Formalization of Termination of Recursive Functions

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Termination

- In computer science, *termination* is the is the quintessential example of a property that is undecidable.
- In 1939, Turing proved that it is impossible to construct an algorithm that decides whether or not another algorithm terminates on a given input [Tur37].
- Turing's proof applies to algorithms written as Turing machines, but the proof extends to other formalisms for expressing computations: λ-calculus, rewriting systems, computer programs.

Termination in Rewriting Systems

- Termination is a fundamental property of rewriting systems, e.g., confluence is decidable in terminating systems.
- Termination is undecidable even when a rewrite system consists of only one rule.
- Several syntactic and semantic techniques are available to prove termination of rewriting systems.
- This lecture focuses on termination of recursive functions specified in proof assistants.

Termination in Proof Assistants

- Termination is a meta-theoretical property in most interactive theorem provers, e.g.,
  - Termination is guaranteed for well-typed functions.
  - Termination is guaranteed for functions satisfying some constraints.
  - Termination is guaranteed for functions satisfying some semantic conditions.
- Once a definition of a function f is accepted by a proof assistant, the statement "For every value a the computation of $f(a)$ terminates" is assumed to hold.
- Our proof assistant of choice: SRI's Prototype Verification System (PVS).[2]

[2] `https://pvs.csl.sri.com`.

PVS

- PVS is an interactive theorem prover based on classical higher-order logic.
- PVS provides a strongly-typed specification language that supports predicate sub-typing, dependent types, inductive data types, parametric theories, etc.
- PVS is extensively used at NASA in the verification of safety-critical and mission-critical systems.[3]
- The NASA PVS Library consists of more than 20K lemmas (including the formalization presented in this lecture).[4]

[3] `https://shemesh.larc.nasa.gov/people/cam/FM`.

[4] `https://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/`.

Recursive Functions in PVS

```
factorial(n:nat): RECURSIVE nat =
  IF n = 0 THEN 1
  ELSE n*factorial(n-1)
  ENDIF
MEASURE M BY R
```

where M is a measure, i.e., a function from the domain of *factorial* into a type T, and R is a well-founded relation on T, e.g.,

- $M =$ `LAMBDA(n:nat):n`,
- $R =$ `LAMBDA(n,m:nat):n < m`.

Recursive Functions in PVS

The well-founded relation R is usually "$<$" on natural numbers. In this case, it can be written

```
factorial(n:nat): RECURSIVE nat =
  IF n = 0 THEN 1
  ELSE n*factorial(n-1)
  ENDIF
MEASURE n
```

PVS produces the following Termination Correctness Condition:[5]

```
factorial_TCC2: OBLIGATION
  FORALL (n: nat): n /= 0 IMPLIES n - 1 < n
```

[5] This TCC corresponds to the so called Turing Termination Criterion.

Recursive Functions in PVS

```
gcd(m,n:nat) : RECURSIVE nat =
  IF m = 0 OR n = 0 THEN m + n
  ELSIF n >= m THEN gcd(m,n-m)
    ELSE gcd(n,m)
  ENDIF
MEASURE ?
```

Recursive Functions in PVS

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gcd(m,n:nat) : RECURSIVE nat =
  IF m = 0 OR n = 0 THEN m + n
  ELSIF n >= m THEN gcd(m,n-m)
    ELSE gcd(n,m)
  ENDIF
MEASURE lex2(m,n)
```

In this case,

- T, the range of `lex2` is `ordinal`.
- R, the well-founded relation, is $<$ on ordinals.

Termination Correctness Conditions for *gcd*

```
gcd_TCC2: OBLIGATION
  FORALL (m, n: nat):
    n >= m AND NOT m = 0 AND NOT n = 0 IMPLIES
    lex2(m, n - m) < lex2(m, n)

gcd_TCC3: OBLIGATION
  FORALL (m, n: nat):
    NOT n >= m AND NOT m = 0 AND NOT n = 0 IMPLIES
    lex2(n, m) < lex2(m, n)
```

- `factorial_TCC2`, `gcd_TCC2`, and `gcd_TCC3` are automatically discharged by PVS.
- In general, the user has to provide the measure, the well-founded relation, and prove the TCCs.

Research Objectives

- Formalize in PVS different termination criteria and prove their equivalence, e.g., Turing termination [Tur89], size change principle [LJB01, TG03, KST$^+$11], calling context graphs [MV06], matrix-weighted graphs [Ave15], and dependency pairs [Art96, YSTK16, AG00].
- Use these criteria to specify terminating recursive functions in PVS and automatically discharge Termination Correctness Conditions.
- Study meta-theoretical properties related to termination and computability of PVS recursive functions.

PVS0: A Simple Computational Model

- PVS0 is a deep embedding of first-order PVS functions of type $T \to T$, where T is a parametric type.
- PVS0 functional expressions consists of
  - Constant values of type T.
  - A variable symbol of type T.
  - Unary and binary "built-in" operators.
  - If-then-else expressions.
  - Recursive calls.
- PVS0 is simple, but not minimal. In particular, it enables the use of arbitrary PVS functions of types $T \to T$ and $T \times T \to T$ as built-in atomic operators.

PVS0 Expressions

- PVS0 expressions e have the following form:

  $\texttt{cnst}(v) \mid \texttt{vr} \mid op1(n, e) \mid op2(n, e, e) \mid rec(e) \mid \texttt{ite}(e, e, e)$,

  where v is a value of type T and $n \in \mathbb{N}$.
- In PVS, it is defined using the following abstract data type.

```
PVS0Expr[T:TYPE+] : DATATYPE
BEGIN
  cnst(get_val:T) : cnst?
  vr : vr?
  op1(get_op:nat,get_arg:PVS0Expr) : op1?
  op2(get_op:nat,get_arg1,get_arg2:PVS0Expr) : op2?
  rec(get_arg:PVS0Expr) : rec?
  ite(get_cond,get_if,get_else:PVS0Expr) : ite?
END PVS0Expr
```

PVS0 Programs

Given a concrete type Val, which instantiates T, a PVS0 program with values in Val is a 4-tuple of the form $(O_1, O_2, \bot, e)$, where

- O_1 is a list of PVS functions of type $Val \to Val$, where $O_1(i)$, i.e., the i-th element of the list O_1, interprets the unary operator indexed by i in the constructor op1,

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- O_2 is a list of PVS functions of type $Val \times Val \to Val$, where $O_2(i)$, i.e., the i-th element of the list O_2, interprets the binary operator indexed by i in the constructor op2,

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- $\bot$ is a constant of type Val representing the Boolean value false in the conditional construction `ite`, and
- e is a `PVS0Expr`$[Val]$, which is the syntactic representation of the program itself.

Example 1: Factorial

```
factorial(n:nat): RECURSIVE nat =
  IF n /= 0 THEN n*factorial(n-1)
  ELSE 1
  ENDIF
```

Let $f \in \texttt{PVSO}[Val] = (O_1, O_2, \bot, e_f)$, where

- $Val = \texttt{nat}$

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- $\text{op2}(0, n, m) = n * m$
- $e_f = \texttt{ite(vr,op2(0,vr,rec(op1(0,vr))),cnst(1))}$

Example 2: GCD

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gcd(m,n:nat) : RECURSIVE nat =
  IF m = 0 OR n = 0 THEN m + n
  ELSIF n >= m THEN gcd(m,n-m)
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Let $g \in \texttt{PVS0}[\mathit{Val}] = (O_1, \texttt{null}, \bot, e_g)$, where

- $\mathit{Val} = [\texttt{nat}, \texttt{nat}]$

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```

- $e_g =$

```
ite(op1(0,vr),
    op1(2,vr),
    ite(op1(1,vr),
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```

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ackermann(m,n:nat) : RECURSIVE nat =
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- $\mathrm{op2}(0,(m,n),(i,j)) = (\max(0,m-1),i)$

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ite(op1(0,vr),
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```

Semantic Evaluation Relation ε

Given a PVS0 program $pvso = (O_1, O_2, \bot, e_{pvso})$ of type $\texttt{PVS0}[Val]$, the predicate ε holds when the semantic evaluation of an expression e of type $\texttt{PVS0Expr}[Val]$ on the input value v_i results in the value v_o.

$$
\begin{array}{rcl}
\varepsilon(pvso)(e, v_i, v_o) & := & \texttt{CASES } e \texttt{ OF} \\
\texttt{cnst}(v) & : & v_o = v; \\
\texttt{vr} & : & v_o = v_i; \\
\texttt{op1}(j, e_1) & : & j < |O_1| \ \wedge \exists\, v' \in Val : \\
 & & \varepsilon(pvso)(e_1, v_i, v') \wedge v_o = O_1(j)(v'); \\
\texttt{op2}(j, e_1, e_2) & : & j < |O_2| \ \wedge \exists\, v', v'' \in Val : \\
 & & \varepsilon(pvso)(e_1, v_i, v') \ \wedge \\
 & & \varepsilon(pvso)(e_2, v_i, v'') \ \wedge \\
 & & v_o = O_2(j)(v', v''); \\
\texttt{rec}(e_1) & : & \exists\, v' \in \mathcal{V}al : \varepsilon(pvso)(e_1, v_i, v') \ \wedge \\
 & & \varepsilon(pvso)(e_{pvso}, v', v_o) \\
\texttt{ite}(e_1, e_2, e_3) & : & \exists\, v' : \varepsilon(pvso)(e_1, v_i, v') \ \wedge \\
 & & \texttt{IF } v' \neq \bot \texttt{ THEN } \varepsilon(pvso)(e_2, v_i, v_o) \\
 & & \texttt{ELSE } \varepsilon(pvso)(e_3, v_i, v_o) \texttt{ ENDIF } .
\end{array}
$$

The Relation ε is Deterministic

Lemma 1.
Let pvso be a PVS0 program of type `PVS0`$[Val]$. *For any expression e of type* `PVS0Expr`$[Val]$ *and all values* $v_i, v_o', v_o'' \in Val$,

$$\varepsilon(pvso)(e, v_i, v_o') \text{ and } \varepsilon(pvso)(e, v_i, v_o'') \text{ implies } v_o' = v_o''.$$

The proof of this lemma uses the induction schema generated for the inductive relation ε.

Semantic Evaluation Function χ

- Given a PVS0 program $pvso = (O_1, O_2, \bot, e_{pvso})$ of type $\texttt{PVS0}[Val]$ and a natural number n, representing a maximum number of recursive calls, the function χ evaluates an expression e of type $\texttt{PVS0Expr}[Val]$ on the input value v_i. The function returns either the undefined value $\diamondsuit$ or a value of type Val.
- The function χ is recursively defined on the structure of e.

The Function χ

$$
\begin{array}{rcl}
\chi(pvso)(e, v_i, n) & := & \texttt{IF } n = 0 \texttt{ THEN } \diamondsuit \texttt{ ELSE CASES } e \texttt{ OF} \\
\texttt{cnst}(v) & : & \quad v; \\
\texttt{vr} & : & \quad v_i; \\
\texttt{op1}(j, e_1) & : & \quad \texttt{IF } j < |O_1| \texttt{ THEN} \\
 & & \qquad \texttt{LET } v' = \chi(pvso)(e_1, v_i, n) \texttt{ IN} \\
 & & \qquad \texttt{IF } v' = \diamondsuit \texttt{ THEN } \diamondsuit \texttt{ ELSE } O_1(j)(v') \texttt{ ENDIF} \\
 & & \quad \texttt{ELSE } \bot \texttt{ ENDIF ;} \\
\texttt{op2}(j, e_1, e_2) & : & \quad \texttt{IF } j < |O_2| \texttt{ THEN} \\
 & & \qquad \texttt{LET } v' = \chi(pvso)(e_1, v_i, n), \\
 & & \qquad\quad v'' = \chi(pvso)(e_2, v_i, n) \texttt{ IN} \\
 & & \qquad \texttt{IF } v' = \diamondsuit \vee v'' = \diamondsuit \texttt{ THEN } \diamondsuit \texttt{ ELSE } O_2(j)(v', v'') \texttt{ ENDIF} \\
 & & \quad \texttt{ELSE } \bot \texttt{ ENDIF ;} \\
\texttt{rec}(e_1) & : & \quad \texttt{LET } v' = \chi(pvso)(e_1, v_i, n) \texttt{ IN} \\
 & & \quad \texttt{IF } v' = \diamondsuit \texttt{ THEN } \diamondsuit \texttt{ ELSE } \chi(pvso)(e_{pvso}, v', n-1) \texttt{ ENDIF ;} \\
\texttt{ite}(e_1, e_2, e_3) & : & \quad \texttt{LET } v' = \chi(pvso)(e_1, v_i, n) \texttt{ IN} \\
 & & \quad \texttt{IF } v' = \diamondsuit \texttt{ THEN } \diamondsuit \\
 & & \quad \texttt{ELSIF } v' \neq \bot \texttt{ THEN } \chi(pvso)(e_2, v_i, n) \\
 & & \quad \texttt{ELSE } \chi(pvso)(e_3, v_i, n) \texttt{ ENDIF ;} \\
 & & \texttt{ENDIF}
\end{array}
$$

Equivalence of ε and χ Evaluation

Theorem 2.
Let pvso be a PVS0 program of type `PVS0`$[Val]$. *For any* $v_i \in Val$ *and e of type* `PVS0Expr`$[Val]$,

$$\varepsilon(pvso)(e, v_i, v_o) \text{ if and only if } v_o = \chi(pvso)(e, v_i, n),$$

for some n, where $v_o \neq \diamondsuit$.

Factorial, GCD, and Ackermann Lemmas

- The PVS0 program f computes the PVS function factorial, i.e., for any $n, k \in \texttt{nat}$,

$$\mathit{factorial}(n) = k \text{ if and only if } \varepsilon(f)(e_f, n, k).$$

- The PVS0 program g computes the PVS function gcd, i.e., for any $n, m, k \in \texttt{nat}$,

$$\mathit{gcd}(m, n) = k \text{ if and only if } \varepsilon(g)(e_g, (m, n), (k, i)),$$

  for some i.

- The PVS0 program a computes the PVS function ackermann, i.e., for any $n, m, k \in \texttt{nat}$,

$$\mathit{ackermann}(m, n) = k \text{ if and only if } \varepsilon(a)(e_a, (m, n), (k, i)),$$

  for some i.

ε-Termination

- The PVS0 program $pvso \in \mathtt{PVS0}[Val]$ ε-terminates for an input $v_i \in Val$ if the following predicate holds

$$T_\varepsilon(pvso, v_i) \equiv \exists\, v_o \in Val \;:\; \varepsilon(pvso)(e_{pvso}, v_i, v_o).$$

- The PVS0 program $pvso$ is *ε-terminating* if for all $v_i \in Val$, $T_\varepsilon(pvso, v_i)$ holds.

χ-Termination

- The PVS0 program $pvso \in \texttt{PVS0}[Val]$ χ-terminates for an input $v_i \in Val$ if the following predicate holds

$$T_\chi(pvso, v_i) \equiv \exists\, n \in \texttt{nat} \,:\, \chi(pvso)(e_{pvso}, v_i, n) \neq \diamondsuit.$$

- The PVS0 program $pvso$ is χ-*terminating* if for all $v_i \in Val$, $T_\chi(pvso, v_i)$ holds.

Equivalence of T_ε and T_χ

Theorem 3.
Let pvso be a PVS0 program of type `PVS0`[*Val*]. *The following conditions hold:*

1. *For any* $v_i \in Val$, $T_\varepsilon(pvso, v_i)$ *if and only if* $T_\chi(pvso, v_i)$.
2. *pvso is* ε*-terminating if and only if pvso is* χ*-terminating.*

A PVS0 program *pvso* that is ε-terminating (or equivalently, χ-terminating) is said to be *terminating*, denoted `terminating`(*pvso*).

Existence of Non-terminating PVS0 Programs

Lemma 4.
Let $\Delta = (O_1, O_2, \bot, \texttt{rec(vr)})$. *For any* $v \in \mathit{Val}$, $\neg T_\chi(\Delta, v)$.

Proof.

1. Define $\mu(\mathit{pvso}, v)$, for any *pvso* and v such that $T_\chi(\mathit{pvso}, v)$, as the minimum n that satisfies

$$\chi(\mathit{pvso})(e_{\mathit{pvso}}, v, n) \neq \diamondsuit.$$

2. By contradiction, assume that $\chi(\Delta)(e_\Delta, v, n) \neq \diamondsuit$ for some n.
3. Therefore, $\chi(\Delta)(e_\Delta, v, \mu(\Delta, v)) \neq \diamondsuit$, by definition of μ.
4. By definition of Δ and χ, it is also the case that $\chi(\Delta)(e_\Delta, v, \mu(\Delta, v) - 1)$.
5. This is a contradiction because $\mu(\Delta, v) - 1 < \mu(\Delta, v)$.

□

Evaluation of Terminating PVS0 Programs

Let $pvso = (O_1, O_2, \bot, e_{pvso})$ be a terminating PVS0 program in $\mathtt{PVS0}[Val]$, i.e., $\mathtt{terminating}(pvso)$. The evaluation of $pvso$ on v can be defined as

$\mathtt{pvs0\_eval}(pvso)(v) \equiv \mathtt{pvs0\_eval\_expr}(pvso)(e_{pvso}, v)$, where

$$
\begin{array}{rcl}
\mathtt{pvs0\_eval\_expr}(pvso)(e, v_i) & := & \mathtt{CASES}\ e\ \mathtt{OF} \\
\mathtt{cnst}(v) & : & v; \\
\mathtt{vr} & : & v_i; \\
\mathtt{op1}(j, e_1) & : & \mathtt{IF}\ j < |O_1|\ \mathtt{THEN} \\
 & & \quad O_1(j)(\mathtt{pvs0\_eval\_expr}(pvso)(e_1, v_i)) \\
 & & \mathtt{ELSE}\ \bot\ \mathtt{ENDIF}\ ; \\
\mathtt{op2}(j, e_1, e_2) & : & \mathtt{IF}\ j < |O_2|\ \mathtt{THEN} \\
 & & \quad O_1(j)(\mathtt{pvs0\_eval\_expr}(pvso)(e_1, v_i), \\
 & & \qquad \mathtt{pvs0\_eval\_expr}(pvso)(e_2, v_i)) \\
 & & \mathtt{ELSE}\ \bot\ \mathtt{ENDIF}\ ; \\
\mathtt{rec}(e_1) & : & \mathtt{pvs0\_eval\_expr}(pvso)(e_{pvso})(\mathtt{pvs0\_eval\_expr}(pvso)(e_1, v_i)) \\
\mathtt{ite}(e_1, e_2, e_3) & : & \mathtt{LET}\ v' = \mathtt{pvs0\_eval\_expr}(pvso)(e_1, v_i)\ \mathtt{IN} \\
 & & \mathtt{IF}\ v' \neq \bot\ \mathtt{THEN}\ \mathtt{pvs0\_eval\_expr}(pvso)(e_2, v_i) \\
 & & \mathtt{ELSE}\ \mathtt{pvs0\_eval\_expr}(pvso)(e_3, v_i)\ \mathtt{ENDIF}\ .
\end{array}
$$

Correctness of `pvs0_eval`

Theorem 5.
For all terminating programs $pvso \in \mathtt{PVS0}[Val]$ *and* $v_i, v_o \in Val$, $\varepsilon(pvso)(e_{pvso}, v_i, v_o)$ *if and only if* $v_o = \mathtt{pvs0\_eval}(pvso)(v_i)$.

A Syntactic Termination Criterion

- T_ε and T_χ are impractical as termination criteria since they require case by case analysis on the input to the function.
- Turing Termination Criterion: If there is a measure on a well-founded relation that strictly decreases at every recursive call, the function is terminating.
- This criterion, which implements a simple static analysis, requires the formalization of several syntactic elements:
  - Recursive calls.
  - Conditions.
  - Paths.

PVS0 Calling Context

A PVS0 Calling Context is triple (r, P, C), where

- r is a `PVS0Expr` of the form `rec`(e), where e is a `PVS0Expr`.
- P is a path, i.e., a list of natural numbers.
- C is a list of Boolean expressions, i.e., a `PVS0Expr` or a negation of a `PVS0Expr`.

A (r, P, C) is a valid calling context of $e \in$ `PVS0Expr` if

- r is a subexpression of e.
- P is the path of r in e,
- C is the set of accumulated conditions for the path P.

The set of valid calling contexts of e are denoted `pvs0_tcc_valid_cc`(e).

Valid Calling Contexts of Ackermann

```
IF m = 0 THEN n+1
ELSIF n = 0 THEN ackermann(m-1,1)
ELSE ackermann(m-1,ackermann(m,n-1))
ENDIF
```

$$\texttt{ite}_{\langle 0\rangle}(\texttt{op1}_{\langle 00\rangle}(\texttt{0},\texttt{vr}_{\langle 000\rangle}),\texttt{op1}_{\langle 10\rangle}(\texttt{2},\texttt{vr}_{\langle 010\rangle}),$$
$$\texttt{ite}_{\langle 20\rangle}(\texttt{op1}_{\langle 020\rangle}(\texttt{1},\texttt{vr}_{\langle 0020\rangle}),\texttt{rec}_{\langle 120\rangle}(\texttt{op1}_{\langle 0120\rangle}(\texttt{3},\texttt{vr}_{\langle 00120\rangle})),$$
$$\texttt{rec}_{\langle 220\rangle}(\texttt{op2}_{\langle 0220\rangle}(\texttt{0},\texttt{vr}_{\langle 00220\rangle},$$
$$\texttt{rec}_{\langle 10220\rangle}(\texttt{op1}_{\langle 0\ldots\rangle}(\texttt{4},\texttt{vr}_{\langle 00\ldots\rangle}))))))$$

- $cc_1 =$
  $(\texttt{rec(op1(3,vr))}, \langle 120\rangle, \{\texttt{op1(1,vr)}, \texttt{!(op1(0,vr))}\})$.

Valid Calling Contexts of Ackermann

```
IF m = 0 THEN n+1
ELSIF n = 0 THEN ackermann(m-1,1)
ELSE ackermann(m-1,ackermann(m,n-1))
ENDIF
```

$$\texttt{ite}_{\langle 0\rangle}(\texttt{op1}_{\langle 00\rangle}(0,\texttt{vr}_{\langle 000\rangle}),\texttt{op1}_{\langle 10\rangle}(2,\texttt{vr}_{\langle 010\rangle}),$$
$$\texttt{ite}_{\langle 20\rangle}(\texttt{op1}_{\langle 020\rangle}(1,\texttt{vr}_{\langle 0020\rangle}),\texttt{rec}_{\langle 120\rangle}(\texttt{op1}_{\langle 0120\rangle}(3,\texttt{vr}_{\langle 00120\rangle})),$$
$$\texttt{rec}_{\langle 220\rangle}(\texttt{op2}_{\langle 0220\rangle}(0,\texttt{vr}_{\langle 00220\rangle},$$
$$\texttt{rec}_{\langle 10220\rangle}(\texttt{op1}_{\langle 0\ldots\rangle}(4,\texttt{vr}_{\langle 00\ldots\rangle}))))))$$

- $cc_1 =$
  $(\texttt{rec(op1(3,vr))}, \langle 120\rangle, \{\texttt{op1(1,vr)}, \texttt{!(op1(0,vr))}\})$.
- $cc_2 =$
  $(\texttt{rec(op2(0,vr,...))}, \langle 220\rangle, \{\texttt{!(op1(1,vr))}, \texttt{!(op1(0,vr))}\})$.

Valid Calling Contexts of Ackermann

```
IF m = 0 THEN n+1
ELSIF n = 0 THEN ackermann(m-1,1)
ELSE ackermann(m-1,ackermann(m,n-1))
ENDIF
```

$$\texttt{ite}_{\langle 0\rangle}(\texttt{op1}_{\langle 00\rangle}(0,\texttt{vr}_{\langle 000\rangle}),\texttt{op1}_{\langle 10\rangle}(2,\texttt{vr}_{\langle 010\rangle}),$$
$$\texttt{ite}_{\langle 20\rangle}(\texttt{op1}_{\langle 020\rangle}(1,\texttt{vr}_{\langle 0020\rangle}),\texttt{rec}_{\langle 120\rangle}(\texttt{op1}_{\langle 0120\rangle}(3,\texttt{vr}_{\langle 00120\rangle})),$$
$$\texttt{rec}_{\langle 220\rangle}(\texttt{op2}_{\langle 0220\rangle}(0,\texttt{vr}_{\langle 00220\rangle},$$
$$\texttt{rec}_{\langle 10220\rangle}(\texttt{op1}_{\langle 0\ldots\rangle}(4,\texttt{vr}_{\langle 00\ldots\rangle}))))))$$

- $cc_1 =$ $(\texttt{rec(op1(3,vr))}, \langle 120\rangle, \{\texttt{op1(1,vr)}, \texttt{!(op1(0,vr))}\})$.
- $cc_2 =$ $(\texttt{rec(op2(0,vr,...))}, \langle 220\rangle, \{\texttt{!(op1(1,vr))}, \texttt{!(op1(0,vr))}\})$.
- $cc_3 =$ $(\texttt{rec(op1(0,vr))}, \langle 10220\rangle, \{\texttt{!(op1(4,vr))}, \texttt{!(op1(0,vr))}\})$.

Evaluation of Conditions

$$
\begin{array}{rcl}
\texttt{eval\_conds}(pvso)(C, v_i) & := & \texttt{CASES } C \texttt{ OF} \\
\texttt{null} & : & \texttt{TRUE}; \\
\texttt{cons}(e, C') & : & (\exists\, v_o \in Val : \varepsilon(pvso)(e, v_i, v_o) \wedge v_o \neq \bot) \wedge \\
 & & \texttt{eval\_conds}(pvso)(C', v_i); \\
\texttt{cons}(!(e), C') & : & (\exists\, v_o \in Val : \varepsilon(pvso)(e, v_i, v_o) \wedge v_o = \bot) \wedge \\
 & & \texttt{eval\_conds}(pvso)(C', v_i).
\end{array}
$$

Termination Correctness Condition

- A PVS0 program $pvso \in \texttt{PVS0}[Val]$ satisfies the predicate $\texttt{pvs0\_tcc\_termination}(pvso)$ for a type M if and only
- There exists a function m from Val into M and a well-founded relation $<$ on M such that for all $v_i, v_o : Val$ and $(rec(e), P, C) \in \texttt{pvs0\_tcc\_valid\_cc}(e_{pvso})$,
  - $\varepsilon(pvso)(e, v_i, v_o)$ and
  - $\texttt{eval\_conds}(pvso)(C, v_i)$

  implies $m(v_o) < m(v_i)$.

TCC Correctness

Theorem 6.
Let pvso be a PVS0 program of type `PVS0`[*Val*]. *The predicate* `pvs0_tcc_termination`(*pvso*) *holds for a type M if and only if* `terminating`(*pvso*), *i.e., for all* $v : Val$, $T_\varepsilon(pvso, v)$ *(or, equivalently,* $T_\chi(pvso, v)$*)* .

- The direction "$\leftarrow$" uses $M = \texttt{nat}$ and the well-founded order $<$ on natural numbers.
- The proof of this statement uses the definition of

$$\Omega_m(v) := \min(\{n : \mathbb{N}^+ \mid \forall\, v' \in V \,:\, \neg(m(v) >^n m(v'))\}).$$

- Intuitively, $\Omega_m(v)$ is the length of the longest path downwards starting from $m(v)$.

Ω and μ

The following lemma states a relation between μ and Ω.

Lemma 7.
Let pvso be a PVS0 program that satisfies `pvs0_tcc_termination`*(pvso) for a well-founded relation* $<$ *over* M *and a measure function* m*. For any value* $v \in Val$*,* $\mu(pvso, v) \leq \Omega_m(v)$.

The PVS0 Ackermann Program is Terminating

Theorem 8.
The PVS0 Ackerman program ▸ a *satisfies* $\mathit{terminating}(a)$.

- By Theorem 6, it suffices to check $\texttt{pvs0\_tcc\_termination}(a)$.
- $\texttt{pvs0\_tcc\_termination}(a)$ can be checked for all the PVS0 calling contexts of a, i.e., ▸ $\{cc_1, cc_2, cc_3\}$ using $M = [\texttt{nat}, \texttt{nat}]$ and $(a, b) < (c, d) \equiv a < c \texttt{ OR } (a = c \texttt{ AND } b < d)$.
- The proofs of these conditions correspond to the proofs of the actual termination correctness conditions generated by the PVS Type Checker for the PVS function *ackermann*.

The Size-Change Principle (SCP)

SCP Termination Criterion

A program terminates on all inputs if *every infinite call sequence* (following program control flow) would cause an infinite descent (over a well-founded relation) in some data values. [LJB01].

Infinite Sequence of Computations

Let $pvso \in \texttt{PVS0}[Val]$, $\mathbf{cc} = (\texttt{rec}(e_0), P_0, C_0), \ldots$ be an infinite sequence of calling contexts of $pvso$, and $\mathbf{V} = v_0, \ldots$ be an infinite sequence of values in Val such that the following predicate holds.

$$\texttt{infinite\_seq\_ccs}(\mathbf{cc}, \mathbf{V}) \equiv \forall(i : nat) : (\texttt{eval\_conds}(pvso)(C_i, v_i)\ \texttt{AND}\ \varepsilon(pvso)(e_i, v_i, v_{i+1})).$$

Let $<$ be a well-founded relation over Val, $\texttt{SCP}_<(pvso)$ holds if for all infinite sequence $\mathbf{cc}$ of calling contexts of $pvso$ and infinite sequence $\mathbf{V}$ of values in Val that satisfy $\texttt{infinite\_seq\_ccs}(\mathbf{cc}, \mathbf{V})$, $v_{i+1} < v_i$ for all i.

SCP Termination Criterion

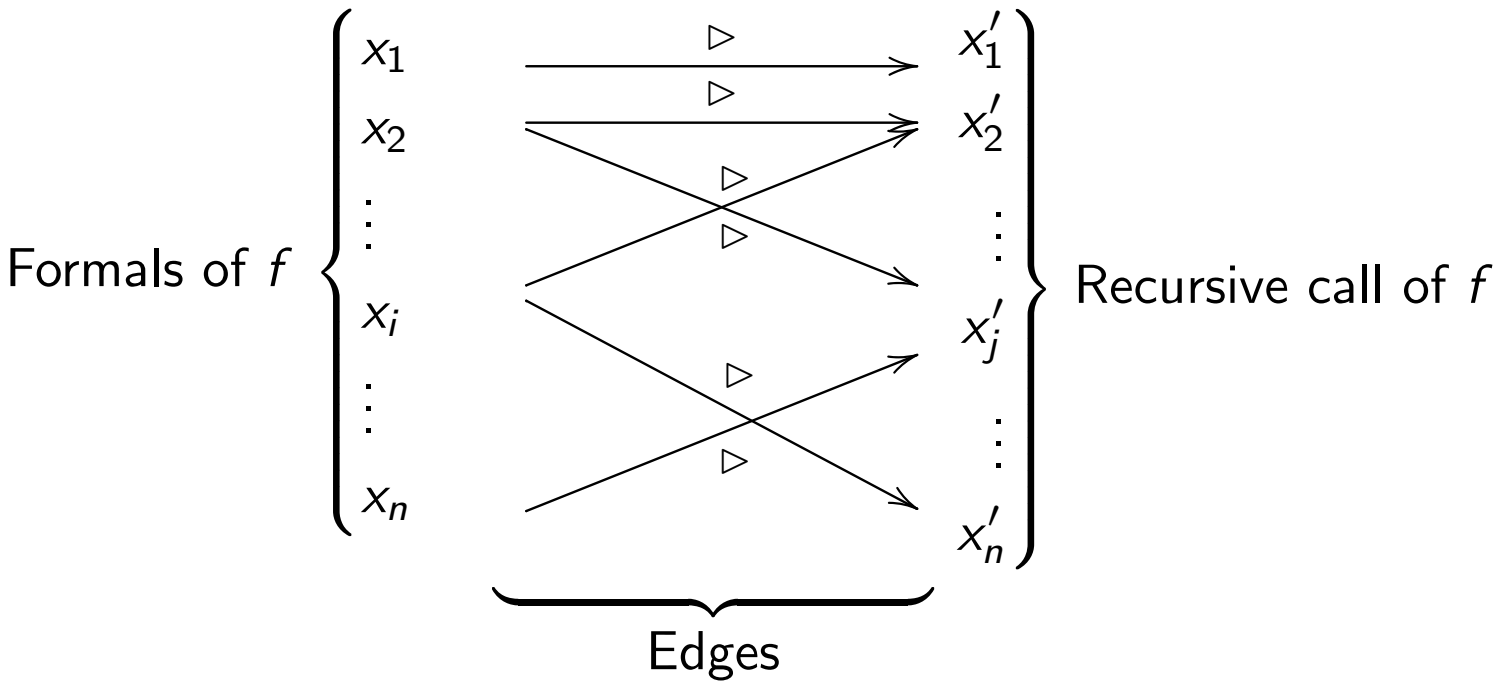
`scp_termination`(*pvso*) holds if there are no infinite sequence **cc** of calling contexts of *pvso* and infinite sequence **V** of values in *Val* that satisfy `infinite_seq_ccs`(**cc**, **V**).

Theorem 9.
For all pvso $\in$ `PVSO`[*Val*], *terminating*(*pvso*) *if and only if* *scp_termination*(*pvso*).

Theorem 10.
For all pvso $\in$ `PVSO`[*Val*], *scp_termination*(*pvso*) *if and only if* $SCP_{<}$(*pvso*) *for a well-founded relation* $<$ *over Val*.

Implementing SCP: Size Change Graph (SG)

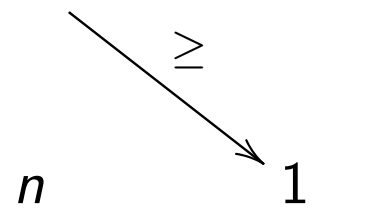
To every call c associated to a function call $f(x_1, ..., x_n) \xrightarrow{c} f(x'_1, ..., x'_m)$, the graph $\mathcal{G}_c$ is defined such that there is an edge $x_i \xrightarrow{\triangleright} x'_j$ if $x_i \triangleright x'_j$, where $\triangleright \in \{>, \geq\}$:



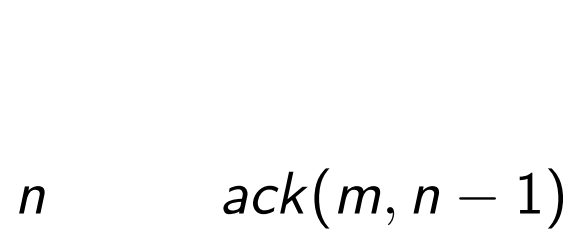
SCGs for Ackermann

$$ack(m,n) = ite(m = 0, n+1, ite(n = 0, \underbrace{ack}_{1}(m-1,1), \underbrace{ack}_{2}(m-1, 3 : \underbrace{ack}_{3}(m, n-1))))$$

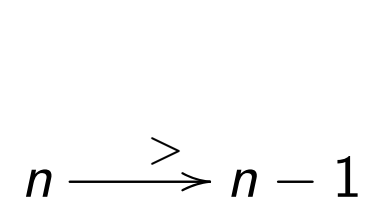
G_1 :



G_2 :

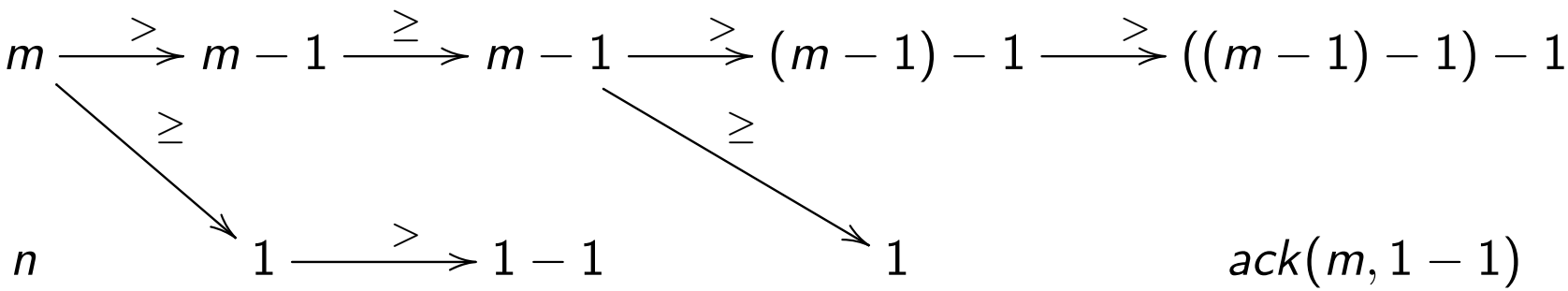


G_3 :



Multipaths and Threads

- A multipath is a sequence, potentially infinite, $G_{c_1}, G_{c_2}, \ldots$ of SCGs. A multipath can be seen as a concatenated graph.
- E.g., the Ackermann sequence of calls $c_1 c_3 c_1 c_2$ yields the multipath G_1, G_3, G_1, G_2:



Threads

- A thread in a multipath is a connected path of arcs $r_1 \xrightarrow{\rhd_1} r_2 \xrightarrow{\rhd_2} \dots$.
- A thread is descending if at least one $\rhd_i$ is $>$. The thread is infinitely descending if it contains infinitely many occurrences of $>$.
- Size Change Principle: A program terminates if every infinite call sequence yields an infinitely descending thread (over a well-founded order $<$).

Calling Context Graphs (CCG)

- CCG a termination analysis technique based on the size-change principle [MV06].
- SCG Multipaths are represented by:
  - A directed graph of calling contexts,
  - where edges are labelled using a family of measures.

Calling Contexts (Reminder)

Calling contexts are a representation of recursive calls and their governing conditions.

$$ack(m, n) = ite(m = 0, n + 1, \\ ite(n = 0, ack(m - 1, 1), \\ ack(m - 1, ack(m, n - 1))))$$

- $cc_1 = (ack(m - 1, 1), m \neq 0 \wedge n = 0)$

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- $cc_1 = (ack(m - 1, 1), m \neq 0 \wedge n = 0)$
- $cc_2 = (ack(m - 1, ack(m, n - 1)), m \neq 0 \wedge n \neq 0)$

Calling Contexts (Reminder)

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$$ack(m, n) = ite(m = 0, n + 1, \\ ite(n = 0, ack(m - 1, 1), \\ ack(m - 1, ack(m, n - 1))))$$

- $cc_1 = (ack(m - 1, 1), m \neq 0 \wedge n = 0)$
- $cc_2 = (ack(m - 1, ack(m, n - 1)), m \neq 0 \wedge n \neq 0)$
- $cc_3 = (ack(m, n - 1), m \neq 0 \wedge n \neq 0)$

Calling Context Graph

Given a PVS0 program $pvso \in \texttt{PVS0}[Val]$, a directed graph G of is built, where

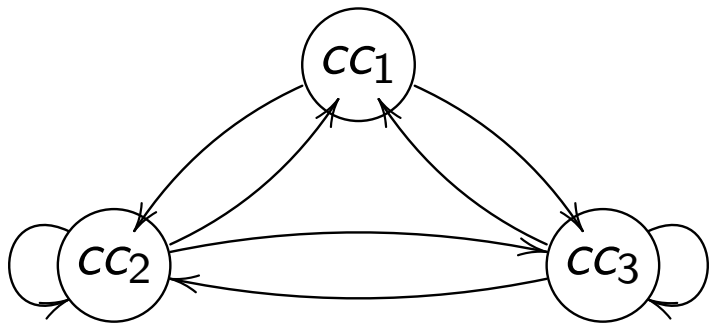
- the nodes of G are the calling contexts in $\texttt{pvs0\_tcc\_valid\_cc}(e_{pvso})$,
- there is an edge between calling contexts $(\texttt{rec}(e_a), P_a, C_a)$ and $(\texttt{rec}(e_b), P_b, C_b)$ if

$$\begin{aligned}\exists(v_a, v_b : Val) :&\texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \\ &\varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{AND} \\ &\texttt{eval\_conds}(pvso)(C_b, v_b).\end{aligned}$$

- Remark: The above is not an "if-and-only-if" condition. Hence, a fully connected graph of calling contexts satisfies this condition.

A CCG for Ackermann

- $cc_1 = (ack(m-1,1), m \neq 0 \wedge n = 0)$
- $cc_2 = (ack(m-1, ack(m,n-1)), m \neq 0 \wedge n \neq 0)$
- $cc_3 = (ack(m,n-1), m \neq 0 \wedge n \neq 0)$



- There is no edge between cc_1 and itself because

$$\nexists(m,n:\mathtt{nat}) : (m \neq 0 \ \wedge \ n = 0) \ \wedge \ (m-1 \neq 0 \ \wedge \ 1 = 0)$$

- There is an edge between cc_2 and cc_1 even although

$$\nexists(m,n:\mathtt{nat}) : (m \neq 0 \wedge n \neq 0) \wedge (m-1 \neq 0 \wedge ack(m,n-1) = 0)$$

Walks, Circuits, and Cycles

Let G_{pvso} be a CCG of a PVS0 program *pvso* in PVS0[*Val*].

- A walk of G_{pvso} is a sequence $cc_{i_1}, \ldots, cc_{i_n}$ of calling contexts such that for all $1 \leq j < n$ there is an edge between cc_{i_j} and $cc_{i_{j+1}}$.
- A circuit is a walk $cc_{i_1}, \ldots, cc_{i_n}$, with $n > 1$, where $cc_{i_1} = cc_{i_n}$.
- A cycle is an elementary circuit, i.e., a circuit $cc_{i_1}, \ldots, cc_{i_n}$ where the only repeating nodes are cc_{i_1} and cc_{i_n}.

Measure Combination

Let $\mathcal{M}$ be a family of N measures $\mu_k : \mathit{Val} \to M$, with $1 \leq k \leq N$, and $<$ be a well-founded relation over M.

- A measure combination of a walk $cc_{i_1}, \ldots, cc_{i_n}$ is a sequence of natural numbers $k_1, \ldots, k_n$, with $1 \leq k_j \leq N$ representing measure μ_{k_j}, such that for all $1 \leq j < n$, $v, v' \in \mathit{Val}$,

$$\texttt{eval\_conds}(\mathit{pvso})(C_j, v) \ \texttt{AND}\ \varepsilon(\mathit{pvso})(e_j, v, v') \\ \texttt{IMPLIES}\ \mu_{k_j}(v) \ \rhd_j \ \mu_{k_{j+1}}(v'),$$

  where $cc_{i_j} = (C_j, P_j, \texttt{rec}(e_j))$ and $\rhd_j \in \{>, \geq\}$.
- A measure combination is descending if at least one $\rhd_j$ is $>$.

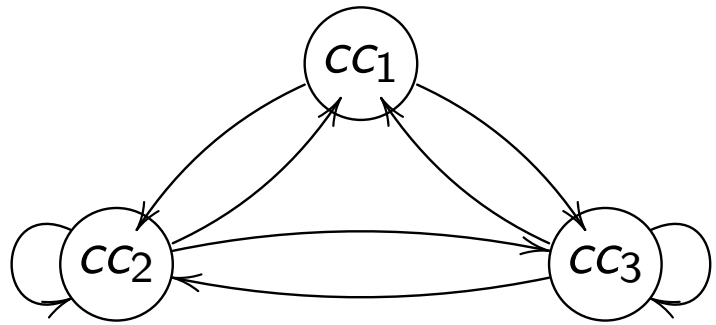
CCG Termination Criterion

- Let G_{pvso} be a CCG of a PVS0 program *pvso* in $\texttt{PVS0}[Val]$ and $\mathcal{M}$ be a family of N measures for a well-founded relation $<$ over a type M.
- $\texttt{ccg\_termination}_{\mathcal{M}}(G_{pvso})$ holds if for all circuits $cc_{i_1}, \ldots, cc_{i_n}$ in G_{pvso} there is a descending measure combination $k_1, \ldots, k_n$, with $k_1 = k_n$.

Finding Descending Measure Combinations

$$\mu_1(m,n) = m$$
$$\mu_2(m,n) = n$$

$\mu_1\mu_1$
$\mu_1\mu_2$
$\mu_2\mu_1$
$\mu_2\mu_2$



$ack(m,n) \equiv \ldots$

$cc_1 = (ack(m-1,1), m \neq 0 \wedge n = 0)$
$cc_2 = (ack(m-1, ack(m,n-1)), m \neq 0 \wedge n \neq 0)$
$cc_3 = (ack(m,n-1), m \neq 0 \wedge n \neq 0)$

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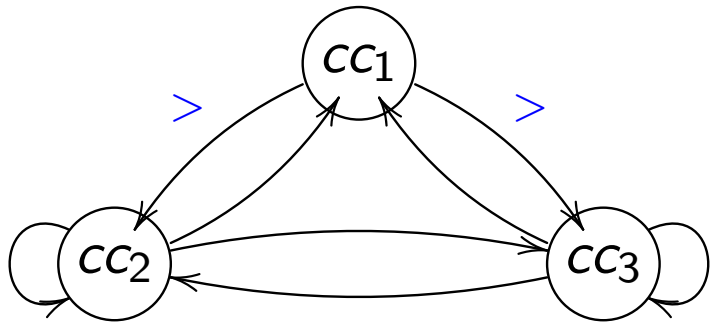

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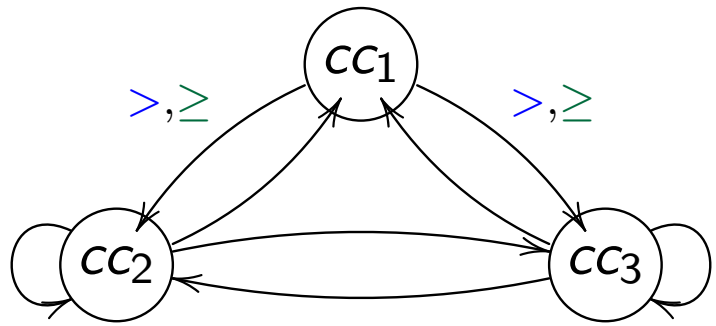

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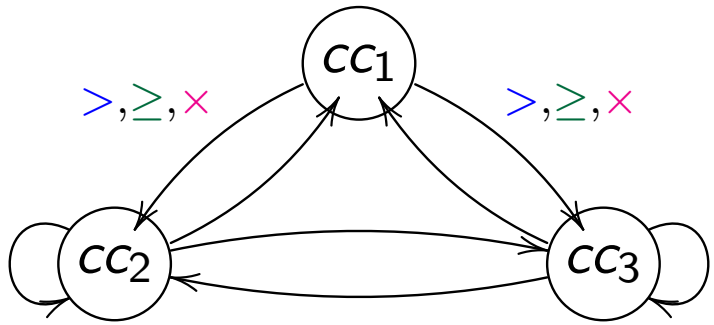
Finding Descending Measure Combinations

$$\mu_1(m,n) = m$$
$$\mu_2(m,n) = n$$

$$\mu_2\mu_1$$



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Finding Descending Measure Combinations

$$\mu_1(m,n) = m$$
$$\mu_2(m,n) = n$$

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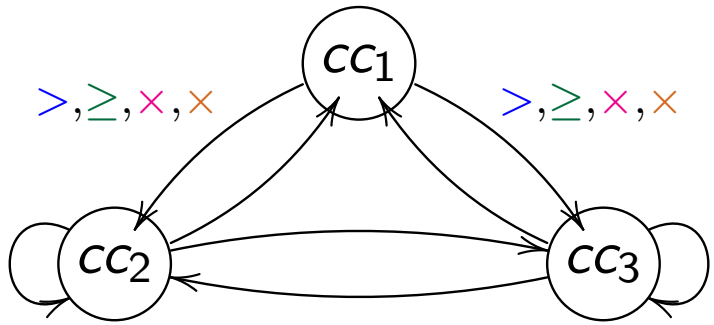

$ack(m,n) \equiv \ldots$

$cc_1 = (ack(m-1,1), m \neq 0 \wedge n = 0)$
$cc_2 = (ack(m-1, ack(m,n-1)), m \neq 0 \wedge n \neq 0)$
$cc_3 = (ack(m,n-1), m \neq 0 \wedge n \neq 0)$

Finding Descending Measure Combinations

$$\mu_1(m,n) = m$$
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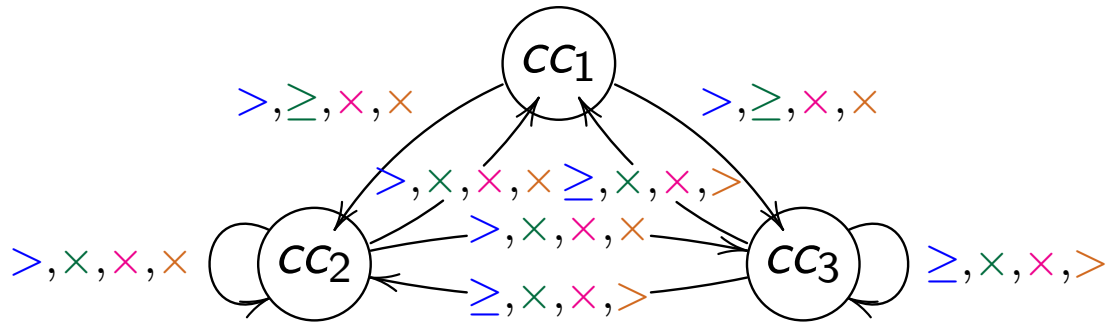

$ack(m,n) \equiv \ldots$

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$cc_2 = (ack(m-1, ack(m,n-1)), m \neq 0 \wedge n \neq 0)$
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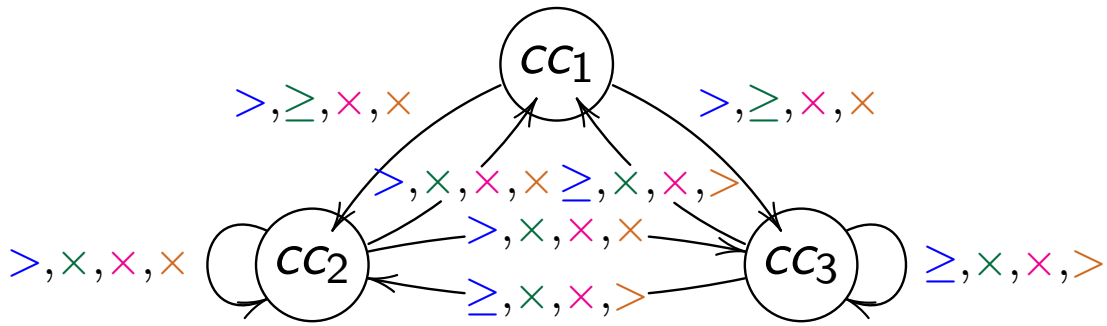
Finding Descending Measure Combinations

$\mu_1\mu_1$ $\mu_1\mu_2$
$\mu_2\mu_1$ $\mu_2\mu_2$



- Consider the circuit $cc_1, cc_3, cc_2, cc_2, cc_1$, the measure combination $1,1,1,1,1$, i.e., $>, \geq, >, >,$ is descending.
- Consider the circuit cc_3, cc_3, the measure combination $2,2$, i.e., $>$, is descending.
- The measure μ_1 is not enough to prove termination because $2,2$ is the only measure combination that is descending for cc_3, cc_3.

CCG Termination Correctness

Theorem 11.
For all pvso $\in \mathtt{PVS0}[Val]$, $scp\_termination(pvso)$ *if and only if* $ccg\_termination_{\mathcal{M}}(G_{pvso})$ *for some CCG* G_{pvso} *of pvso and family* $\mathcal{M}$ *of measures.*

- The number of cycles in a graph is finite. However, the number of circuits is potentially infinite.
- It's not enough to check for decreasing measure combinations in cycles (see [Ave15]).

Matrix Weighted Graphs

- Matrix Weighted Graphs is an effective technique to check for descending measure combinations in a CCG using an algebra over matrices [Ave15].
- Every edge in a CCG is labeled with a N$\times$N-matrix of values in $\{-1, 0, 1\}$, where N is the number of measures in $\mathcal{M}$.

Matrix Weighted Graph (MWG)

- Let G_{pvso} be a CCG of a PVS0 program *pvso* in $\mathtt{PVS0}[Val]$ and $\mathcal{M}$ be a family of N measures.
- A matrix weighted graph (MWG) W_{pvso} of a PVS0 program *pvso* in $\mathtt{PVS0}[Val]$ consists of a CCG G_{pvso} whose edges are correctly labeled by $\mathbb{N}\times\mathbb{N}$-matrices of values $\{-1, 0, 1\}$.

Correct Labels

$\mathbf{M}_{ab}$ is a correct label of an edge cc_a, cc_b in G_{pvso} if and only if for all $1 \leq i, j \leq N$ one of the following cases holds:

- $\mathbf{M}_{ab}(i,j) = 1$ only if for all $v_a, v_b \in Val$,

$$\texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \ \varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{IMPLIES} \ \mu_i(v_a) > \mu_j(v_b).$$

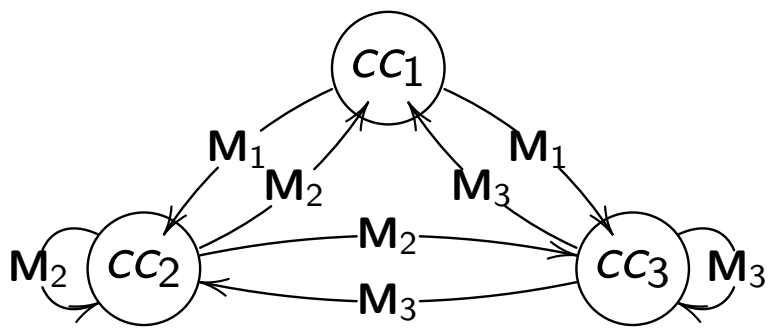
- $\mathbf{M}_{ab}(i,j) = 0$ only if for all $v_a, v_b \in Val$,

$$\texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \ \varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{IMPLIES} \ \mu_i(v_a) \geq \mu_j(v_b).$$

- $\mathbf{M}_{ab}(i,j) = -1$.

A MWG for Ackermann

- $cc_1 = (ack(m-1,1), m \neq 0 \wedge n = 0)$
- $cc_2 = (ack(m-1, ack(m, n-1)), m \neq 0 \wedge n \neq 0)$
- $cc_3 = (ack(m, n-1), m \neq 0 \wedge n \neq 0)$
- $\mu_1(m,n) = m$
- $\mu_2(m,n) = n$



$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad \mathbf{M}_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$$

Basic Operations

- Let x, y be in $\{-1, 0, 1\}$.

$$x \times y = \begin{cases} -1 & \text{if } x = -1 \vee y = -1 \\ 1 & \text{if } x = 1 \vee y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$x + y = \max(x, y).$$

- Matrix multiplication of values in $\{-1, 0, 1\}$ is defined as usual where addition and multiplication of values is defined as above.

Weight of Walk

- The weight of a walk $cc_{i_1}, \ldots, cc_{i_n}$ of a matrix weighted graph W_{pvso} is defined as $\Pi_{j=1}^{n-1} \mathbf{M}_{i_j i_{j+1}}$.
- A weight $\mathbf{M}$ is positive is there exists $1 \leq j \leq N$ such that $\mathbf{M}(i, i) > 0$.

MWG Termination Criterion

- Let W_{pvso} be a MWG of a PVS0 program *pvso* in $\texttt{PVS0}[Val]$ and $\mathcal{M}$ be a family of N measures for a well-founded relation $<$ over a type M.
- $\texttt{mwg\_termination}_{\mathcal{M}}(W_{pvso}$ holds if all circuits in W_{pvso} have a positive weight.
- Dutle's Procedure:
  - A sound and complete effective procedure to decide positive weight of all circuits in a CCG.
  - Formally verified in PVS.

Theorem 12.
For all pvso $\in \texttt{PVS0}[Val]$, $\texttt{ccg\_termination}_{\mathcal{M}}(G_{pvso})$ *for some CCG* G_{pvso} *if and only if* $\texttt{mwg\_termination}_{\mathcal{M}}(W_{pvso})$ *for some MWG* W_{pvso}.

Termination Analysis by CCG

- How to build a CCG?
- Which well-founded relation?
- Which family of measures?
- How to build the matrices?
- How to check for positive weight of all circuits in a CCG?
- How is CCG integrated into PVS?

How to Build a CCG

- The set of calling contexts is finite and can be extracted from the program by syntactic analysis.
- A fully connected CCG is sound (of course, the more edges the more inefficient the method).
- The theorem prover can be used to soundly remove edges from the graph, i.e., an edge cc_a, cc_b can be removed if

$$\begin{aligned} \vdash \forall(v_a, v_b : Val) : & \texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \\ & \varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{IMPLIES} \\ & \texttt{NOT eval\_conds}(pvso)(C_b, v_b). \end{aligned}$$

  can be discharged.

Which well-founded relation? Which family of measures?

- The order relation $<$ over natural numbers is a good starting point.
- Since *CCG* allows for a family of measures, it is sound to add as many measures as possible (of course the more measures the more inefficient the method).
- Predefined functions can be used, e.g., parameter projections (in the case of natural numbers), natural size of parameters (in the case of data types), maximum/minimum of parameters, etc. More complex recursions may need heuristics based on static analysis.
- Manolios and Vroon report that "[CCG] was able to automatically prove termination for over 98% of the more than 10,000 functions in the regression suite [of ACL2s]" [MV06].

How to Build the Matrices

- All edges starting in a given a calling context cc_a are labelled with the same matrix $\mathbf{M}_a$.
- To build a matrix $\mathbf{M}_a$ for the edges starting in cc_a, it is sound to set all its entries to -1.
- The theorem prover can be used to soundly reset the entries in $\mathbf{M}_a(i,j)$ to either 0 or 1 as follows,
  - If

$$\vdash \forall(v_a, v_b : Val) : \texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \ \varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{IMPLIES} \ \mu_i(v_a) > \mu_j(v_b),$$

    can be discharged then set $M_a(i,j)$ to 1.
  - If

$$\vdash \forall v_a, v_b : Val) : \texttt{eval\_conds}(pvso)(C_a, v_a) \ \texttt{AND} \ \varepsilon(pvso)(e_a, v_a, v_b) \ \texttt{IMPLIES} \ \mu_i(v_a) \geq \mu_j(v_b),$$

    can be discharged, then set $M_a(i,j)$ to 0.

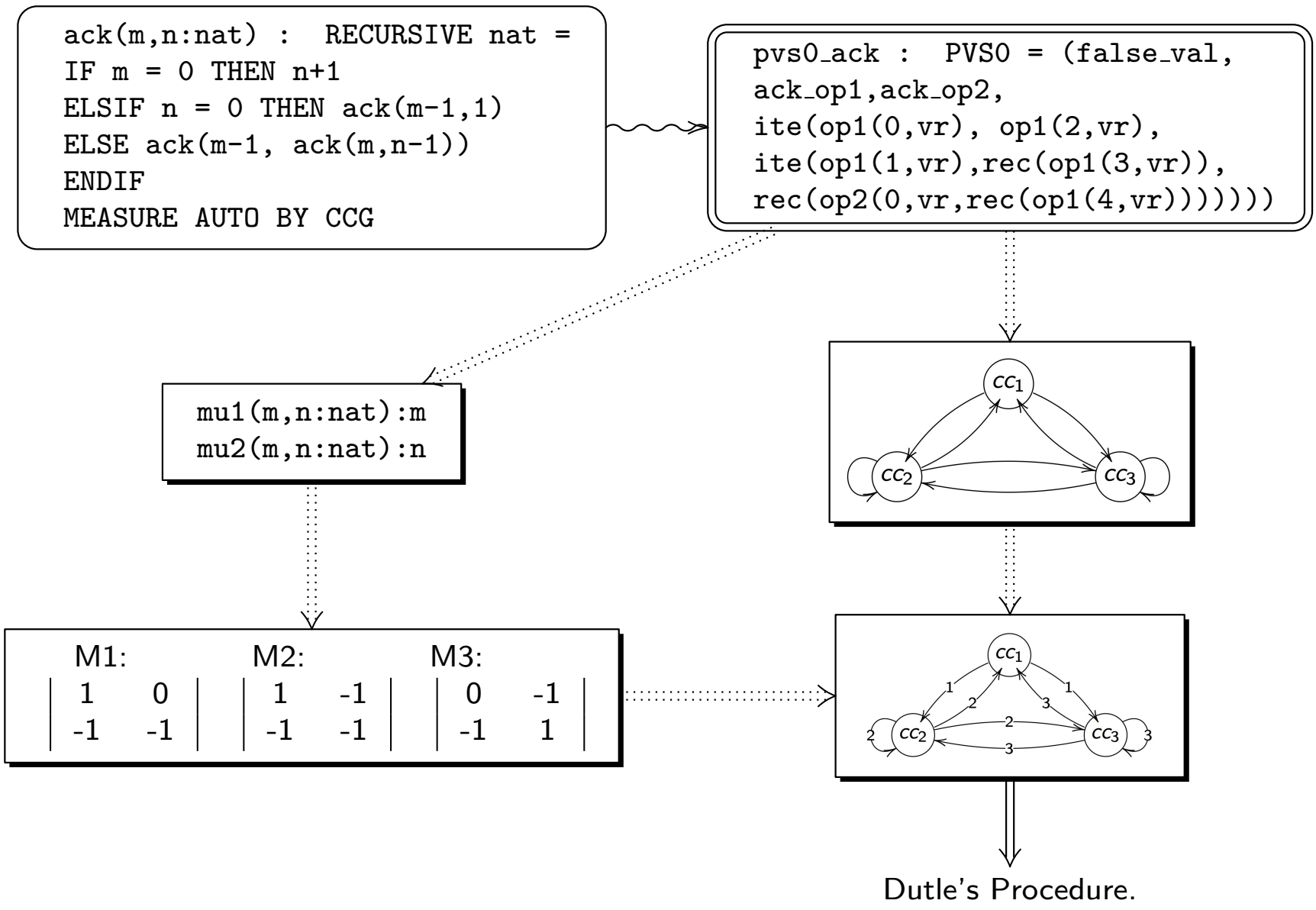
How to check for positive weight of all circuits in a CCG

Use Dutle's procedure!

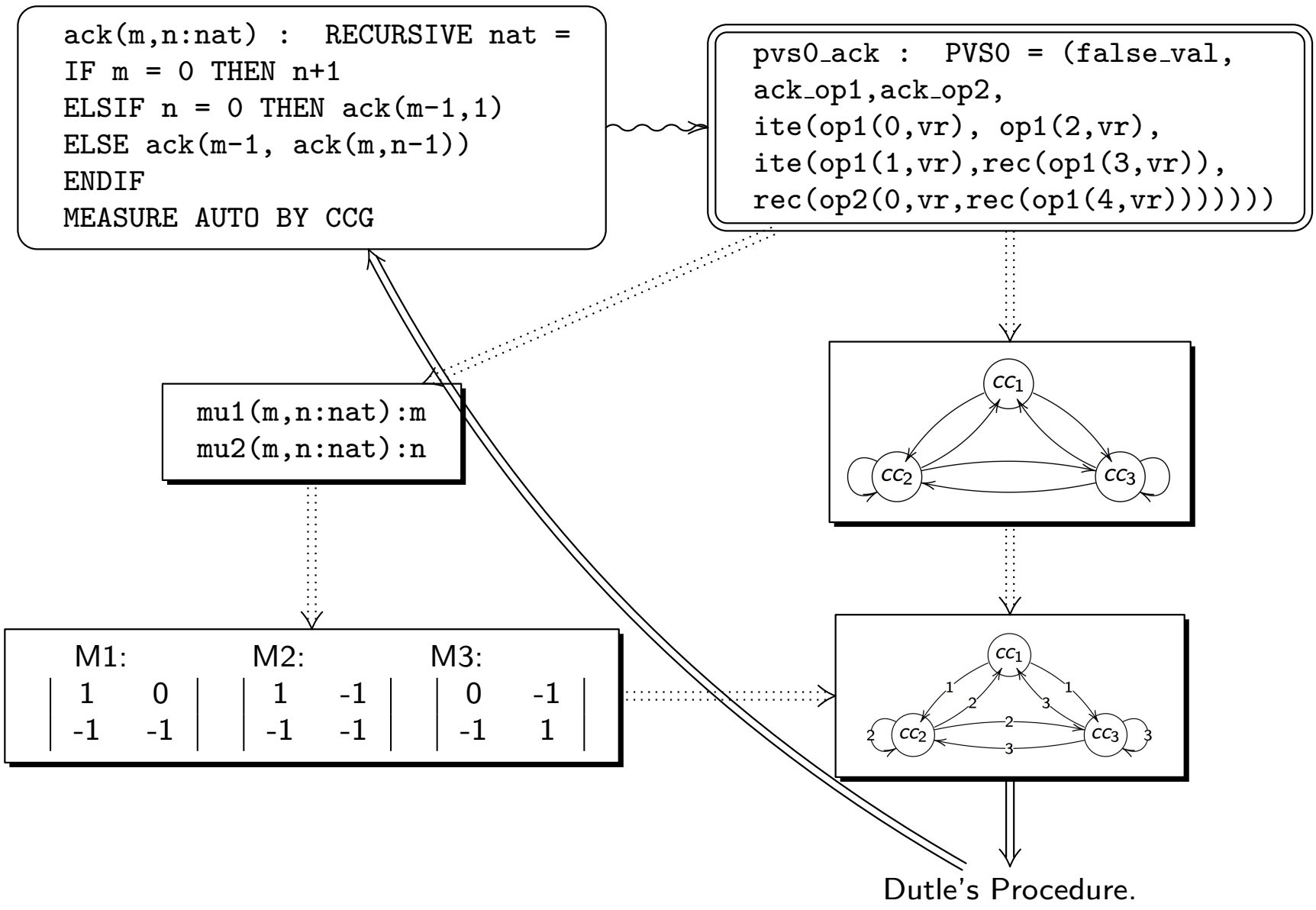
PVS Development

- The development presented in this lecture is fully formalized in PVS (including Dutle's procedure).
- For reference, see `CCG` and `PVS0` in NASA PVS Library (`https://github.com/nasa/pvslib`).
- In particular, it's formally verified that the following termination criteria are all equivalent: ε-termination, χ-termination, `pvs0_tcc_termination`, $\text{SCP}_{<}$, `scp_termination`, $\texttt{ccg\_termination}_{\mathcal{M}}$, $\texttt{mwg\_termination}_{\mathcal{M}}$, and Dutle's procedure.

Howe is CCG Being Integrated into PVS



Howe is CCG Being Integrated into PVS



PVS $\subseteq$ PVS0

By design, the PVS0 language can directly encode any PVS function f of type $T \to T$, where T is an arbitrary PVS type.

Lemma 13.
Let f be a PVS function of type $T \to T$. The program $mk\_pvs0(f) = (O_1, O_2, \bot, e_f)$ of type `PVS0`*, where $e_f = \texttt{op1}(0, \texttt{vr})$ and $O_1(0)(t) = f(t)$, satisfies the following properties:*

- $\mathit{terminating}(mk\_pvs0(f))$*, i.e., for any $t \in T$, $T_\varepsilon(mk\_pvs0(f), t)$.*
- $f = \mathit{pvs0\_eval}(mk\_pvs0(f))$*, i.e., for any $t \in T$, $f(t) = \mathit{pvs0\_eval}(mk\_pvs0(f))(t)$.*

PVS0 $\not\subseteq$ PVS

Some PVS0 programs cannot be embedded as PVS functions.

Lemma 14.
Let T be non-empty type. There is no PVS function f of type $T \to T$ such that for any $t \in T$, $\varepsilon(\Delta)(e_\Delta, t, f(t))$, where Δ is the PVS0 program defined in Lemma 4.

PVS0 $\subseteq$ PVS $\cup\{\diamondsuit\}$

However, any PVS0 program, even non-terminating ones, can be encoded as a PVS function of type $T \to T \cup \{\diamondsuit\}$.

Lemma 15.
Let pvso be a, possibly non-terminating, program of type `PVS0` *and* e_{pvso} *the* `PVS0` *expression of pvso. The PVS function*

$$f(t) := \texttt{IF } T_\chi(pvso, t) \texttt{ THEN } \chi(pvso)(e_{pvso}, t, \mu(pvso, t)) \\ \texttt{ELSE } \diamondsuit \texttt{ ENDIF },$$

satisfies the following property for any $t_i, t_o \in T$*:*

$$\varepsilon(pvso)(e_{pvso}, t_i, t_o) \text{ if and only if } f(t_i) = t_o.$$

An Oracle for PVS0 Programs

It is possible to define an oracle of type $\texttt{PVS0}[\texttt{PVS0}[\mathit{Val}]]$ that decides if a program of type $\texttt{PVS0}[\mathit{Val}]$ is terminating or not.

Theorem 16.
The program $\mathit{Oracle} = (O_1, O_2, \bot, e)$ *of type* $\texttt{PVS0}[\texttt{PVS0}[\mathit{Val}]]$, *where*

$$e = \mathit{mk\_pvs0}(\texttt{LAMBDA}(\mathit{pvso} : \texttt{PVS0}[\mathit{Val}]) : \texttt{IF } \mathit{terminating}(\mathit{pvso}) \texttt{ THEN } \top \texttt{ ELSE } \bot \texttt{ENDIF }),$$

with $\bot \neq \top$, *has the following properties.*

- *Oracle is a terminating* $\texttt{PVS0}[\texttt{PVS0}[\mathit{Val}]]$ *program, i.e., forall pvso of type* $\texttt{PVS0}[\mathit{Val}]$, $T_\varepsilon(\mathit{Oracle}, \mathit{pvso})$.
- *Oracle decides termination of any* $\texttt{PVS0}[\mathit{Val}]$ *program, i.e., for all pvso of type* $\texttt{PVS0}[\mathit{Val}]$,

$$\chi(\mathit{Oracle})(e, \mathit{pvso}, \mu(\mathit{Oracle}, \mathit{pvso})) = \top \text{ if and only if } T_\varepsilon(\mathit{pvso}).$$

An Oracle for PVS0 Programs

- This counterintuitive result is possible because PVS, in contrast to proof assistants based on constructive logic, allows for the definition of total functions that are non-computable, e.g.,

$$\texttt{LAMBDA}(\mathit{pvso} : \texttt{PVS0}[\mathit{Val}]) :$$
$$\texttt{IF terminating}(\mathit{pvso})\ \texttt{THEN}\ \top\ \texttt{ELSE}\ \bot\texttt{ENDIF}\ .$$

- These non-computable functions can be used in the construction of terminating PVS0 programs through the built-in operators.

Partial Recursive PVS0 Programs

Formalizing the notion of partial recursive functions in PVS0 requires to restrict the way in which programs are built.

1. The parametric type T is set to $\mathbb{N}$, i.e., $Val = \mathbb{N}$, where the number 0 represents the value false, i.e., $\bot = 0$. Any value different from 0 represents a true value, in particular $\top = 1$.
2. The built-in operators used in the construction of programs are restricted by a hierarchy of levels:
   - Operators in the first level can only be defined using
     - projections ($\Pi_1(x, y : \mathbb{N}) := x$ and $\Pi_2(x, y : \mathbb{N}) := y$),
     - successor ($succ(x : \mathbb{N}) : x + 1$), and
     - greater or equal than functions.
   - Operators in higher levels can only be constructed using programs from the previous level.

Partial Recursive PVS0 Programs

A hierarchy of PVS0[*nat*] program levels is formalized by the following predicate.

$$
\begin{array}{l}
\mathit{pvs0\_level}(n)(O_1, O_2, \bot, e) := \\
\quad \texttt{IF } n = 0 \texttt{ THEN } O_1 = \langle \mathit{succ} \rangle \ \wedge O_2 = \langle \Pi_1, \Pi_2, \mathit{ge} \rangle \\
\quad \texttt{ELSE } (\ \ \exists p' \in \texttt{PVS0}[\mathbb{N}] \ : \ \mathit{pvs0\_level}(n-1)(p') \ \wedge \\
\qquad\qquad \texttt{LET } ({O_1}', {O_2}', \bot', e') = p', \ {l_1}' = |{O_1}'| \texttt{ IN} \\
\qquad\qquad\ |O_1| = {l_1}' + 1 \ \wedge \\
\qquad\qquad\ (\ \forall i \in \mathbb{N} : i < {l_1}' \Rightarrow O_1(i) = {O_1}'(i) \ ) \wedge \\
\qquad\qquad\ (\ \forall v \in \mathbb{N} : \varepsilon(p')(e', v, O_1({l_1}')(v)) \ ) \ ) \wedge \\
\qquad\quad (\ \ \exists p' \in \texttt{PVS0}[\mathbb{N}] : \ \mathit{pvs0\_level}(n-1)(p') \ \wedge \\
\qquad\qquad \texttt{LET } ({O_1}', {O_2}', \bot', e') = p', {l_2}' = |{O_2}'| \texttt{ IN} \\
\qquad\qquad\ |O_2| = {l_2}' + 1 \ \wedge \\
\qquad\qquad\ (\ \forall i \in \mathbb{N} : i < {l_2}' \Rightarrow O_2(i) = {O_2}'(i) \ ) \wedge \\
\qquad\qquad\ (\ \forall v_1, v_2 \in \mathbb{N} : \varepsilon(p')(e', \kappa_2(v_1, v_2), O_2({l_2}')(v_1, v_2)) \ ) \ ),
\end{array}
$$

where the function κ_2 is an encoding of pairs of natural numbers onto natural numbers.

Undecidability of the Halting Problem

- The type `PartialRecursive` is defined to be a subtype of `PVS0`$[\mathbb{N}]$ containing all the programs *pvso* such that there is a natural n for which *pvs0_level*(n)(*pvso*) holds.
- `Computable` is a subtype of `PartialRecursive` containing those elements that are also terminating.
- `PartialRecursive`, and thus `Computable`, are enumerable.

Theorem 17.
There exists a PVS function of type $\mathbb{N} \to$ `PartialRecursive` *that is surjective.*

- The inverse of this surjective function, denoted as κ_P, is an injective function of type `PartialRecursive` $\to \mathbb{N}$.

Undecidability of the Halting Problem

Theorem 18.
There is no program oracle $= (O_1, O_2, \bot, e_o)$ *of type* `Computable` *such that for all pvso* $= (O_1', O_2', \bot, e)$ *of type* `PartialRecursive` *and for all* $n \in \mathbb{N}$,

$$T_\varepsilon(pvso, n) \text{ if and only if } \neg\varepsilon(oracle)(e_o, \kappa_2(\kappa_P(pvso), n), \bot).$$

Proof.
See [RMAR+18]. □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