}{\langle A \wedge B \rangle, \Delta \Rightarrow \Lambda} (\wedge^*a)}{\Gamma, \Delta \Rightarrow \Theta, \Lambda} (cut)$$

⇓

$$\frac{\Gamma \Rightarrow \Theta, \langle A \rangle \quad \langle A \rangle, \Delta \Rightarrow \Lambda}{\Gamma, \Delta \Rightarrow \Theta, \Lambda} (cut)$$

Cut Elimination

- Cut Elimination for Sequent Calculi for Specific Logics:

Warning: This cut rule application cannot be eliminated.

$$\frac{\frac{A, A \leftrightarrow C \wedge D \Rightarrow C \wedge D \quad C \wedge D, A \leftrightarrow C \wedge D \Rightarrow A}{\langle A \rangle, A \leftrightarrow C \wedge D \Rightarrow \langle C \wedge D \rangle} (\Downarrow) \quad \frac{\langle D \rangle \Rightarrow \langle D \rangle}{\langle C \wedge D \rangle \Rightarrow \langle D \rangle} (\wedge^* a)}{\langle A \rangle, A \leftrightarrow C \wedge D \Rightarrow \langle D \rangle} (cut)$$

Conclusion and Future Work

- Sequent Calculus for Intuitionistic Logic + Basic Logic
- Sequent Calculus for Intuitionistic Logic + Specific Logics
- Characterization of the Derivation Structure
- Future work:
Analytical Tableau Methods for Logics of ‘Generally’

Logics for ‘generally’: Semantics

Module description: common properties of its complexes

Name	Property
universe	$A \in \mathcal{K}$
non-void	$\emptyset \notin \mathcal{K}$
superset	$T \cap S \in \mathcal{K} \Rightarrow T \in \mathcal{K} \text{ and } S \in \mathcal{K}$
intersection	$T \in \mathcal{K} \text{ and } S \in \mathcal{K} \Rightarrow T \cap S \in \mathcal{K}$
union	$T \in \mathcal{K} \text{ and } S \in \mathcal{K} \Rightarrow T \cup S \in \mathcal{K}$
prime	$S \cup T \in \mathcal{K} \Rightarrow S \in \mathcal{K} \text{ or } T \in \mathcal{K}$
rejection	$\bar{S} \in \mathcal{K} \Rightarrow S \notin \mathcal{K}$
absorption	$S \notin \mathcal{K} \Rightarrow \bar{S} \in \mathcal{K}$

Module: characteristic property & axioma/schema

Module Property	Schema	Notation
universe	$\nabla x(A \rightarrow A)$	$[\nabla \top]$
non-void	$\neg \nabla x \perp$	$[\perp \nabla]$
intersection	$\nabla x A \wedge \nabla x B \rightarrow \nabla x(A \wedge B)$	$[\nabla \wedge]$
union	$\nabla x A \wedge \nabla x B \rightarrow \nabla x(A \vee B)$	$[\nabla \vee]$
superset	$\nabla x(A \wedge B) \rightarrow (\nabla x A \wedge \nabla x B)$	$[\wedge \nabla]$
prime	$\nabla x(A \vee B) \rightarrow \nabla x A \vee \nabla x B$	$[\vee \nabla]$
rejection	$\nabla x \neg A \rightarrow \neg \nabla x A$	$[\neg \nabla]$
absortion	$\neg \nabla x A \rightarrow \nabla x \neg A$	$[\nabla \neg]$