The Double Transpose of the Ruelle Operator

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Abstract In this paper we study the double transpose of the $L^1(X, \mathscr{B}(X), \nu)$-extensions of the Ruelle transfer operator $\mathscr{L}_f$ associated to a general real continuous potential $f \in C(X)$, where $X = E^{\mathbb{N}}$, the alphabet E is any compact metric space and ν is a maximal eigenmeasure. For this operator, denoted by $\mathbb{L}_f^{**}$, we prove the existence of some non-negative eigenfunction, in the Banach lattice sense, associated to $\rho(\mathscr{L}_f)$, the spectral radius of the Ruelle operator acting on $C(X)$. As an application, we obtain a sufficient condition ensuring that the extension of the Ruelle operator to $L^1(X, \mathscr{B}(X), \nu)$ has an eigenfunction associated to $\rho(\mathscr{L}_f)$. These eigenfunctions agree with the usual maximal eigenfunctions, when the potential f belongs to the Hölder, Walters or Bowen class. We also construct solutions to the classical and generalized variational problem, using the eigenvector constructed here.

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1 Introduction

The Ruelle transfer operator, or simply the Ruelle operator, $\mathscr{L}_f$, has its roots
in the transfer matrix method introduced by Kramers and Wannier [39] and
(independently) by Montroll [47], to study the famous Ising model. This op-
erator, acting on an infinite-dimensional vector space, was introduced in 1968
by David Ruelle [52] to give a rigorous mathematical description of a relation
between local and global properties of a one-dimensional system composed of
infinitely many particles, subject to an infinite-range potential. In particular,
under an appropriate decay condition on the interaction (a local condition),
uniqueness of the Gibbs measure (a global property) was proved. The Ruelle
operator is one of the fundamental tools in Ergodic Theory/Thermodynamic
Formalism, and one of the most important results about this operator is the
so-called Ruelle-Perron-Frobenius theorem. Among other things, the Ruelle-
Perron-Frobenius theorem generalizes the classical Perron-Frobenius theorem
for matrices to a class of positive operators acting on a suitable infinite-
dimensional real vector space.

With the advent of the Markov partitions due to Adler and Weiss [2], Sinai
[60] and Bowen [12], remarkable applications of this operator to hyperbolic
dynamical systems on compact manifolds were further obtained by Ruelle,
Sinai and Bowen, see [13, 52, 59]. Since its creation, this operator has remained
a major tool, which has had a great influence in many fields of pure and applied
mathematics. In particular, it has been a powerful tool to study topological
dynamics, invariant measures for Anosov flows, statistical mechanics in one
dimension, meromorphy of the Selberg and Ruelle dynamical zeta functions,
multifractal analysis, Lyapunov exponents for product of random matrices,
conformal dynamics in one dimension and fractal dimensions of horseshoes,
just to name a few. Regarding these topics we refer the reader to [7, 14, 15, 43,
46, 48, 49, 53, 59] and references therein.

The spectral analysis of these transfer operators is deeply connected with
fundamental problems in Ergodic Theory, and Classical and Quantum Sta-
tistical Mechanics on the one-dimensional lattice. For example, the maximal
eigendata (eigenvalue, eigenfunction, eigenmeasure and so on) of the Ruelle
transfer operator can be used to compute and determine uniqueness of the
solutions of a central problem in Thermodynamic Formalism, introduced by
Ruelle [51] and Walters [64], which is a variational problem of the following
form:

$$\sup_{\mu \in \mathscr{M}_\sigma(X)} \{h_\mu(\sigma) + \int_X f \, \mathrm{d}\mu\}, \tag{1}$$

where $h_\mu(\sigma)$ is the measure-theoretical (or Kolmogorov-Sinai) entropy of μ
and $\mathscr{M}_\sigma(X)$ is the set of all σ-invariant Borel probability measures on X, see
also [5, 37, 40, 48, 66].

Before presenting the precise definition of the Ruelle operator, we need to
introduce some more notation. Let (E, d_E) denote a general compact metric
space which is sometimes called the state space, and fix a Borel probability

measure $p : \mathscr{B}(E) \to [0,1]$ defined on E, having full support. This condition will be important later, when we will talk about the extension of the Ruelle operator to $L^1(\nu) \equiv L^1(X, \mathscr{B}(X), \nu)$. We refer to this measure p as the *a priori* measure. Consider the infinite product space (on the half-line) $X = E^{\mathbb{N}}$, endowed with any metric $d : X \times X \to [0, \infty)$ inducing the product topology, and let $\sigma : X \to X$ be the left shift map. As usual we write $C(X) \equiv C(X, \mathbb{R})$ to denote the space of all real continuous functions defined on X and always assume that it is endowed with its standard norm $\| \cdot \|_\infty$.

Finally, for a fixed potential $f \in C(X)$, we define the Ruelle operator $\mathscr{L}_f : C(X) \to C(X)$ as being the linear operator that sends a continuous function φ to another continuous function $\mathscr{L}_f \varphi$, which is given by the following expression

$$\mathscr{L}_f \varphi(x) = \int_E \exp(f(ax)) \varphi(ax) \, \mathrm{d}p(a), \qquad \text{where} \quad ax \equiv (a, x_1, x_2, \ldots). \quad (2)$$

The spectral radius of $\mathscr{L}_f$ acting on $(C(X), \| \cdot \|_\infty)$ is denoted by $\rho(\mathscr{L}_f)$.

In the sequel we summarize some of the classical results about the Ruelle operator and its maximal eigendata in both finite and compact state space cases.

Before we proceed, we present some of the definitions of the most used regularity conditions in Thermodynamic Formalism. Recall that a function $\omega : [0, +\infty) \to [0, +\infty)$ is called a *modulus of continuity*, if ω is a continuous, increasing and concave function, such that $\omega(0) = 0$. Given a modulus of continuity we say that a potential f satisfies the generalized Hölder's condition, if there is a constant $C > 0$ and a modulus of continuity ω such $|f(x) - f(y)| \leq C\omega(d(x,y))$, for all $x, y \in X$. We say that the potential f satisfies the Dini's condition, if f has a modulus of continuity ω such that $\int_0^1 \frac{\omega(t)}{t} \, dt < \infty$.

The n-th variation of a function $f : X \to \mathbb{R}$ is defined by

$$\mathrm{var}_n(f) \equiv \sup\{|f(x) - f(y)| : \ x_i = y_i, \ i = 1, \ldots, n\}.$$

For the Birkhoff sum we write $S_n(f) = f + f \circ \sigma + \ldots + f \circ \sigma^n$.

1. Walters' condition holds, if $\lim_{k \to \infty} \sup_{n \in \mathbb{N}} \mathrm{var}_{n+k}(S_n(f)) = 0$;
2. Bowen's condition holds, if there is $k \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} \mathrm{var}_{n+k}(S_n(f)) < +\infty$.

The problems of finding the maximal positive eigenvalue for $\mathscr{L}_f$ and its respective eigenfunction and eigenmeasure are already solved if the potential f satisfies one of the four conditions presented above. For finite state spaces see, for example, $[5, 11, 15, 37, 48, 52, 54, 65, 67, 68]$ and for the case of general compact metric state space see, for example, the references $[6, 18, 21, 26, 27, 38, 42, 45]$. On the other hand, in $[31]$ Hofbauer gives an example of a continuous potential f defined on $X = \{0, 1\}^{\mathbb{N}}$ for which there is no continuous eigenfunction for $\mathscr{L}_f$. Later, Walters in $[68]$ generalized Hofbauer's result for a large

class of potentials. Of course, the potentials in this class do not satisfy the above regularity conditions.

Historically, the results on the existence of eigenfunctions for the Ruelle operator started with the investigation of Lipschitz and Hölder potentials defined over symbol spaces with finite-state space, see [5, 15, 52, 54]. The literature about the Ruelle operator associated to such potentials is vast, and it lies at the heart of the most important applications of the Ruelle operator in several branches of pure and applied mathematics. We also mention here that the investigation of the basic properties of this operator for Hölder potentials is a very important chapter in the theory of Thermodynamic Formalism. In the nineties and the beginning of the two-thousands the interest in this operator, defined on shifts with infinite alphabets, increased, motivated in part by applications to non-uniformly hyperbolic dynamical systems, see [1, 30, 44, 55, 56, 70] and references therein. Simultaneously, investigations about this operator associated to potentials belonging to more general function spaces, like Walters and Bowen spaces, were carried out. Nowadays we can say that we have a well-developed theory on this subject, but in both cases of finite-state space and infinite-state space, the problem of determining necessary and sufficient conditions on the potential ensuring the existence of positive eigenfunctions for the Ruelle operator associated to its spectral radius remains open.

When considering potentials $f \in C(X)$ living outside the Bowen space, even for finite-state spaces, in very few cases the maximal eigendata of the Ruelle operator can be obtained. The obstacle one faces to perform the spectral analysis of the operator in this case, by using current mathematical technology, is not only of a technical nature. For such potentials a phase transition, ($|\mathscr{G}^*(f)| > 1$, see Proposition 1 for the definition of $\mathscr{G}^*(f)$), can take place and this introduces a bunch of new difficulties. The classical example, in the Thermodynamic Formalism, of a continuous potential where such phenomena can occur is given by a Hofbauer-type potential see [31]. Basically, in such generality no general mathematical theory exists, and as far as we know, things can only be handled on a case-by-case basis. Other one-dimensional examples of systems which have phase transitions are the Dyson models, the Bramson-Kalikow and Berger-Hoffman-Sidoravicius g-measure examples, and the Fisher-Felderhof renewal-type examples, see [8, 16, 19, 25, 28, 36].

The goal of this paper is to initiate the study of the double transpose of some suitable extensions $\mathbb{L}_f : L^1(\nu) \to L^1(\nu)$, where $\nu \in \mathscr{G}^*(f)$, of the Ruelle transfer operator $\mathscr{L}_f$, associated to a general real continuous potential $f : X \to \mathbb{R}$, and the main results obtained here are Theorems 3 and 5.

The approach taken here is novel, in so far as it focuses on the bidual of either $C(X)$ or $L^1(\nu)$ and the extension of the Ruelle operator to these spaces. It allows us to find a new sufficient condition to solve the problem of existence of maximal eigenfunctions for continuous potentials having low regularity properties.

As mentioned above, one of the main ideas of this paper is to carefully study the double transpose $\mathbb{L}_f^{**}$, where $\mathbb{L}_f : L^1(\nu) \to L^1(\nu)$ is an extension of the classical Ruelle operator $\mathscr{L}_f$. We prove that $\rho(\mathscr{L}_f)$ is an eigenvalue of $\mathbb{L}_f^{**}$ and has associated to it a non-negative (in the Banach lattice sense) eigenvector ξ_f, for any continuous potential f in $C(X)$. Afterward, we show how we can use this eigenvector ξ_f to construct equilibrium states.

When working with potentials with very low regularity properties, to prove the existence of non-negative eigenfunctions for $\mathbb{L}_f$ is in general a hard task, see [17, 19, 31].

1.1 Directions for future research

One of our main results, Theorem 4, states that for any continuous potential f, and $\nu \in \mathscr{G}^*(f)$, there is a positive element $\xi_f \in L^1(\nu)^{**}$ which determines a σ-invariant Borel finitely additive measure given by $A \longmapsto \xi_f(1_A\nu) \equiv \mu(A)$, where $1_A\nu$ denotes the bounded linear functional on $L^1(\nu)$ given by $1_A\nu(\varphi) = \int_X 1_A\varphi \, d\nu$.

By Theorem 3.1 in [62] we have

$$L^1(\nu)^{**} \cong \left\{\gamma : \mathscr{B}(X) \to \mathbb{R} : \gamma \in \mathrm{ba}(\mathscr{B}(X)) \text{ satifying } \nu(Z)=0 \Longrightarrow \gamma(Z) = 0\right\},$$

where $\mathrm{ba}(\mathscr{B}(X))$ stands for the space of all Borel bounded finitely additive signed measures. By Theorem 3.1 in [62], we can identify the finitely additive measure μ, constructed above, with a bounded linear functional $F_\mu : L^\infty(\nu) \to \mathbb{R}$. The restriction of F_μ to $C(X)$ determines a countably additive σ-invariant measure, because of the Riesz-Markov Theorem. It would be very interesting to know whether such a Borel probability measure is actually an equilibrium state associated to f, and how to determine the support of such a measure in terms of its Yosida-Hewitt decomposition $\mu = \mu_a + \mu_c$, see [62, 69] for details on this decomposition.

Another interesting question related to Theorem 4 is the following. Consider ν the barycenter of the convex set $\mathscr{G}^*(f)$ and let $\mathscr{H} \equiv \{\xi \in L^1(\nu)^{**} : \mathbb{L}_f^{**}\xi = \rho(\mathscr{L}_f)\xi\}$. Some of the results in [20] suggest that $\dim_{\mathbb{R}} \mathscr{H} = \#\mathrm{ex}(\mathscr{G}^*(f))$ (the set of extreme points in $\mathscr{G}^*(f)$). If such a relation indeed holds, it has as a consequence a new criterion for the existence of phase transitions for one-dimensional one-sided lattice statistical mechanical spin systems, which in turn would reduce this problem to a problem of spectral analysis of Markov operators.

Yet another question, which seems to be more puzzling, is the following. When the Yosida-Hewitt decomposition of μ is trivial, meaning that $\mu = \mu_a$ is purely finitely additive, can any statistics along typical orbits of observables be obtained, similarly to the countably additive setting? Are there any relations between this decomposition and the existence of sigma-finite shift-invariant measures which are not finite?

The connection between $\mathrm{Eq}(f)$, the set of equilibrium states associated to f, and $\mathscr{G}^*(f)$ is still mysterious when f has low regularity properties. For

finite alphabets it is very well known that any probability measure $\nu \in \mathscr{G}^*(f)$ is fully supported and the same holds in the metric compact case, see [20]. On the other hand, Jenkinson, and Israel and Phelps showed in [34,35] that for every non-empty collection $\mathscr{C}$ of ergodic measures which is a closed subset of $\mathscr{M}_\sigma(X)$ (in the weak-$*$-topology), there is a continuous potential f such that the closure of the convex hull of $\mathscr{C}$ is precisely the set of the equilibrium states for f. In particular, if $x \in X$ is such that $\sigma(x) = x$ and δ_x denotes the Dirac delta measure concentrated in x, then there is a continuous potential f such that $\mathrm{Eq}(f) = \{\delta_x\}$. On the other hand, any $\nu \in \mathscr{G}^*(f)$ is such that $\mathrm{supp}(\nu) = X$. In addition, if ν is atomless then $\nu \perp \delta_x$ and in such case they are not linked, as in the classical cases (Hölder, Walters and Bowen), by an eigenfunction. Could they be linked by the generalized eigenfunctions $\xi \in \mathscr{H}$? The example in the last section suggests that the answer to this question is affirmative, at least in the particular cases where the set of equilibrium states is a singleton and the link is provided by the restriction of F_μ as described above.

1.2 Organization of the paper

The paper is organized as follows. In Section 2 we study the pressure functional for general continuous potentials defined on $E^{\mathbb{N}}$, where E is any compact metric space. In such a general setting we prove the existence of an eigenmeasure associated to $\rho(\mathscr{L}_f)$, the spectral radius of the Ruelle operator acting on $C(X)$. In particular, we prove that the set $\mathscr{G}^*(f) \equiv \{\nu \in \mathscr{M}_1(X) : \mathscr{L}_f^* \nu = \rho(\mathscr{L}_f)\nu\}$ is always non-empty, for any continuous potential f. In Section 3 we study the double transpose of the Ruelle operator, associated to a general continuous potential, acting on both the spaces $C(X)$ and $L^1(\nu)$, where $\nu \in \mathscr{G}^*(f)$. For any continuous potential f, the existence of a positive eigenvector ξ_f for the double transpose of the Ruelle operator acting on $L^1(\nu)$, associated to $\rho(\mathscr{L}_f)$, is established. In Section 4 we show how to construct for each eigenmeasure $\nu \in \mathscr{G}^*(f)$ a shift-invariant finitely additive measure μ by using the eigenvector ξ_f. When this finitely additive measure is a probability measure (countably additive) we show that $\mu \ll \nu$, and consequently how to construct an eigenfunction for the natural extension of the Ruelle operator to $L^1(\nu)$, associated to $\rho(\mathscr{L}_f)$. In Section 5, we present an example where the shift-invariant measure induced by the eigenvector ξ_f is a finitely additive but not countably additive measure. In Section 6 we study a generalization of the classical variational problem (1) for continuous potentials defined on $E^{\mathbb{N}}$, where E is any compact metric space. We introduce a generalization of the Kolmogorov-Sinai entropy in order to obtain a meaningful definition of entropy for shifts in symbol spaces having uncountable alphabets. The problem (1) is then reformulated, in a natural way, in this setting and next we show how to solve this variational problem using the shift-invariant probability measures constructed in Section 4. In Section 7, we provide some concluding remarks.

2 Pressure Functional and Eigenmeasures

In this section we prove the existence of an eigenmeasure for the transpose of the Ruelle operator, associated to $\rho(\mathscr{L}_f)$. To this end we recall that $C(X)^*$ is isometrically isomorphic to $\mathscr{M}_s(X)$ (the space of all finite Borel signed measures endowed with the total variation norm), so we can think of $\mathscr{L}_f^*$ as a bounded linear operator acting on $\mathscr{M}_s(X)$ sending $\nu \mapsto \mathscr{L}_f^* \nu$, where $\mathscr{L}_f^* \nu$ is the unique finite signed Borel measure satisfying

$$\int_X \mathscr{L}_f \varphi \, \mathrm{d}\nu = \int_X \varphi \, \mathrm{d}[\mathscr{L}_f^* \nu], \qquad \forall \varphi \in C(X).$$

Theorem 1 (Pressure Functional) *Let $X = E^{\mathbb{N}}$, where E is a general compact metric space and $f \in C(X)$ a continuous potential. Then there is a real number $P(f)$, called the pressure of the potential f, such that*

$$\lim_{n \to \infty} \sup_{x \in X} \left| \frac{1}{n} \log \mathscr{L}_f^n(1)(x) - P(f) \right| = 0. \tag{3}$$

For finite-state space and Hölder potentials the proof can be found in [5, 15, 48, 54]. For general compact metric spaces and continuous potentials, see [18, 43, 58]. Note that this theorem does not contradict the main result in [57]. Since for each $f \in C(X)$, the expression $n^{-1} \log \mathscr{L}_f^n(1)(x)$, can always be written as a finite-volume pressure of a translation invariant interaction $\Phi = (\Phi_\Lambda)_{\Lambda \in \mathbb{N}}$ on the lattice $\mathbb{N}$, satisfying the following regularity condition: $\sum_{\Lambda \ni 1} \|\Phi_\Lambda\|_\infty < +\infty$.

In view of this comment we think that a one-sided ($\mathbb{N}$) continuous potential appears to provide substantially stronger regularity conditions on the associated Gibbs measures, than a continuous potential in the two-sided ($\mathbb{Z}$) case does.

The next result relates the pressure functional to the logarithm of the spectral radius for continuous potentials defined over general metric compact symbolic spaces.

Corollary 1 *For any $f \in C(X)$ we have $P(f) = \log \rho(\mathscr{L}_f)$.*

Proof The idea is to use Gelfand's Formula for the spectral radius. Since $\mathscr{L}_f$ is a positive operator, f is continuous and X is compact, for each $n \in \mathbb{N}$ we can ensure the existence of some $x_n \in X$ for which

$$\|\mathscr{L}_f^n\|_{\mathrm{op}} \equiv \sup_{\|\varphi\|_\infty = 1} \|\mathscr{L}_f^n(\varphi)\|_\infty = \|\mathscr{L}_f^n(1)\|_\infty = \mathscr{L}_f^n(1)(x_n)$$

$$= \int_{E^n} \exp\left(\sum_{j=0}^{n-1} f \circ \sigma^j(a_1 \dots a_n x_n) \right) \prod_{i=1}^n dp(a_i)$$

$$\geq \int_{E^n} \exp\left(-n\|f\|_\infty \right) \prod_{i=1}^n dp(a_i)$$

$$\geq \exp(-n\|f\|_\infty).$$

The above inequality implies that the spectral radius is non-null. By taking the n-th root of both sides above and then the logarithm, we get from Theorem 1 that

$$\log \|\mathscr{L}_f^n\|_{\mathrm{op}}^{\frac{1}{n}} = \frac{1}{n} \log \mathscr{L}_f^n(1)(x_n) \xrightarrow{n \to \infty} P(f) \geq -\|f\|_\infty.$$

On the other hand, from the boundedness of $\mathscr{L}_f$ and Gelfand's formula, it follows that the limit, when n goes to infinity, of the lhs above is precisely the logarithm of the spectral radius of $\mathscr{L}_f$. Therefore $\log \rho(\mathscr{L}_f) = P(f)$.

We use the notation $\mathscr{M}_1(X)$ for the space of all Borel probability measures on X, endowed with the subspace topology defined from the weak-$*$-topology on $\mathscr{M}_s(X)$.

Now we are ready to prove the main result of this section.

Proposition 1 *For any continuous potential f we have*

$$\mathscr{G}^*(f) \equiv \{\nu \in \mathscr{M}_1(X) : \mathscr{L}_f^* \nu = \rho(\mathscr{L}_f)\nu\}$$

is non-empty.

Proof Let $F : \mathscr{M}_1(X) \to \mathscr{M}_1(X)$ be given by $F(\gamma) = (\mathscr{L}_f^*(\gamma)(X))^{-1} \mathscr{L}_f^*(\gamma)$. Since $\mathscr{M}_1(X)$ is convex and compact, in the weak-$*$-topology which is Hausdorff when X is metric and compact, it follows from the continuity of $\mathscr{L}_f^*$ and the Tychonov-Schauder Theorem that this mapping has at least one fixed point ν. Note that this fixed point is an eigenmeasure for the transpose of the Ruelle operator, i.e., $\mathscr{L}_f^*(\nu) = (\mathscr{L}_f^*(\nu)(X)) \nu$.

We claim that the following bounds,

$$\exp(-\|f\|_\infty) \leq \mathscr{L}_f^*(\nu)(X) \leq \exp(\|f\|_\infty),$$

hold for any $\nu \in \mathscr{M}_1(X)$. In fact, from the inequality $\exp(-\|f\|_\infty) \leq \mathscr{L}_f(1)(x) \leq \exp(\|f\|_\infty)$, which holds for all $x \in X$, we obtain

$$\exp(-\|f\|_\infty) \leq \int_X \mathscr{L}_f(1) \,\mathrm{d}\nu \leq \exp(\|f\|_\infty), \ \forall \nu \in \mathscr{M}_1(X).$$

Since $\int_X \mathscr{L}_f(1) \,\mathrm{d}\nu = \mathscr{L}_f^*(\nu)(X)$ the claim follows.

From the claim it follows that

$$\overline{\rho(\mathscr{L}_f)} \equiv \sup\{\mathscr{L}_f^*(\nu)(X) : \mathscr{L}_f^*(\nu) = (\mathscr{L}_f^*(\nu)(X)) \nu\} < +\infty.$$

A simple compactness argument shows that there is $\nu \in \mathscr{M}_1(X)$ so that $\mathscr{L}_f^* \nu = \overline{\rho(\mathscr{L}_f)}\nu$. Indeed, let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence such that $\mathscr{L}_f^*(\nu_n)(X) \uparrow \overline{\rho(\mathscr{L}_f)}$, when n goes to infinity. Since $\mathscr{M}_1(X)$ is a compact metric space in the weak-$*$-topology, we can assume, up to subsequence convergence, that $\nu_n \rightharpoonup \nu$. This convergence, together with the continuity of $\mathscr{L}_f^*$, provides

$$\mathscr{L}_f^* \nu = \lim_{n \to \infty} \mathscr{L}_f^* \nu_n = \lim_{n \to \infty} \mathscr{L}_f^*(\nu_n)(X)\nu_n = \overline{\rho(\mathscr{L}_f)}\,\nu.$$

Therefore the set $\{\nu \in \mathscr{M}_1(X) : \mathscr{L}_f^* \nu = \overline{\rho(\mathscr{L}_f)}\,\nu\} \neq \emptyset$. It remains to show that $\overline{\rho(\mathscr{L}_f)}$ is the spectral radius of $\mathscr{L}_f$. In order to prove this statement we first observe that

$$\overline{\rho(\mathscr{L}_f)}^{\,n} = \int_X \mathscr{L}_f^n(1)(x)\,\mathrm{d}\nu \leq \|\mathscr{L}_f^n\|_{\mathrm{op}}$$

and therefore $\overline{\rho(\mathscr{L}_f)} \leq \rho(\mathscr{L}_f)$. From the uniform convergence provided by Theorem 1, Corollary 1 and Jensen's inequality we get

$$\log \rho(\mathscr{L}_f) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \mathscr{L}_f^n(1)\,\mathrm{d}\nu \leq \lim_{n\to\infty} \frac{1}{n} \log \int_X \mathscr{L}_f^n(1)\,\mathrm{d}\nu = \log \overline{\rho(\mathscr{L}_f)},$$

thus proving that $\overline{\rho(\mathscr{L}_f)} = \rho(\mathscr{L}_f)$.

3 The Double Transpose and Its Eigenfunctions

In this section we establish some elementary properties of the double transpose of the Ruelle operator associated to a continuous potential f.

By using the isomorphism $C^{**}(X) \simeq \mathscr{M}_s(X)^*$, we can consider the double transpose of $\mathscr{L}_f : C(X) \to C(X)$ as the unique linear operator $\mathscr{L}_f^{**} : \mathscr{M}_s(X)^* \to \mathscr{M}_s(X)^*$ sending $\xi \mapsto \mathscr{L}_f^{**}(\xi)$, defined for each $\mu \in \mathscr{M}_s(X)$ by

$$\mathscr{L}_f^{**}(\xi)(\mu) \equiv \xi(\mathscr{L}_f^* \mu).$$

By identifying $C(X)$ with the image of the natural map $J : C(X) \to \mathscr{M}_s(X)^*$, defined by $J(\varphi)(\mu) = \mu(\varphi)$, we can think of $\mathscr{L}_f^{**}$ as a bounded linear extension of the Ruelle operator

$$
\begin{array}{ccc}
\mathscr{M}_s(X)^* & \xrightarrow{\;\mathscr{L}_f^{**}\;} & \mathscr{M}_s(X)^* \\
{\scriptstyle J}\big\uparrow & & \big\uparrow{\scriptstyle J} \\
C(X) & \xrightarrow{\;\mathscr{L}_f\;} & C(X).
\end{array}
$$

Let $\mathscr{M}_+(X)$ denote the set of all finite positive Borel measures over X. We say that an element $\xi \in \mathscr{M}_s(X)^*$ is positive if $\xi(\mathscr{M}_+(X)) \subset [0, +\infty)$.

Before proceeding, let us first present some basic relations between the eigenfunction of the transfer operator and the eigenvectors of its the double transpose.

Note that, if h_f is a positive continuous eigenfunction of $\mathscr{L}_f$ associated to the maximal eigenvalue $\mathscr{L}_f$, then for any finite signed measure μ we have

$$
\begin{aligned}
\mathscr{L}_f^{**}(J(h_f))(\mu) &= J(h_f)(\mathscr{L}_f^*(\mu)) \\
&= \mathscr{L}_f^*(\mu)(h_f) \\
&= \mu(\mathscr{L}_f(h_f)) = \rho(\mathscr{L}_f)\mu(h_f) = \rho(\mathscr{L}_f)J(h_f)(\mu),
\end{aligned}
$$

moreover, if $\mu \in \mathscr{M}_+(X)$, then $J(h_f)(\mu) = \mu(h_f) > 0$. Therefore $J(h_f)$ is a positive eigenvector of $\mathscr{L}_f^{**}$ associated to $\rho(\mathscr{L}_f)$.

Although the existence of an eigenvector of $\mathscr{L}_f^{**}$ can be obtained sometimes by an abstract argument based on the Hahn-Banach Theorem in general, we can not ensure its positivity. Let us elaborate on this comment.

Let us assume first that there is an eigenvector ξ, which might not be necessarily positive, for $\mathscr{L}_f^{**}$, associated to $\rho(\mathscr{L}_f)$. Then for any signed measure μ we have

$$0 = \mathscr{L}_f^{**}(\xi)(\mu) - \rho(\mathscr{L}_f)\xi(\mu) = \xi(\mathscr{L}_f^*(\mu)) - \rho(\mathscr{L}_f)\xi(\mu) = \xi((\mathscr{L}_f^* - \rho(\mathscr{L}_f))\mu).$$

The above equation implies, when such an eigenvector exists, that the range of the operator $\mathscr{L}_f^* - \rho(\mathscr{L}_f)$ is contained in the kernel of ξ, i.e., $\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f)) \subset \ker(\xi)$. On the other hand, if for some $\nu \in \mathscr{G}^*(f)$ we have

$$\overline{\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f))} \cap \langle \nu \rangle = \{0\}, \qquad \text{where } \langle \nu \rangle \text{ is the subspace generated by } \nu,$$

then, as a consequence of the Hahn-Banach theorem, we can guarantee the existence of at least one continuous functional $\xi \in \mathscr{M}_s^*$ so that $\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f)) \subset \ker(\xi)$ and $\xi(\nu) = 1$. Such a functional is clearly an eigenvector for $\mathscr{L}_f^{**}$ associated to $\rho(\mathscr{L}_f)$. We remark that the eigenvector ξ is not necessarily positive, nor necessarily an element of $J(C(X))$, the image of the natural map.

As long as ψ is a $\mathscr{B}(X)$-measurable real function satisfying $\|\psi\|_\infty < +\infty$, we can naturally define $\mathscr{L}_f \psi$, since for every $x \in X$ the following integral is well defined and finite:

$$\int_E \exp(f(ax))|\psi(ax)| \, \mathrm{d}p(a).$$

Definition 1 Let (X, d_X) a compact metric space and $f : X \to \mathbb{R}$ be a function. We say that f is a **Baire-class-one** function if there is a sequence of continuous functions converging to f pointwise.

Proposition 2 *Let f be a continuous potential and suppose that there exists a Baire-class-one real function $\psi : X \to \mathbb{R}$ satisfying $0 < m \leq \psi \leq M < +\infty$ and $\mathscr{L}_f \psi = \rho(\mathscr{L}_f)\psi$. Then*

$$\overline{\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f))} \cap \mathscr{M}_1(X) = \emptyset.$$

Proof The first step is to show that $(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)(\psi) = 0$, for any signed measure μ. Since we are not assuming that $\psi \in C(X)$, there is a small issue in using the duality relation for the Ruelle operator and its transpose. But this issue can be easily overcome as follows. Let $(\psi_n)_{n \in \mathbb{N}}$ a sequence of continuous functions pointwise converging to ψ. Then by the Dominated Convergence

Theorem, for any finite signed measure μ we have

$$(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)(\psi) = \lim_{n \to \infty} (\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)(\psi_n)$$

$$= \lim_{n \to \infty} \mu(\mathscr{L}_f \psi_n) - \rho(\mathscr{L}_f)\mu(\psi_n)$$

$$= \mu(\mathscr{L}_f \psi) - \rho(\mathscr{L}_f)\mu(\psi)$$

$$= 0.$$

Suppose by contradiction that $\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f)) \cap \mathscr{M}_1(X) \neq \emptyset$. Then there are $\mu \in \mathscr{M}_s(X)$ and $\nu \in \mathscr{M}_1(X)$ such that $(\mathscr{L}_f^* - \rho(\mathscr{L}_f))\mu = \nu$. By using the above equalities and the bounds on ψ, we get $0 < m \leq \min(\psi)\nu(X) \leq \nu(\psi) = (\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)(\psi) = 0$ and therefore $\mathscr{R}(\mathscr{L}_f^* - \rho(\mathscr{L}_f)) \cap \mathscr{M}_1(X) = \emptyset$.

We now prove that there is no sequence $(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu_n)$ converging to a probability measure ν in the strong topology. Suppose by contradiction that

$$\|(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu_n) - \nu\|_T \to 0,$$

when $n \to \infty$. We have already shown that $(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu_n)(\psi) = 0$, and therefore we have

$$m \leq \nu(\psi) = |(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu_n)(\psi) - \nu(\psi)| \leq \|(\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu_n) - \nu\|_T \, \|\psi\|_\infty.$$

Since the rhs converges to zero we reach a contradiction.

Theorem 2 *Let f be a continuous potential and suppose that there exists a Baire-class-one real function $\psi : X \to \mathbb{R}$, satisfying $0 < m \leq \psi \leq M < +\infty$ and $\mathscr{L}_f \psi = \rho(\mathscr{L}_f)\psi$. Then there is a positive element $\xi_f \in \mathscr{M}_s^*(X)$ such that $\mathscr{L}_f^{**}\xi_f = \rho(\mathscr{L}_f)\xi_f$.*

Proof Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence in $C(X)$ such that $\psi_n \to \psi$ pointwise. Since the sequence $\max\{\min\{\psi_n, M\}, m\}$ is continuous and converges pointwise to ψ we can assume that $m \leq \psi_n \leq M$. Now we consider the sequence of linear functionals $(J(\psi_n))_{n \in \mathbb{N}}$ in the bidual of $C(X)$. The natural map is an isometry and so $\|J(\psi_n)\| \leq M$. From the Banach-Alaoglu theorem it follows that the closed ball $\overline{B(0, M)}$ is compact in the weak-$*$ topology, therefore the sequence $(J(\psi_n))_{n \in \mathbb{N}}$ viewed as a topological net has at least one convergent subnet $(J(\psi_{i(d)}))_{d \in D}$ so that

$$\lim_{d \in D} J(\psi_{i(d)}) = \xi_f.$$

We claim that ξ_f is a positive eigenfunction of $\mathscr{L}_f^{**}$. The positivity of ξ_f is trivial, because for any $\mu \in \mathscr{M}_+(X)$ we have $J(\psi_{i(d)})(\mu) \geq m$. In particular, ξ_f is not the null vector. To finish the proof it is enough to show that $\mathscr{R}(\mathscr{L}_f^* -$

$\rho(\mathscr{L}_f)) \subset \ker(\xi_f)$. Indeed, for any $\mu \in \mathscr{M}_s^*(X)$ we have

$$
\begin{aligned}
\xi_f((\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)) &= \lim_{d \in D} J(\psi_{i(d)})((\mathscr{L}_f^* - \rho(\mathscr{L}_f))(\mu)) \\
&= \lim_{d \in D} \mu(\mathscr{L}_f(\psi_{i(d)}) - \rho(\mathscr{L}_f)\psi_{i(d)}) \\
&= \mu(\mathscr{L}_f(\psi) - \rho(\mathscr{L}_f)\psi) \\
&= 0.
\end{aligned}
$$

Note that the function ψ in Theorem 2, up to normalization, can be identified with an element $\xi \in C^{**}(X)$, by using that the unit ball in $C^{**}(X)$ is a dense in the weak-$*$ topology. The positiveness is then a consequence of $0 < \rho(\mathscr{L}_f)$. We chose not prove Theorem 2 by using this abstract argument because our proof presents in a simple case the ideas involved in the proof of Theorem 3.

In what follows we consider the extension $\mathbb{L}_f : L^1(\nu) \to L^1(\nu)$ of the classical Ruelle operator $\mathscr{L}_f : C(X) \to C(X)$. We say that a bounded positive linear operator $\mathbb{L}_f : L^1(\nu) \to L^1(\nu)$ is an extension of the transfer operator $\mathscr{L}_f : C(X) \to C(X)$ if the vector space $C(X)$ embeds in $L^1(\nu)$ and for any $\varphi \in C(X)$ we have $\mathbb{L}_f([\varphi]_\nu) \cap C(X) = \{\mathscr{L}_f \varphi\}$, where $[\varphi]_\nu$ consist of the set of all ν-integrable functions which are ν-equivalent to φ.

When E is a finite set, we can apply either Proposition 2.2 of [23] or Corollary 4 in [65] to obtain this $L^1(\nu)$-extension. For uncountable alphabets this extension is obtained in [20]. The key point in obtaining the extension is to show that any $\nu \in \mathscr{G}^*(f)$ is fully supported, that is, $\operatorname{supp}(\nu) = X$. In [20] the authors prove that whenever $\operatorname{supp}(p) = E$ we have that $\operatorname{supp}(\nu) = X$. Therefore from now on we will assume that $\operatorname{supp}(p) = E$.

One of the advantages of working with the extension $\mathbb{L}_f$ is that we can compute explicitly its operator norm. In fact, note that for any $\varphi \in L^1(\nu)$ we have

$$
\int_X \varphi \, d[\mathscr{L}_f^* \nu] = \int_X \mathbb{L}_f \varphi \, d\nu
$$

and so

$$
\begin{aligned}
\|\mathbb{L}_f\|_{\mathrm{op}} &\equiv \sup_{\|\varphi\|_1 \leq 1} \int_X |\mathbb{L}_f \varphi| \, d\nu \\
&\leq \sup_{\|\varphi\|_1 \leq 1} \int_X \mathbb{L}_f \varphi^+ + \mathbb{L}_f \varphi^- \, d\nu \\
&\leq \sup_{\|\varphi\|_1 \leq 1} \rho(\mathscr{L}_f) \int_X \varphi^+ + \varphi^- \, d\nu \\
&\leq \rho(\mathscr{L}_f)
\end{aligned}
$$

and the supremum is attained if we take the test function $\varphi \equiv 1$.

Let $\tilde{J}$ denote the cannonical map from $L^1(\nu)$ to its bidual. As usual, by identifying $L^1(\nu)$ with $\tilde{J}(L^1(\nu))$, we can look at the double transpose $\mathbb{L}_f^{**}$ as an linear bounded extension of $\mathbb{L}_f$. We recall that the double transpose $\mathbb{L}_f^{**} : L^1(\nu)^{**} \to L^1(\nu)^{**}$ sends $\xi \in L^1(\nu)^{**}$ to $\mathbb{L}_f^{**}\xi$, which is defined for each functional $\ell \in L^1(\nu)^*$ by

$$\mathbb{L}_f^{**}\xi(\ell) \equiv \xi(\mathbb{L}_f^*\ell).$$

Next, we prove that $\mathbb{L}_f^{**}$ always has a positive eigenvector associated to $\rho(\mathscr{L}_f)$.

$$
\begin{array}{ccc}
(L^1(\nu)^{**}, \|\cdot\|_{\mathrm{op}}) & \xrightarrow{\;\mathbb{L}_f^{**}\;} & (L^1(\nu)^{**}, \|\cdot\|_{\mathrm{op}}) \\[4pt]
{\scriptstyle \tilde{J}}\big\uparrow & & \big\uparrow{\scriptstyle \tilde{J}} \\[4pt]
(L^1(\nu), \|\cdot\|_1) & \xrightarrow{\;\mathbb{L}_f\;} & (L^1(\nu), \|\cdot\|_1). \\[4pt]
{\scriptstyle q}\big\uparrow & & \big\uparrow{\scriptstyle q} \\[4pt]
(C(X), \|\cdot\|_1) & \xrightarrow{\;\mathscr{L}_f\;} & (C(X), \|\cdot\|_1).
\end{array}
$$

Now we are ready to prove one of the main results of this paper.

Theorem 3 *Let $X = E^{\mathbb{N}}$, where E is a compact metric space, $f : X \to \mathbb{R}$ be a continuous potential, $\nu \in \mathscr{G}^*(f)$ and $\mathbb{L}_f : L^1(\nu) \to L^1(\nu)$ be the natural extension of the Ruelle operator. Then there exists a positive element $\xi_f \in L^1(\nu)^{**}$ such that*

$$\mathbb{L}_f^{**}\xi_f = \rho(\mathscr{L}_f)\xi_f.$$

Proof For each $n \in \mathbb{N}$ we define

$$\xi_n \equiv \frac{1}{\rho(\mathscr{L}_f)^n} J(\mathbb{L}_f^n(1)),$$

here J denotes the natural map from $L^1(\nu)$ to its bidual. Since the mapping J is an isometry and $\|\mathbb{L}_f^n(1)\|_{L^1(\nu)} \le \rho(\mathscr{L}_f)^n$, it follows that $\|\xi_n\| \le 1$. We observe that the mapping $L^1(\nu) \ni \varphi \mapsto \nu(\varphi)$ is a norm-one element in $L^1(\nu)^*$. Indeed, the norm of this linear functional satisfies $\sup\{|\nu(\varphi)| : \|\varphi\|_1 = 1\} = \sup\{|\nu(\varphi)| : \nu(|\varphi|) = 1\} \le 1$, and the supremum is attained by taking $\varphi \equiv 1$. Moreover, if this functional is denoted by ℓ, then we have for any $n \in \mathbb{N}$, that $\xi_n(\ell) = 1$. In fact,

$$\xi_n(\ell) = \frac{1}{\rho(\mathscr{L}_f)^n} J(\mathbb{L}_f^n(1))(\ell) = \frac{1}{\rho(\mathscr{L}_f)^n} \ell(\mathbb{L}_f^n(1))$$

$$= \frac{1}{\rho(\mathscr{L}_f)^n} \int_X \mathbb{L}_f^n(1)\, d\nu = \frac{1}{\rho(\mathscr{L}_f)^n} \int_X 1\, d[(\mathscr{L}_f^*)^n\nu] = 1.$$

As we did before, if we look at the sequence $(\xi_n)_{n\in\mathbb{N}}$ as a topological net, it follows from the Banach-Alaoglu theorem that this net has at least one convergent subnet $(\xi_{i(d)})_{d\in D}$. Let ξ_f the limit of such a subnet. Clearly this

functional is non-null, positive and $\xi_f(\nu) = 1$. Note that $\rho(\mathscr{L}_f)\xi_{n+1} = \mathbb{L}_f^{**}\xi_n$, for all $n \in \mathbb{N}$. By using the weak-$*$ to weak-$*$ continuity of $\mathbb{L}_f^{**}$ and a similar compactness argument as used for ξ_n, but instead for $\rho(\mathscr{L}_f)\xi_{n+1}$ and $\mathbb{L}_f^{**}\xi_n$ separately, we get

$$\mathbb{L}_f^{**}\xi_f = \rho(\mathscr{L}_f)\widetilde{\xi}_f,$$

for some $\widetilde{\xi}_f \in L^1(\nu)^{**}$. Note that $\xi_{n+1}(\ell) = 1$ for all $n \in \mathbb{N}$ so $\widetilde{\xi}_f(\ell) = 1$. Since $\langle \ell \rangle \oplus \ker(\xi_f) = L^1(\nu)^* = \langle \ell \rangle \oplus \ker(\widetilde{\xi}_f)$ follows that $\ker(\xi_f) = \ker(\widetilde{\xi}_f)$ and therefore $\widetilde{\xi}_f$ is non-zero multiple of ξ_f. The images of these functionals evaluated at ℓ coincide, so they are equal, which allows us to conclude that ξ_f is an eigenvector of $\mathbb{L}_f^{**}$. To complete the proof we observe that for all positive functional $\ell \in L^1(\nu)^*$, we have $\xi_f(\ell) \geq 0$.

Remark 1 One might think that the above result is only a particular case of a more abstract theorem on positive bounded operators in ordered Banach spaces. The problem is that, in general, the space $C(X)$ can not be embedded in $L^1(\nu)$. A sufficient condition for this is that the support of the measure ν is the whole space X. Note that if the support of ν is a singleton, then $L^1(\nu)$ is a one-dimensional vector space over $\mathbb{R}$, while $C(X)$ has infinite dimension. Under our assumptions the support of ν is equal to X, see [20]. And so we can embed $C(X)$ into $L^1(\nu)$. Moreover we also need to compare the spectral radius $\rho(\mathscr{L}_f)$ (which is defined by the action of the transfer operator in $C(X)$) with $\rho(\mathbb{L}_f)$ (the spectral radius of the extension acting on $L^1(\nu)$). In this case we have $\rho(\mathscr{L}_f) = \rho(\mathbb{L}_f) = \|\mathbb{L}_f\|_{\mathrm{op}}$, see [20]. However here, we only needed $\rho(\mathscr{L}_f) = \|\mathbb{L}_f\|_{\mathrm{op}}$ to ensure that the sequence $(\xi_n)_{n \in \mathbb{N}}$ lives on the unit sphere, and consequently has a cluster point. It would be also natural to consider an extension $\mathbb{L}_f^p$ of the transfer operator $\mathscr{L}_f$ to the $L^p(\nu)$-space, where $1 < p < \infty$, since it is a reflexive space. But in this case, where $p > 1$, we have $\rho(\mathbb{L}_f^p) < \|\mathbb{L}_f^p\|_{\mathrm{op}}$. Due to this strict inequality it is not possible anymore to employ the techniques presented above to guarantee that $(\xi_n)_{n \in \mathbb{N}}$ has a cluster point.

It also worth pointing out that the expression for ξ_n appearing in the proof of Theorem 3 could be used to obtain, for example, linear response and statistical properties of random dynamical systems.

4 Invariant Measures and Integrable Eigenfunctions

The next result is an important application of the existence of ξ_f. Before presenting this result let us introduce some more notation. For each $A \in \mathscr{B}(X)$ and $\nu \in \mathscr{M}_1(X)$ we define a non-negative measure in $\mathscr{B}(X)$ so that $B \mapsto \nu(A \cap B)$. This measure will be simply denoted by $1_A\nu$. It will be convenient to identify the measure $1_A\nu$ with the element of $L^1(\nu)^*$ given by $\varphi \mapsto \nu(1_A\varphi)$.

Recall that a dual pair $\langle X, Y \rangle$ is a pair of vector spaces X and Y with a bilinear map $X \times Y \ni (x, y) \longmapsto \langle y, x \rangle \in \mathbb{R}$, which is nondegenerate in the sense that $\forall x, \exists y \ \langle y, x \rangle \neq 0$ and $\forall y, \exists x \ \langle y, x \rangle \neq 0$. Here we are interested in the case where $X = L^1(\nu)$, $Y = L^1(\nu)^*$, and $\langle y, x \rangle = y(x)$.

Theorem 4 *Let f be a continuous potential, $\nu \in \mathscr{G}^*(f)$ and let ξ_f be a eigenfunction of $\mathbb{L}_f^{**}$ as constructed above. Then the set function μ given by $\mathscr{B}(X) \ni A \mapsto \xi_f(1_A\nu)$ is a non-negative additive shift-invariant set function.*

Proof We first show that $\mu(\sigma^{-1}(A)) = \mu(A)$. By the definition of μ we have $\mu(\sigma^{-1}(A)) = \xi_f((1_A \circ \sigma) \cdot \nu)$. From the definition of the transpose of the Ruelle operator, the Riesz L^p duality theorem and basic properties of the Ruelle operator, we have for any $\varphi \in L^1(\nu)$ that

$$\langle (1_A \circ \sigma) \cdot \nu, \varphi \rangle = \int_X 1_A \circ \sigma \cdot \varphi \, \mathrm{d}\nu = \rho(\mathscr{L}_f)^{-1} \int_X 1_A \mathbb{L}_f \varphi \, \mathrm{d}\nu$$
$$= \rho(\mathscr{L}_f)^{-1} \langle 1_A \cdot \nu, \mathbb{L}_f \varphi \rangle$$

so that

$$\xi_f((1_A \circ \sigma) \cdot \nu) = \rho(\mathscr{L}_f)^{-1} \xi_f(\mathbb{L}_f^*(1_A\nu)) = \xi_f(1_A\nu).$$

It is obvious that μ is a positive finitely additive set function, thus concluding the proof.

We remark that the set $\mathscr{G}^*(f) \equiv \mathscr{G}^*(f, p)$, where p is the a priori measure fixed at the beginning of this paper. Consequently, the measure μ defined above also depends on p.

Theorem 5 *Let f be a continuous potential and $\nu \in \mathscr{G}^*(f)$. Assume that the measure μ induced by $\xi_f \in L^1(\nu)^{**}$, as in Theorem 4, is a probability measure (countably additive). Then $\mu \ll \nu$ and the Radon-Nikodym derivative $d\mu/d\nu$ is a eigenfunction associated to $\rho(\mathscr{L}_f)$*

$$\mathbb{L}_f \frac{d\mu}{d\nu} = \rho(\mathscr{L}_f) \frac{d\mu}{d\nu}. \tag{4}$$

Proof Since $\mu(A) = \xi_f(1_A\nu)$, for each Borel set A it follows that $\mu \ll \nu$. The proof of equation (4) is identical to the one given in [18]. Since it is simple, for the convenience of the reader, we repeat the arguments below.

For any continuous function φ we have

$$\int_X \varphi \, \mathbb{L}_f \left(\frac{d\mu}{d\nu} \right) d\nu = \int_X \mathbb{L}_f \left(\varphi \circ \sigma \cdot \frac{d\mu}{d\nu} \right) d\nu$$
$$= \rho(\mathscr{L}_f) \int_X \varphi \circ \sigma \cdot \frac{d\mu}{d\nu} \, d\nu = \rho(\mathscr{L}_f) \int_X \varphi \circ \sigma \cdot d\mu$$
$$= \rho(\mathscr{L}_f) \int_X \varphi \, d\mu = \rho(\mathscr{L}_f) \int_X \varphi \cdot \frac{d\mu}{d\nu} \, d\nu.$$

The above theorem is one of the most important results of this paper. It ensures that the spectral radius of the Ruelle operator $\mathbb{L}_f$ is always an eigenvalue, as long as the finitely additive set function μ provided by Theorem 4 is a probability measure (countably additive). We remark that in general, there is no $L^1(\nu)$ eigenfunction e.g. for a generic expanding C^1 map of the circle if we consider the geometric potential then there is no absolutely continuous

invariant measure with respect to the Lebesgue measure which in this case plays the role of the eingenmeasure. Actually, this result can be reformulated in our setting, shift spaces, through a Markov coding, see Section 5 for details.

Note that the above Radon-Nikodym derivative is not in general a continuous function. This derivative is continuous when the potential f has good regularity properties, but for some continuous potentials f we know that there is no continuous eigenfunction for $\mathscr{L}_f$ associated to the spectral radius $\rho(\mathscr{L}_f)$, see for example reference [17].

These previous results show that the existence of the maximal eigenfunctions for $\mathbb{L}_f$ can be established by showing that the set function $\mu(A) = \xi_f(1_A\nu)$ is countably additive. This problem can overcome if one proves that μ is a regular measure because of Alexandroff's theorem. Let us elaborate on this. We say that a Borel additive signed measure μ on a topological space $X = (X, \tau)$ is called regular, if given $A \in \mathscr{B}(\tau)$ and $\varepsilon > 0$, there exists a closed set $F \subset A$ and an open set $O \supset A$ such that for all Borel sets $C \subset (O \setminus F)$ we have $|\mu(C)| < \varepsilon$. The Alexandroff theorem ensures that if μ is a bounded regular complex-valued additive set function defined on the Borel sigma-algebra of a compact topological space, then μ is countably additive. For a proof see [24, III.5.13].

We remark that the regularity required by the Alexandroff theorem will in general not be satisfied. To show this we provide an example where μ can not be a probability measure in Section 5.

Alternatively, we can study the Yosida-Hewitt decomposition [69] of $\mu = \mu_a + \mu_c$. The term μ_a corresponds to the purely finitely additive part and μ_c is the countably additive part. This decomposition is constructed in such way that the pure finitely additive part is characterized by the following property: if γ is a non-negative countably additive measure such that $0 \leqslant \gamma \leqslant \mu_a$, then $\gamma = 0$. Note that $\mu_c \ll \nu$ and so, by the Radon-Nikodym theorem, there is some h such that $d\mu = d\mu_a + h\nu$. Of course, if $\mu_c = 0$ then $h = 0$. In what follows we present an example where $\mu_c = 0$.

5 Manneville-Pomeau Maps and the Yosida-Hewitt Decomposition

Fix a real number $\alpha \geq 2$ and consider the dynamics given by a Manneville-Pomeau map $T \equiv T_\alpha : [0, 1] \to [0, 1]$ which is defined for each $x \in [0, 1]$ by the following expression: $T(x) = x + x^\alpha \pmod 1$. Note that T induces a continuous map on the one-dimensional torus $\mathbb{T}$, which will be denoted by $T : \mathbb{T} \to \mathbb{T}$. For any $x \in \mathbb{T}$ we have that $T'(x) = 1 + \alpha x^{\alpha-1}$, where T' denotes the left derivative. The cover $\{[0, x_0(\alpha)], [x_0(\alpha), 1]\}$ of $\mathbb{T}$ defines a Markov partition and we can prove that the Manneville-Pomeau maps we are considering here are semi-conjugated to the full-shift on two-symbol sequences

$$
\begin{array}{ccc}
\{0,1\}^{\mathbb{N}} & \xrightarrow{\ \sigma\ } & \{0,1\}^{\mathbb{N}} \\
\pi_\alpha \downarrow & & \downarrow \pi_\alpha \\
\mathbb{T} & \xrightarrow{\ T\ } & \mathbb{T}
\end{array}
$$

that is, there is a surjective (not injective) map $\pi_\alpha : \{0,1\}^{\mathbb{N}} \to \mathbb{T}$ so that $\pi_\alpha \circ \sigma = T \circ \pi_\alpha$. The semi-conjugacy π_α is 1 to 1 except for countably many points.

Consider the potential $g : \mathbb{T} \to \mathbb{R}$ given by $g(x) \equiv -\log |T'(x)|$. In this setting the Ruelle operator $\mathcal{L}_g : C(\mathbb{T}, \mathbb{R}) \to C(\mathbb{T}, \mathbb{R})$ is defined as follows

$$\mathcal{L}_g(\varphi)(x) = \sum_{y \in T^{-1}(x)} \exp(g(y))\varphi(y).$$

One can prove that the Lebesgue measure λ is an eigenmeasure associated to the spectral radius of $\mathcal{L}_g$, which is in this case equal to one. Now we consider the continuous potential $f \equiv -\log T' \circ \pi_\alpha$ defined on the symbol space $\{0,1\}^{\mathbb{N}}$. It well-known that $\rho(\mathscr{L}_f) = 1$, and $\nu \circ \pi_\alpha^{-1} = \lambda$, where $\mathscr{L}_f^* \nu = \nu$. If we assume that the set function $\mu(A) = \xi_f(1_A \nu)$ is regular, then μ is a shift-invariant probability measure satisfying $\mu \ll \nu$. This implies the existence of a T-invariant probability measure $\rho = \mu \circ \pi_\alpha^{-1}$ which is absolutely continuous with respect to the Lebesgue measure. But for $\alpha \geq 2$ this is a contradiction with some known results about the non-existence of such measures, see [41, 50,61]. Actually, the absence of an invariant probability measure obtained in [61] should be related in our setting to the Yosida-Hewitt decomposition of μ. Moreover, if we consider the Yosida-Hewitt decomposition of $\mu = \mu_a + \mu_c$, a small variation of the previous argument implies that $\mu_c = 0$. The way μ is constructed, suggests that $F_{\mu_a}|_{C(X)} = \delta_0$, which is an equilibrium state for this system.

This example shows that the hypotheses in Theorem 5 can not be weakened in general, that is, for some continuous potential f it is possible that the set function $A \longmapsto \xi_f(1_A \nu)$ is not countably additive, but only a finitely additive measure.

Note that Alexandroff's Theorem [24, III.5.13] does not apply here because the inner and outer regularity can be broken. So in general the set function defined in Theorem 5 can really be only a finitely additive measure.

Note that our result implies a curious result about such Manneville-Pomeau maps. For these dynamical systems there is always at least one finitely additive T-invariant "probability" measure μ such that $\mu(Z) = 0$, for all $Z \in \mathscr{B}([0,1])$ satisfying $\text{Leb}(Z) = 0$. It is not clear whether such objects can be used to get quantitative information on the asymptotic behavior of means of observables evaluated on typical orbits of the dynamical system.

On the other hand, for f in a dense subset of $C(X)$, there is a positive eigenfunction $h_f \in C(X)$ for the Ruelle operator $\mathscr{L}_f$ associated to $\rho(\mathscr{L}_f)$. Therefore $\mathscr{L}_f^{**} J(h_f) = \rho(\mathscr{L}_f) J(h_f)$, where J is Jordan canonical mapping. By taking $\xi_f = J(h_f)$ one can see that

$$\int_X 1_A h_f \, d\nu = \xi_f(1_A \nu),$$

thus proving that the regularity of the set function $A \longmapsto \xi_f(1_A \nu)$ is verified for potentials in a dense subset of $C(X)$.

6 The Variational Problem

Our next concern will be solving a generalization of the variational problem (1) using the theory developed here. To guide the discussion, we start by recalling the common strategy normally employed to solve this problem when the state space is finite and the potential f is Hölder. After this discussion we turn our attention to general state spaces and continuous potentials.

Let us assume for a moment that $E = \{0, 1, \ldots, d-1\}$ and $X = E^{\mathbb{N}}$. We recall that a probability measure $\mu \in \mathscr{M}_\sigma(X)$ is said to be an *equilibrium state* for the potential f if the supremum in (1) is attained at μ, i.e.,

$$h_\mu(\sigma) + \int_X f \, \mathrm{d}\mu = \sup_{\nu \in \mathscr{M}_\sigma(X)} \{h_\mu(\sigma) + \int_X f \, \mathrm{d}\nu\}. \qquad (5)$$

In this finite-state space it is very well known how to construct an equilibrium state μ by means of the maximal eigendata of the Ruelle operator. For example, if f is a Hölder potential, then the classical Ruelle-Perron-Frobenius theorem implies the existence of h_f and ν_f, the maximal eigenfunction and eigenmeasure, of the Ruelle operator and its transpose, respectively. By taking a suitable normalization one can prove that the probability measure $h_f \nu_f$ is the unique equilibrium state for f, see [48].

When working with a possible uncountable state space E and a dynamics given by the left shift mapping, to avoid trivialities in the variational problem one needs to avoid using the Kolmogorov-Sinai entropy in its formulation. To obtain a generalization of the finite-state space we adopt here the setting usually considered in Statistical Mechanics, see [29,33,63].

If μ and ν are two arbitrary finite measures over X and $\mathscr{A}$ is a sub-σ-algebra of $\mathscr{B}(X)$ we define

$$\mathscr{H}_\mathscr{A}(\mu|\nu) = \begin{cases} \int_X \dfrac{d\mu|_\mathscr{A}}{d\nu|_\mathscr{A}} \log\left(\dfrac{d\mu|_\mathscr{A}}{d\nu|_\mathscr{A}}\right) \, \mathrm{d}\nu, & \text{if } \mu \ll \nu \text{ on } \mathscr{A}; \\[2ex] \infty, & \text{otherwise.} \end{cases}$$

The extended real number $\mathscr{H}_\mathscr{A}(\mu|\nu)$ is the negative of the *relative entropy* of μ with respect to ν on $\mathscr{A}$. Consider the product measure $\boldsymbol{p} = \prod_{n \in \mathbb{N}} p$, where the probability measure p still is the a priori measure used to construct the Ruelle operator. For each $\mu \in \mathscr{M}_\sigma(X)$ it is proved in [29, Theorem 15.12] that the following limits exist

$$\mathtt{h}(\mu) \equiv -\lim_{n \to \infty} \frac{1}{n} \mathscr{H}_{\mathscr{B}(X)_n}(\mu|\boldsymbol{p}),$$

where $\mathscr{B}(X)_n$ is the σ-algebra generated by the projections $\{\pi_j : X \to E : 1 \le j \le n\}$. If $\#E = d$, we can show that $\mathtt{h}(\mu) + \log d = h_\mu(\sigma)$. Therefore both entropies determine the same set of equilibrium states in the finite-alphabet setting.

As usual, we define the pressure of $f \in C(X)$ by

$$P(f) \equiv \sup_{\mu \in \mathscr{M}_\sigma(X)} \{ \mathtt{h}(\mu) + \int_X f \, \mathrm{d}\mu \}.$$

If f is an arbitrary continuous potential and $\rho(\mathscr{L}_f)$ is the spectral radius of $\mathscr{L}_f$, then we can show that $P(f) = \log \rho(\mathscr{L}_f)$, see [42] for Hölder potentials and [18] for continuous potentials. If f is a Hölder potential, it follows from one of the main results of [3] that the probability measure μ given by $\mathscr{B}(X) \ni A \mapsto \xi_f(1_A \nu)$ has to be the unique equilibrium state for f. This fact follows from Theorem 4, Theorem 5 and the uniqueness of the equilibrium states for Hölder potentials. If f is a more general continuous potential we do not know what are necessary and sufficient conditions on the potential ensuring that the set function μ still is an equilibrium state for f. If ξ_f does not belong to the image of the natural map, the set function μ may neither be a probability measure nor a quasilocal measure. In that case the equilibrium measure does not necessarily satisfy the DLR conditions, and, using the Israel-Bishop-Phelps Theorem, many types of phase transitions and pathologies can be shown to occur, for more details see [10, 22, 32, 34].

7 Concluding Remarks

If the Yosida-Hewitt decomposition of μ is not trivial, meaning $d\mu = d\mu_a + h d\nu$, for some non-trivial h, it is natural to ask whether $h d\nu$, up to a normalization, is or not an equilibrium state for f, even if f has low regularity properties. This could be obtained by generalizing the Rokhlin formula to this context. We also remark that in [9] the authors investigate thoroughly dynamical properties of finitely additive equilibrium states.

A positive operator $T : L^\infty(\nu) \to L^\infty(\nu)$ satisfying $T(1_X) = 1_X$ is called a Markov operator, see [20]. We can show that $\mathbb{L}^*_{(f - \log \rho(\mathscr{L}_f))}$ is a Markov operator. And so we can use Theorem 18.4 in [4] to obtain a similar result as our Theorem 3. The difference of this approach compared with ours is that it uses the Brouwer-Schauder-Tychonoff fixed point theorem, while in our proof we used the Banach-Alaoglu theorem. Although both proofs are based on a compactness argument, our proof is more constructive because the eigenvector is obtained by means of an explicit sequence $(\xi_n)_{n \in \mathbb{N}}$.

As mentioned before, the maximal eigenfunctions of the Ruelle operator are useful in getting information on equilibrium states, ergodic optimization and large deviations. Here we study them in a weaker sense. We proved, under appropriate conditions, that the weak solutions (bidual solution) can be "regularized" to classic solutions (integrable functions). Of course, to exploit all the benefits of the previous results in Thermodynamic Formalism, one would like to take this regularization proceeding from

$$L^1(\nu)^{**} \to L^1(\nu) \to L^2(\nu) \to \ldots \to L^p(\nu) \to L^\infty(\nu) \to C(X).$$

To know what are all the continuous potentials for which the bidual solutions, for the maximal eigenvalue problem, can be embedded in $C(X)$, beyond being an interesting mathematical problem, would have several applications. In particular, some examples where this regularization is not possible from $L^1(\nu)^{**}$ to $C(X)$ involves potentials where we do have some sort of phase transition. It is not clear whether there is a connection between such obstructions and phase transition phenomena. Of course, if f is either a Hölder or Walters potential then it is possible to go all the way from the bidual space $L^1(\nu)^{**}$ to $C(X)$.

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