THERMODYNAMIC FORMALISM FOR TOPOLOGICAL MARKOV CHAINS ON STANDARD BOREL SPACES

L. CIOLETTI, E. A. SILVA, AND M. STADLBAUER

Abstract. We develop a Thermodynamic Formalism for bounded continuous potentials defined on the sequence space $X \equiv E^{\mathbb{N}}$, where E is a general standard Borel space. In particular, we introduce meaningful concepts of entropy and pressure for shifts acting on X and obtain the existence of equilibrium states as finitely additive probability measures for any bounded continuous potential. Furthermore, we establish convexity and other structural properties of the set of equilibrium states, prove a version of the Perron-Frobenius-Ruelle theorem under additional assumptions on the regularity of the potential and show that the Yosida-Hewitt decomposition of these equilibrium states does not have a purely finite additive part.

We then apply our results to the construction of invariant measures of time-homogeneous Markov chains taking values on a general Borel standard space and obtain exponential asymptotic stability for a class of Markov operators. We also construct conformal measures for an infinite collection of interacting random paths which are associated to a potential depending on infinitely many coordinates. Under an additional differentiability hypothesis, we show how this process is related after a proper scaling limit to a certain infinite-dimensional diffusion.

1. Introduction

One of the principal motivations of Ergodic Theory is to understand the statistical behavior of a deterministic dynamical system $T : X \to X$ by studying invariant probability measures of the system. In this context, ergodic theorems provide quantitative information on the asymptotic behavior of typical orbits of T. However, if T is continuous, X is compact and the dynamical system has some sort of mixing behavior, then there exists a plethora of these invariant measures. In these cases, the theory of Thermodynamic Formalism is nowadays a recognized method for making a canonical choice of an invariant measure. That is, one fixes a continuous potential $f : X \to \mathbb{R}$ which encodes some qualitative behaviour of the system and considers those invariant probability measures, the so-called *equilibrium states*, which satisfy a certain variational problem with respect this potential and which exist by compactness. However, if X is not a compact space, additional hypotheses on f are required in order to ensure the existence of such canonical ergodic probability measures.

For example, if X is a shift-invariant, closed subset of $E^{\mathbb{N}}$, where E is an infinite countable set, the existence of equilibrium states is non-trivial and has been

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intensively investigated, due to its applications to the Gauss map, to partially hyperbolic dynamical systems and to unbounded spin systems in Statistical Mechanics on one-dimensional one-sided lattices. From the viewpoint of abstract Thermodynamic Formalism, Mauldin & Urbanski, Sarig and many others developed a rather complete theory for potentials on $X \subset E^{\mathbb{N}}$, where E is an infinite countable alphabet (see, for example, [BS03, CS09, Sar99, Sar01a, Sar01b, Sar03, Sar09, MU99, MU01, MU03]). From the viewpoint of Statistical Mechanics, there is also a vast literature about unbounded spins systems, they could be either the set of integer numbers $\mathbb{Z}$ or continuous $\mathbb{R}$, but usually the interactions are unbounded as in SOS, discrete Gaussian, Φ^4 models and so on. See, for example, [BH99, COPP78, DP83, Geo11, KKP12, LP76, LS17, Zeg96, FZ91]. Even though in all these references, the concepts of pressure, entropy and thermodynamic limit play a major role, we do not yet have a unified framework which relates these concepts across areas. For example, the potentials considered in the Thermodynamic Formalism literature usually depend on infinitely many coordinates (which can be seen as infinite-body interactions) and satisfy suitable regularity conditions, and the alphabet is countable. On the other hand, in the Statistical Mechanics literature, the potential is typically less regular, infinite-range, the potential might be translation invariant or not, some times quasi-periodic potentials are considered and the spins might take values in an uncountable set (or uncountable alphabet), but are usually defined in terms of finite-body interactions. The theory developed in this article now allows to consider potentials given by infinite-body interactions, uncountable alphabets and general bounded and continuous potentials. These three theories are related as shown in the following diagram. We shall remark that no proper inclusion on the diagram below is possible.

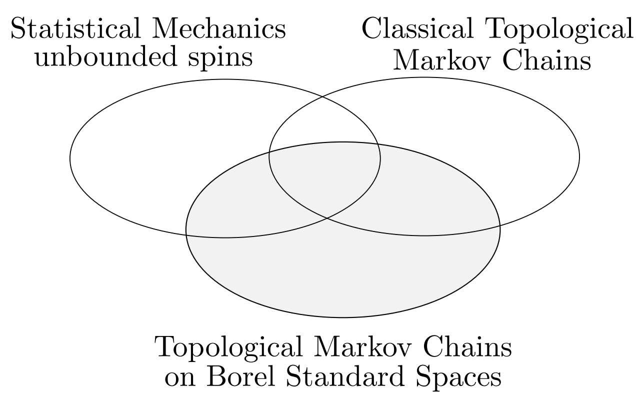

Figure 1. Relations among the three theories.

The classical Thermodynamic Formalism has its starting point in the seminal work by David Ruelle [Rue68] on the lattice gas model and was subsequently developed for subshifts of finite type, which are subsets of $M^{\mathbb{N}}$ with $M = \{1, \ldots, k\}$ (see, for example, [Bal00, Bow08, PP90, Rue04]) and are nowadays well-known tools in the context of hyperbolic dynamical systems. By considering a notion of pressure based on local returns, the Gurevic pressure, Sarig was able to extend the principles of Thermodynamic Formalism in [Sar99] to countable alphabets and obtained, among other things, a classification of the underlying dynamics into positively recurrent, null recurrent or transient behaviour through convergence of the transfer operator in [Sar01b] or a proof of Katok's conjecture on the growth of periodic points of surface diffeomorphisms ([Sar13]).

However, from the viewpoint of Statistical Mechanics, it is also of interest to consider shift spaces with a compact metric alphabet which was done, for example, in [ACR18, BCL$^+$11, CL16, CS16, CvER17, LMMS15, SdSS14, Sil17]. In [BCL$^+$11], a Ruelle operator formalism was developed for the alphabet $M = \mathbb{S}^1$ and extended to general compact metric alphabets in [LMMS15]. As uncountable alphabets do not fit in the classical theory, as the number of preimages under the shift map is uncountable, the authors considered an *a priori* measure μ defined on M which allows to define a generalized Ruelle operator and prove a Perron-Frobenius-Ruelle Theorem. We would like to point out that the use of an *a priori* measure is a standard procedure in Equilibrium Statistical Mechanics in order to deal with continuous spin systems, see [Geo11, vEFS93], and, in combination with the given potential function, is also closely related to the notion of a transition kernel from probability theory.

In this setting, it is necessary to propose new concepts of entropy, the so-called *variational entropy*, and pressure. So an *equilibrium state* for a continuous potential f is an element of $\mathscr{M}_\sigma(X)$, the set of all shift-invariant Borel probability measures, such that this measure realizes the supremum

$$\sup_{\mu \in \mathscr{M}_\sigma(X)} \{\mathtt{h}^{\mathtt{v}}(\mu) + \langle \mu, f \rangle\},$$

where $\mathtt{h}^{\mathtt{v}}(\mu)$ is the variational entropy of μ as introduced in [LMMS15]. The associated variational principle was obtained in [LMMS15] and the uniqueness of the equilibrium state in the class of Walters potentials in [ACR18]. In there, the authors also showed that the variational entropy defined in [LMMS15] equals the *specific entropy* commonly used in Statistical Mechanics (see [Geo11]). As a corollary, a variational formulation for the specific entropy is derived. It is also worth noting that several results for countable alphabets can be recovered by choosing a suitable a priori measure on the one-point compactification of $\mathbb{N}$ (see [LMMS15]) and that the concepts of Gibbs measures and equilibrium states are equivalent if one considers potentials which are Hölder continuous or in Walters' class ([BFV18, CL17, CL16, FGM11]).

The aim of this article is to develop a Thermodynamic Formalism for continuous and bounded potentials and alphabets which are standard Borel spaces. In this very general setting, one has to consider *ergodic finitely additive probability measures* instead of ergodic probability measures as it will turn out in Theorem 4.4 and Corollary 5.2 below that the following holds.

Main Results. **(Equilibrium States)**. Let f be a bounded and continuous potential. Then there exists a shift invariant and finitely additive measure which attains the supremum

$$\sup_{\mu \in \mathscr{M}^a_\sigma(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle \mu, f \rangle.$$

(Ergodic Optimization). Let E be a non-compact space, then there exists a bounded and continuous potential f, having a unique maximizing measure

$$m(f) = \sup_{\mu \in \mathscr{M}^a_\sigma(X)} \langle \mu, f \rangle$$

which is finitely but not necessarily countably additive.

Although finitely additive measures lead to a very abstract setting, we shall mention that these objects have been for a long time important mathematical objects in several branches of pure and applied Mathematics, and naturally occur, for example, in the Fundamental Theorem of Asset Pricing under the absence of arbitrages of the first kind ([Kar10]).

This paper is organized as follows. In Section 2 we introduce the basic notation and recall the definition of the space $rba(X)$ as well as some of its basic properties. After that, the Ruelle operator acting on $C_b(X,\mathbb{R})$ is introduced, where $X = E^{\mathbb{N}}$ is a cartesian product of a general standard Borel space E. In Section 3 we prove a Perron-Frobenius-Ruelle (PFR) theorem for bounded Hölder potentials defined on X and obtain a Central Limit Theorem as a corollary. Thereafter, we use PFR theorem to motivate the definition of the entropy and pressure. This leads to a natural definition of an equilibrium state as an element of $rba(X)$. We prove its existence for general bounded continuous potentials and also show that the supremum in the variational problem is attained by some shift-invariant regular finitely additive Borel probability measure. As a complement, it is proven that the set of equilibrium states is convex and compact and that bounded Hölder potentials admit equilibrium states whose Yosida-Hewitt decomposition does not have a purely finitely additive part. In Section 5, we then prove a characterization of the extremal mesures in order to obtain the second part of our main theorem. Thereafter, in Section 6, the above Perron-Frobenius-Ruelle theorem is applied in the context of ergodic optimization and asymptotic stability of stochastic processes, and, we show in part 6.4 how to use this theorem in order to construct an equilibrium state for infinite interacting random paths subject to an infinite-range potential. We briefly discuss how their scaling limits are connected to some diffusions in infinite dimensions.

2. Preliminaries

A measurable space $(E,\mathscr{E})$ is a standard Borel space if there exists a metric d_E such that (E,d_E) is a complete separable metric space and $\mathscr{E}$ is the Borel sigma-algebra. Good examples to have in mind in order to compare our results with the classical ones in the literature are a finite set $\{1,\ldots,d\}$, the set of positive integers $\mathbb{N}$, a compact metric space K or the Euclidean space $\mathbb{R}^d$. Throughout this paper, X denotes the product space $E^{\mathbb{N}}$ and $\sigma : X \to X$, $(x_1,x_2,\ldots) \mapsto (x_2,x_3,\ldots)$ is the left shift. The space X is regarded as a metric space with metric

$$d_X(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{d_E(x_n,y_n),1\}.$$

As easily can be verified, X is always a bounded, complete and separable metric space, even though it may not be compact. Furthermore, we refer to $C_b(X,\mathbb{R})$ as the Banach space of all real-valued bounded continuous functions endowed with its standard supremum norm.

A Borel finitely additive signed measure on a topological space $X = (X,\tau)$ is an extended real valued set-function $\mu : \mathscr{B}(\tau) \to \mathbb{R}\cup\{-\infty,+\infty\}$ which satisfies *(i)* μ assumes at most one of the values $-\infty$ and ∞, *(ii)* $\mu(\emptyset) = 0$, *(iii)* for each finite family $\{A_1,\ldots,A_n\}$ of pairwise disjoint sets in $\mathscr{B}(\tau)$, we have $\mu(A_1\cup\ldots\cup A_n)=\mu(A_1)+\ldots+\mu(A_n)$. If $\sup_{A\in\mathscr{B}(\tau)}|\mu(A)|<+\infty$ for all $A\in\mathscr{B}(\tau)$, then we say that μ is bounded. A Borel finitely additive signed measure μ is called regular if for any $A\in\mathscr{B}(\tau)$ and $\varepsilon>0$, there exists a closed set $F\subset A$ and an open set $O\supset A$ such that for all Borel sets $C\subset O\setminus F$ we have $|\mu(C)|<\varepsilon$. The total variation norm of a Borel finitely additive signed measure μ is defined by

$$\|\mu\|_{TV}\equiv\sup\left\{\sum_{k=1}^{n}|\mu(A_k)|:\ \{A_1,\ldots,A_n\}\subset\mathscr{B}(\tau)\text{ is a partition of }X\right\}.$$

It is known that the space of all regular bounded Borel finitely additive signed measures on a topological space X endowed with the total variation norm is a Banach space and that, since X is a metric space, the topological dual $C_b(X,\mathbb{R})^*$ is isometrically isomorphic to $(rba(X),\|\cdot\|_{TV})$ (see IV - Th. 6 in [DS58] or Th. 14.9 in [AB06]). By [DS58, p. 261], every $f\in C_b(X,\mathbb{R})$ is integrable with respect to every $\mu\in rba(X)$, and its integral will be denoted by either $\mu(f)$, $\int_X f\,d\mu$ or $\langle\mu,f\rangle$. A countably additive Borel measure is an element $\mu\in rba(X)$ which is both countably additive and non-negative, that is, $\mu(A)\geq 0$ for all $A\in\mathscr{B}(\tau)$. If, in addition, $\mu(X)=1$ then μ is called a countably additive Borel probability measure, and we will make use of $\mathscr{M}_1(X)$ for the subset of $rba(X)$ of all countably additive Borel probability measures. Furthermore, a regular finitely additive bounded Borel signed measure is said to be shift-invariant if $\mu(f)=\mu(f\circ\sigma)$ for all $f\in C_b(X,\mathbb{R})$.

In this paper, a generalized version of the Ruelle transfer operator will play a major role. Therefore, we first fix a Borel probability measure p on E and a potential $f\in C_b(X,\mathbb{R})$. The Ruelle operator is defined as the positive linear operator $\mathscr{L}_f: C_b(X,\mathbb{R})\to C_b(X,\mathbb{R})$ sending $\varphi\longmapsto\mathscr{L}_f\varphi$ defined by

$$\mathscr{L}_f\varphi(x)\equiv\int_E e^{f(ax)}\varphi(ax)\,dp(a),\quad\text{where } ax\equiv(a,x_1,x_2,\ldots).$$

In particular, it follows by induction that, for all $n\in\mathbb{N}$, $dp^n(a_1,\ldots,a_n)\equiv dp(a_1)\cdots dp(a_n)$ and $f_n(x)\equiv\sum_{k=0}^{n-1}f(\sigma^k(x))$,

$$\begin{aligned}\mathscr{L}_f^n(\varphi)(x)&=\int_{E^n}e^{f_n(a_1\ldots a_nx)}\varphi(a_1\ldots a_nx)\,dp(a_1)\cdots dp(a_n)\\&\equiv\int_{E^n}e^{f_n(ax)}\varphi(ax)\,dp^n(a).\end{aligned}$$

Since $\|\mathscr{L}_f1\|_\infty<+\infty$ the Ruelle operator is bounded and the action of its dual (or Banach transpose) $\mathscr{L}_f^*$ on a generic element $\mu\in rba(X)$ is determined by

$$\int_X\varphi\,d[\mathscr{L}_f^*\mu]=\int_X\mathscr{L}_f(\varphi)\,d\mu,\quad\forall\varphi\in C_b(X,\mathbb{R}).$$

Remark 2.1. Assume $E=\{1,\ldots,d\}$, the a priori measure p is the normalized counting measure on E, and f is a continuous potential. Then we have, for all $\varphi\in C(X,\mathbb{R})$

$$\mathscr{L}_f(\varphi)(x)=\int_E e^{f(ax)}\varphi(ax)\,dp(a)=\sum_{y\in\sigma^{-1}(x)}e^{\widetilde{f}(y)}\varphi(y),$$

where $\widetilde{f}\equiv f-\log d$. This shows thus that, in this particular setting, the Ruelle operator associated to a potential f considered here coincides with the classical

Ruelle operator but associated to a potential that differs from the original one by a constant.

In order to motivate the concepts of pressure and entropy introduced in Section 4, we prove in the sequel a Perron-Frobenius-Ruelle theorem for bounded Hölder potentials.

3. Perron-Frobenius-Ruelle Theorem

In this section we are interested in the space of bounded Hölder continuous functions $\mathrm{Hol}(\alpha) \equiv \mathrm{Hol}_\alpha(X,\mathbb{R})$, for $0 < \alpha < 1$, which is defined as the space $\{f \in C_b(X,\mathbb{R}) : \mathrm{D}_\alpha(f) < \infty\}$, where

$$\mathrm{D}_\alpha(f) \equiv \sup_{x\neq y} \frac{|f(x)-f(y)|}{d_X(x,y)^\alpha}.$$

Combining Hölder continuity of f with $d(\sigma^n(x),\sigma^n(y)) = 2^n d(x,y)$, which is valid for points having the same first n coordinates, it follows from a standard argument that there exists C_f such that

$$\left|1 - e^{f_n(ay)-f_n(ax)}\right| \leq C_f\, d(x,y)^\alpha \quad \forall x,y \in X \text{ and } a \in E^n. \tag{1}$$

By (1), it is now easy to see that $\mathscr{L}_f^n$ maps $\mathrm{Hol}(\alpha)$ to itself. Namely, for $f,\varphi \in \mathrm{Hol}(\alpha)$, and $x,y \in X$, we have

$$\begin{aligned}|\mathscr{L}_f^n(\varphi)(x) - \mathscr{L}_f^n(\varphi)(y)| \quad &\leq \textstyle\int_{E^n} e^{f_n(ax)} \left|\left(1 - e^{f_n(ay)-f_n(ax)}\right)\varphi(ax)\right| dp^n(a)\\ &\qquad + \textstyle\int_{E^n} e^{f_n(ay)}\, |\varphi(ax)-\varphi(ay)|\, dp^n(a)\\ &\leq \left(C_f \|\mathscr{L}_f^n(\varphi)\|_\infty + 2^{-n}\mathrm{D}_\alpha(\varphi)\|\mathscr{L}_f^n 1\|_\infty\right) d(x,y)^\alpha.\end{aligned}$$

Instead of constructing an $\mathscr{L}_f$-invariant function through application of the Arzelà-Ascoli theorem and then normalizing $\mathscr{L}_f$, we consider the family of operators $\{\mathbb{P}_n^m\}$ defined by, for $m \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$,

$$\mathbb{P}_n^m(\varphi) \equiv \frac{\mathscr{L}_f^m(\varphi\mathscr{L}_f^n(1))}{\mathscr{L}_f^{m+n}(1)}.$$

Observe that, by construction, $\mathbb{P}_n^m(1) = 1$ and $\mathbb{P}_{k+l}^m \circ \mathbb{P}_l^k = \mathbb{P}_l^{k+m}$. Furthermore, the proof of Lemma 2.1 in [BS16] is also applicable to the situation in here and gives that

$$\mathrm{D}_\alpha(\mathbb{P}_n^m(\varphi)) \leq C_f(2\|\varphi\|_\infty + 2^{-m}\mathrm{D}_\alpha(\varphi)).$$

As shown in [BS16, SZ17], this estimate and the fact that X is a full shift allows to deduce the following. With respect to the equivalent metric

$$\overline{d}(x,y) \equiv \max\{1, 4C_f d_X(x,y)^\alpha\},$$

the space $(X,\overline{d})$ is separable and complete. In particular, as the diameter of $(X,\overline{d})$ is finite, the space $\mathscr{M}_1(X)$ is separable and complete with respect to the Wasserstein metric d ([AGS08, Bol08, Rac91]), which is equal to, through Kantorovich's duality,

$$d(\mu,\nu) \equiv \sup\left\{\int f d(\mu-\nu) : f \text{ with } \sup \frac{|f(x)-f(y)|}{\overline{d}(x,y)} \leq 1\right\}. \tag{2}$$

The action of the operators $\{\mathbb{P}_n^m\}$ on the space of $\overline{d}$-Lipschitz functions then allows to deduce, following in verbatim the proof of Theorem 2.1 in [BS16], that their dual action on the space of probability measures strictly contracts the Wasserstein metric for some $m \in \mathbb{N}$ and uniformly in n.

Since $\mathbb{P}_n^m$ contracts d, it immediately follows from the composition rule that, for any probability measure $\nu_0 \in \mathscr{M}_1(X)$, the sequence $((\mathbb{P}_0^m)^*(\nu_0))_{m\in\mathbb{N}}$ is a Cauchy sequence and therefore converges to a probability measure ν, which, again by contraction, is independent from ν_0. It then follows as in [BS16] that ν is a conformal measure, that is, $\mathscr{L}_f^*(\nu) = \lambda\nu$ for some $\lambda > 0$. Observe that, by conformality, $\lambda = \int \mathscr{L}(1)d\nu$. Moreover, by contraction, ν is the unique measure with this property ([BS16, Prop. 2.1]). Furthermore, it follows from (1) that

$$\sup_{x,y,n} \frac{\mathscr{L}_f^n(1)(x)}{\mathscr{L}_f^n(1)(y)} < \infty,$$

which implies, again by the contraction property, that, with δ_x referring to the Dirac measure in x,

$$h(x) \equiv \lim_{n\to\infty} \frac{\int_X \mathscr{L}_f^n(1)d\delta_x}{\int_X \mathscr{L}_f^n(1)d\nu} = \lim_{n\to\infty} \frac{\mathscr{L}_f^n(1)(x)}{\int_X \mathscr{L}_f^n(1)d\nu} \tag{3}$$

exists for each $x \in X$ and is bounded away from 0 and ∞. Similar to ν, it follows that $\mathscr{L}_f(h) = \lambda h$ and that, up to a multiplication by a scalar, h is the unique Hölder function with this property ([BS16, Prop. 2.2]). Moreover, the following version of exponential decay holds ([BS16, Th. A]).

Theorem 3.1. *There exist $C > 0$ and $s \in (0,1)$ such that, for $\varphi, \psi \in \mathrm{Hol}(\alpha)$ and $\psi > 0$,*

$$D_\alpha\left(\frac{\mathscr{L}^n(\varphi)}{\mathscr{L}^n(\psi)} - \frac{\nu(\varphi)}{\nu(\psi)}\right) \leq Cs^n\left(\mathrm{D}_\alpha(\varphi) + \left|\frac{\nu(\varphi)}{\nu(\psi)}\right| \mathrm{D}_\alpha(\psi)\right)\|1/\psi\|_\infty.$$

We remark that Theorem 3.1 applied to $\psi = h$ (normalized eigenfunction in the sense that $\nu(h) = 1$) and $\varphi = 1$ give the following estimates $D_\alpha(\mathscr{L}_f^n(1)/(\lambda^n h) - 1) \leq 2CD_\alpha(h)\|h\|_\infty^{-1}s^n$. Therefore we have, uniformly in $x \in X$

$$1 - 2Cs^n \leq \frac{\mathscr{L}_f^n(1)(x)}{\lambda^n h(x)} \leq 1 + 2Cs^n. \tag{4}$$

Since $0 < s < 1$ and $\|\log h\|_\infty < \infty$, it follows that $n^{-1}\log \mathscr{L}_f^n(1)(x) \to \log\lambda$. Furthermore, $\lambda = \rho(\mathscr{L}_f|_{\mathrm{Hol}(\alpha)})$ the spectral radius of the action of $\mathscr{L}_f$ on $\mathrm{Hol}(\alpha)$.

As an another application of the above theorem, one obtains almost immediately quasi-compactness of the normalized operator. In order to define the relevant operators and norms, let h refer to the function as constructed above and, for $\varphi : X \to \mathbb{R}$ bounded and measurable, set $\|\varphi\|_\alpha \equiv \|\varphi\|_\infty + D_\alpha(\varphi)$ and

$$Q(\varphi)(x) \equiv \frac{\mathscr{L}_f(h\varphi)(x)}{\lambda h(x)}, \quad \Pi(\varphi)(x) \equiv \int_X \varphi h\, d\nu.$$

Proposition 3.2. *Π and Q act on $\mathrm{Hol}(\alpha)$ as bounded operators, and $\Pi Q = Q\Pi = \Pi$. Furthermore, $\|(Q-\Pi)^n\|_\alpha \leq Cs^n$, where s is as in Theorem 3.1, and the splitting $\mathrm{Hol}(\alpha) \equiv \mathbb{R}\oplus\ker(\Pi)$ into closed subspaces, with $\mathbb{R}$ standing for the constant functions, is invariant under Q and Π. Furthermore, $Q|_\mathbb{R} = \Pi|_\mathbb{R} = id$.*

Proof. Observe that $h \in \mathrm{Hol}(\alpha)$ is bounded from above and below. Hence, $\varphi h \in \mathrm{Hol}(\alpha)$ and $\varphi/h \in \mathrm{Hol}(\alpha)$ for any $\varphi \in \mathrm{Hol}(\alpha)$ which implies that Q acts on $\mathrm{Hol}(\alpha)$. Furthermore, using conformality of ν and invariance of h,

$$\begin{aligned} \Pi \circ Q(\varphi) &= \int_X \lambda^{-1}\mathscr{L}_f(h\varphi)\, d\nu = \int_X h\varphi\, d\nu = \Pi(\varphi) \\ Q \circ \Pi(\varphi) &= \frac{\mathscr{L}_f(h\Pi(\varphi))}{\lambda h} = \Pi(\varphi). \end{aligned}$$

Hence, $\Pi Q = Q\Pi = \Pi$, and, in particular, $(Q-\Pi)^n = Q^n - \Pi$. Hence, by Theorem 3.1 applied to $h\varphi$ in the numerator and h in the denominator,

$$\begin{aligned} D_\alpha((Q-\Pi)^n(\varphi)) &= D_\alpha(Q^n(\varphi) - \Pi(\varphi)) \\ &= D_\alpha\left(Q^n\left(\varphi - \Pi(\varphi)\right)\right) \\ &\leq Cs^n\left(D_\alpha(h\varphi) + |\Pi(\varphi - \Pi(\varphi))| \mathrm{D}_\alpha(h)\right) \|1/h\|_\infty \\ &\leq C\|1/h\|_\infty s^n\left(\|h\|_\infty D_\alpha(\varphi) + \|\varphi\|_\infty \mathrm{D}_\alpha(h)\right) \\ &\leq C^* s^n \|\varphi\|_\alpha. \end{aligned}$$

As $\int_X (Q-\Pi)^n(\varphi)h\, d\nu = 0$, it follows from $\sup_{x,y} d_X(x,y) = 1$ that

$$\|(Q-\Pi)^n(\varphi)\|_\infty \leq D_\alpha((Q-\Pi)^n(\varphi)).$$

Hence, $\|(Q-\Pi)^n(\varphi)\|_\alpha \leq C^* s^n \|\varphi\|_\alpha$. The remaining assertion is obvious. □

Provided that $\mathscr{L}_f(1) = 1$, the above splitting now allows to apply the very general version of Nagaev's method by Hennion and Hervé in [HH01] as follows. As the space of complex-valued Hölder continuous functions $\mathfrak{B}$ is a Banach algebra, condition $\mathcal{H}[1]$ of Hennion and Hervé is satisfied. Furthermore, condition $\mathcal{H}[2]$ in there follows from Proposition 3.2. Now assume that ξ is a real-valued Hölder continuous function and that $t \in \mathbb{R}$. By Lemma VIII.10 in [HH01], the operator $\mathscr{L}_{f+i\xi t}$ acts as bounded operator on $\mathfrak{B}$ and is analytic in t. Hence, also $\mathcal{H}[3]$ and $\hat{\mathcal{D}}$ are satisfied and Theorems A, B and C in [HH01] are applicable.

In order to state the result, set $S_n(\xi) \equiv \sum_{k=0}^{n-1} \xi \circ \sigma^k$ and recall that ξ is referred to as a non-arithmetic observable if the spectral radius of $\mathscr{L}_{f+i\xi t}$ is smaller than 1 for each $t \neq 0$.

Proposition 3.3. *Assume that ξ is a real valued Hölder continuous function such that $\int \xi d\nu = 0$. Then $s^2 = \lim_n \frac{1}{n}\int (S_n(\xi))^2 d\nu$ exists and the following versions of central limit theorems (CLTs) hold. In there, Z refers to a $N(0,s)$-distributed random variable.*

(1) *(CLT with rate). If $s > 0$, then there exists $C > 0$ such that*

$$\sup_{u \in \mathbb{R}} \left|\nu\left(\{x \in X : S_n(\xi)(x) \leq u\sqrt{n}\}\right) - P(Z \leq u)\right| \leq Cn^{-\frac{1}{2}}.$$

(2) *(Local CLT). If $s > 0$ and ξ is non-arithmetic, then for any $g : \mathbb{R} \to \mathbb{R}$ continuous with $\lim_{|u|\to\infty} u^2 g(u) = 0$,*

$$\lim_{n\to\infty} \sup_{t \in \mathbb{R}} \left|\sqrt{2\pi n}s \int g(S_n(\xi) - u)d\nu - e^{-\frac{u^2}{2ns^2}} E(g(Z))\right| = 0.$$

4. Pressure, entropy and their equilibrium states

In this section we define the concepts of entropy and pressure considered here. Before proceeding we recall that in the context of uncountable alphabets, both entropy and pressure are usually introduced as p-dependent concepts, see for example [BCL$^+$11, vEFS93, Geo11, LMMS15].

We say that a potential $f \in C_b(X, \mathbb{R})$ is normalized if $\mathscr{L}_f 1 = 1$. Consider the set $\mathscr{G} \equiv \{\mu \in \mathscr{M}_1(X) : \mathscr{L}_f^* \mu = \mu$ for some normalized potential $f \in \mathrm{Hol}(\alpha)\}$. Following [LMMS15], we define the entropy of $\mu \in \mathscr{G}$ as $\mathrm{h}^{\mathrm{v}}(\mu) \equiv -\langle \mu, f \rangle$, where f is some normalized potential in $\mathrm{Hol}(\alpha)$ arbitrarily chosen so that $\mathscr{L}_f^* \mu = \mu$. Actually, similarly to [LMMS15] we can prove that for any $\mu \in \mathscr{G}$ we have

$$\mathrm{h}^{\mathrm{v}}(\mu) = \inf_{g \in \mathrm{Hol}(\alpha)} -\langle \mu, g \rangle + \log \lambda_g, \tag{5}$$

where λ_g is the eigenvalue obtained in the last section.

Since the above expression makes sense for any $\mu \in rba(X)$ we have a natural way to define the entropy of a bounded finitely additive measure.

Next we obtain a generalization of the classical variational principle. Before we will make a few observations and introduce some notations. We first observe that the constant function $f = 1$ is in $C_b(X, \mathbb{R})$ and so the set of all finitely additive probability measures

$$\mathscr{M}_1^a(X) \equiv \bigcap_{\substack{f \in C_b(X,\mathbb{R}) \\ f \geq 0}} \{\mu \in rba(X) : \mu(1) = 1 \text{ and } \mu(f) \geq 0\}$$

is a closed subset of the closed unit ball $\{\mu \in rba(X) : \|\mu\|_{TV} \leq 1\}$ in the weak-$*$-topology. This fact together with the Banach-Alaoglu theorem implies that $\mathscr{M}_1^a(X)$ is a compact space.

Note that the space of all non-negative shift-invariant finitely additive measures

$$\mathscr{M}_\sigma^a(X) \equiv \{\mu \in \mathscr{M}_1^a(X) : \ \mu(f \circ \sigma) = \mu(f), \ \ \forall f \in C_b(X, \mathbb{R})\}$$

is also a compact, with respect to the weak-$*$-topology. Indeed, let $(\mu_d)_{d \in D}$ a topological net in $\mathscr{M}_\sigma^a(X)$ and suppose that $\mu_d \to \mu$, in the weak-$*$-topology. Then for any $g \in C_b(X, \mathbb{R})$ we have $\mu(g \circ \sigma) = \lim_{d \in D} \mu_d(g \circ \sigma) = \lim_{d \in D} \mu_d(g) = \mu(g)$, where the last equality follows from the weak-$*$ continuity of $\mu \in rba(X)$. Of course, $\mu(g) \geq 0$ whenever $g \geq 0$ and $\mu(1) = 1$.

Definition 4.1. (Pressure Functional) The functional $P : C_b(X, \mathbb{R}) \to \mathbb{R}$ given by

$$P(f) \equiv \sup_{\mu \in \mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle \mu, f \rangle$$

is called the pressure functional and the real number $P(f)$ is called topological pressure of f.

Proposition 4.2. *The pressure functional P is a convex function on $C_b(X, \mathbb{R})$.*

Proof. The convexity follows immediately from Definition 4.1. □

Before we proceed we would like to explain why the theory, that will be developed below, is not comprised in [IP84]. In there, Phelps and Israel developed an abstract theory of generalized pressure and presented some applications to lattice gases. In their work, the space X is supposed to be a metric compact space, and a pressure functional is any real-valued convex function $\mathscr{P}$ defined on $C_b(X,\mathbb{R}) = C(X,\mathbb{R})$ satisfying the conditions

(1) $\mathscr{P}(f+c) = \mathscr{P}(f) + c$,
(2) if $f \leq 0$, then $\mathscr{P}(f) \leq \mathscr{P}(0)$,
(3) if $f \geq 0$, then $\|q(f)\| \leq \mathscr{P}(f)$,
(4) if $g \in \mathscr{I}$, then $\mathscr{P}(f+g) = \mathscr{P}(f)$,

where $c \in \mathbb{R}$ is a constant, $\mathscr{I}$ denotes the subspace of $C(X,\mathbb{R})$ generated by the set $\{g - g\circ\sigma : g \in C(X,\mathbb{R})\}$ and $q : C(X,\mathbb{R}) \to C(X,\mathbb{R})/\mathscr{I}$ is the quotient map. In [IP84], when the authors introduce entropy, condition *(3)* is replaced by a stronger one. This new condition, which we call *(3')*, is a kind of coercivity condition. To be more precise, it requires $\|f\|_\infty \leq \mathscr{P}(f)$ whenever $f \geq 0$. Afterwards, for a given pressure functional $\mathscr{P}$ satisfying *(3')*, the authors define the entropy $\mathfrak{h} \equiv \mathfrak{h}(\mathscr{P})$ as the Legendre-Fenchel transform of $\mathscr{P}$. Condition *(3')* is then employed in [IP84, Prop 2.2] to show that the entropy of any shift-invariant probability measure μ is bounded by $0 \leq \mathfrak{h}(\mu) \leq P(0)$. Although the results in [IP84] can be applied in several contexts, condition *(3')* does not hold in general in Statistical Mechanics and Thermodynamic Formalism. For example, the specific entropy considered in [Geo11] is not bounded from below. Actually, it is well-known in Statistical Mechanics that the ground state entropy can go to minus infinity for uncountable (even compact) spin spaces.

If $X = E^{\mathbb{N}}$, where E is an uncountable infinite compact metric space, the entropy considered in [LMMS15] does neither satisfy *(3)* nor *(3')*, and the authors show that their entropy of a Dirac measure concentrated on a periodic orbit is not finite, see the remark to Proposition 5 in page 1939 in [LMMS15]. Note that the pressure functional introduced above in Definition 4.1 is another instance where condition *(3')* might not hold. We also remark that in here, X is not necessarily compact.

Our pressure functional, likewise in classical equilibrium Statistical Mechanics, depends on the Ruelle operator, which in turn depends on the *a priori* measure p, so the reader should keep in mind that our pressure functional is a p-dependent concept as well as will be our concept of entropy. It is also worth noting that by taking a suitable *a priori* measure, we recover the usual concept of topological pressure in a finite-alphabet setting.

Definition 4.3 (Equilibrium States)**.** Given a continuous potential $f \in C_b(X,\mathbb{R})$, we say that $\mu \in \mathscr{M}_\sigma^a(X)$ is a (generalized) equilibrium state for f if

$$\mathrm{h}^{\mathrm{v}}(\mu) + \mu(f) = \sup_{\mu\in\mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle \mu, f\rangle \equiv P(f).$$

The set of all equilibrium states for f will be denoted by $\mathrm{Eq}(f)$.

Theorem 4.4. *Given a continuous potential $f \in C_b(X,\mathbb{R})$ there is $\mu_f \in \mathscr{M}_\sigma^a(X)$ such that*

$$\mathrm{h}^{\mathrm{v}}(\mu_f) + \langle \mu_f, f\rangle = \sup_{\mu\in\mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle \mu, f\rangle.$$

Proof. From Definition 5 follows that the mapping $\mathscr{M}_\sigma^a(X) \ni \mu \longmapsto \mathrm{h}^{\mathrm{v}}(\mu) + \mu(f)$ is upper semi-continuous with respect to the weak-$*$-topology. Since $\mathscr{M}_\sigma^a(X)$ is compact and convex follows from the Bauer maximum principle that there exists some $\mu_f \in \mathscr{M}_\sigma^a(X)$ such that

$$\mathrm{h}^{\mathrm{v}}(\mu_f) + \mu_f(f) = \sup_{\mu\in\mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \mu(f).$$

Moreover, the Bauer maximum principle ensures that we can take the finitely additive measure μ_f, attaining the above supremum, in such a way that μ_f is in the set of extreme points of $\mathscr{M}_\sigma^a(X)$. □

An equilibrium state μ_f as in the previous theorem is not necessarily a countably additive measure. On the other hand, the Yosida-Hewitt decomposition [YH52, Theorem 1.23] states that $\mu_f = (\mu_f)_c + (\mu_f)_a$, where $(\mu_f)_c$ is a non-negative countably additive measure and $(\mu_f)_a$ is a non-negative purely finitely additive measure. That is, if μ is a non-negative countably additive measure such that $\mu \leq (\mu_f)_a$, then $\mu = 0$.

At this point, we do not have complete information on how the regularity properties or the shape of the graph of the potential are linked to this decomposition this seems to be a relevant and interesting problem. On the other hand, we can prove other important properties about the set $\mathrm{Eq}(f)$ consisting of all equilibrium states associated to a bounded continuous potential f.

If $p = P|_{\mathrm{Hol}(\alpha)}$ then Theorem 4.4 ensures that the subdifferential

$$\partial p(f) \equiv \{\mu \in rba(X) : p(g) \geq p(f) + \langle \mu, g - f\rangle,\ \forall g \in \mathrm{Hol}(\alpha)\},$$

at every $f \in \mathrm{Hol}(\alpha)$ is not empty and it is easy to see that $\mathrm{Eq}(f) = \partial p(f)$. The next proposition is a trivial observation showing that the restriction of h^{v} to a subdifferential $\partial p(f)$ at any $f \in \mathrm{Hol}(\alpha)$ is an affine function.

Proposition 4.5. *Let $f \in \mathrm{Hol}(\alpha)$ be a given potential and $\partial p(f)$ the subdifferential of p at f. Then the restriction $\mathrm{h}^{\mathrm{v}}|_{\partial p(f)}$ is an affine function. In particular, any $\mu \in \partial p(f)$ is an equilibrium state for f.*

Proof. From the definition for any $\mu \in \partial p(f)$ we have $p(f) - \langle \mu, f\rangle \leq p(g) - \langle \mu, g\rangle$ for all $g \in \mathrm{Hol}(\alpha)$. Therefore

$$\partial p(f) \ni \mu \longmapsto \mathrm{h}^{\mathrm{v}}(\mu) = \inf_{g\in\mathrm{Hol}(\alpha)} p(g) - \langle \mu, g\rangle = p(f) - \langle \mu, f\rangle. \qquad \square$$

Proposition 4.6. *For any $f \in C_b(X, \mathbb{R})$ we have that $\mathrm{Eq}(f)$ is a compact and convex subspace of $\mathscr{M}_\sigma^a(X)$.*

Proof. Let $\mu, \nu \in \mathrm{Eq}(f)$ and $\lambda \in [0, 1]$. From elementary properties of the infimum and (5) follows that h^{v} is concave function. Hence,

$$\begin{aligned}\mathrm{h}^{\mathrm{v}}(\lambda\mu + (1-\lambda)\nu) &+ \langle \lambda\mu + (1-\lambda)\nu, f\rangle \\ &\geq \lambda\mathrm{h}^{\mathrm{v}}(\mu) + (1-\lambda)\mathrm{h}^{\mathrm{v}}(\nu) + \lambda\langle \mu, f\rangle + (1-\lambda)\langle \nu, f\rangle = P(f),\end{aligned}$$

thus proving that $\mathrm{Eq}(f)$ is a convex set. The compactness of $\mathrm{Eq}(f)$ follows from compactness of $\mathscr{M}_\sigma^a(X)$ and the upper semi-continuity of h^{v}. □

Remark 4.7. It follows from the last proposition and the Krein-Milman theorem that the set of extreme points of $\mathrm{Eq}(f)$, denoted by $\mathrm{ex}(\mathrm{Eq}(f))$, is not-empty. In particular, it is natural to conjecture that any element in $\mathrm{ex}(\mathrm{Eq}(f))$ is an ergodic finitely additive measure. This usually is established by showing that $\mathrm{ex}(\mathrm{Eq}(f)) = \mathrm{Eq}(f) \cap \mathrm{ex}(\mathscr{M}_\sigma^a(X))$ using that the the entropy is an affine continuous function on $\mathscr{M}_\sigma^a(X)$. However, this approach does not work in our setting for general *a priori* measures and non-compact spaces as h^{v} restricted to $\mathscr{M}_\sigma^a(X)$ might no longer be affine. Actually, $\mathscr{M}_\sigma^a(X)$ contains infinitely many elements whose entropy is equal to minus infinity. Of course, in particular cases, e.g. if the potential is Hölder continuous, there are other techniques to establish the ergodicity of the extreme equilibrium states.

Remark 4.8. Since $(C_b(X,\mathbb{R}), rba(X))$ is a dual pair and P is a proper convex function (that is, if its effective domain is nonempty and P never takes the value $-\infty$) it follows from Corollary 7.17 in [AB06] that $\mathrm{Eq}(f)$ is a singleton if and only if P is Gâteaux differentiable at f. The differentiability of the pressure restricted to $\mathrm{Hol}(\alpha)$ was recently obtained when X is compact (see [Sil17, SdSS14]) and is a classical result for finite alphabets, see for example [Bal00, PP90, Rue68, Wal75, Wal78].

On the other hand, if the potential is Hölder continuous, then the following result shows that $hd\nu$, with h and ν as in Section 3, is a countably additive equilibrium state.

Theorem 4.9. *Let f be a bounded Hölder potential. Then there is at least one equilibrium state μ_f, associated to f, such that its Yosida-Hewitt decomposition has only the countably additive part. More precisely, this equilibrium state is given by $\mu_f = h\nu$, where h is a suitable normalized eigenfunction associated to λ_f and ν is the eigenmeasure of the dual of the Ruelle operator.*

Proof. Let f be a Hölder potential. By the definition of entropy we have

$$\begin{aligned}\sup_{\mu\in\mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle \mu, f\rangle &= \sup_{\mu\in\mathscr{M}_\sigma^a(X)} \Big[\inf_{g\in\mathrm{Hol}(\alpha)} -\langle\mu,g\rangle + \log\lambda_g\Big] + \langle\mu,f\rangle \\ &\le \sup_{\mu\in\mathscr{M}_\sigma^a(X)} -\langle\mu,f\rangle + \log\lambda_f + \langle\mu,f\rangle \\ &= \log\lambda_f.\end{aligned}$$

Since f is a Hölder potential, we can use the Perron-Frobenius-Ruelle Theorem of Section 3 to find a normalized potential $\bar{f} \in \mathrm{Hol}(\alpha)$ cohomologous to f, that is, $\bar{f} = f + \log h - \log h\circ\sigma - \log\lambda_f$. It is easy to see that h, up to a positive constant, can be chosen so that $\mu_f \equiv h\nu \in \mathscr{M}_1(X)$ and $\mathscr{L}_{\bar{f}}^*\mu_f = \mu_f$. Therefore $\mu \in \mathscr{G}\cap\mathscr{M}_\sigma^a(X)$ and by definition we have $\mathrm{h}^{\mathrm{v}}(\mu_f) = -\langle\mu_f,\bar{f}\rangle = -\langle\mu_f, f\rangle + \log\lambda_f$. This equality implies

$$\log\lambda_f = \mathrm{h}^{\mathrm{v}}(\mu_f) + \langle\mu,f\rangle \le \sup_{\mu\in\mathscr{M}_\sigma^a(X)} \mathrm{h}^{\mathrm{v}}(\mu) + \langle\mu,f\rangle$$

which together with the last inequality ensures that μ_f is an equilibrium state. □

5. Extreme Positive $rba(X)$ measures in the Closed Unit Ball are uniquely Maximizing

The aim of this section is to obtain a result similar to the main result of [Jen06] in a non-compact setting. The techniques developed in [Jen06] are not applicable here mainly because $C_b(X,\mathbb{R})$ may not be separable and the induced weak-$*$-topology on the closed unit ball of its dual is not necessarily metrizable.

Theorem 5.1. *Let S be an arbitrary topological space, $C_b(S,\mathbb{R})$ denote the Banach space of all real-valued bounded continuous functions on S endowed with the supremum norm, and $\mathscr{M}_1^a(S)$ the subset of the topological dual $C_b(S,\mathbb{R})^*$, consisting of those functionals which have norm one and are mapping the positive cone $C_b(S,\mathbb{R})_+$ into $[0,\infty)$. For each $\mu\in\mathscr{M}_1^a(S)$, the following assertions are equivalent.*

i) *μ is an extreme point of $\mathscr{M}_1^a(S)$, i.e. μ can not be written as a convex combination of two functionals in $\mathscr{M}_1^a(S) \setminus \{\mu\}$.*
ii) *μ is an* exposed point *of $\mathscr{M}_1^a(S)$, that is, there exists a functional ξ in the bi-dual $C_b(S, \mathbb{R})^{**}$ which attains its strict minimum on the set $\mathscr{M}_1^a(S)$ at the point μ.*
iii) *There exists a functional ξ in the bi-dual $C_b(S, \mathbb{R})^{**}$ which is zero at μ and strictly positive on $\mathscr{M}_1^a(S) \setminus \{\mu\}$.*
iv) *μ is a* lattice homomorphism*, i.e. we have $|\langle \mu, f \rangle| = \langle \mu, |f| \rangle$ for all $f \in C_b(S, \mathbb{R})$.*
v) *μ is an algebra homomorphism, i.e. we have $\langle \mu, f_1 f_2 \rangle = \langle \mu, f_1 \rangle \langle \mu, f_2 \rangle$ for all $f_1, f_2 \in C_b(S, \mathbb{R})$.*

The proof of above theorem can be found in [Glu].

For the next corollary we assume that $X = E^{\mathbb{N}}$, where (E, d_E) is a non-compact standard Borel space satisfying the following property. There exists $a_0 \in E$ and a sequence $(a_n)_{n \geq 1}$ of distinct points such that $d_E(a_0, a_{n-1}) < d_E(a_0, a_n)$ and $d(a_0, a_n) \to \operatorname{diam}(E)$. For the sake of simplicity, we also assume that $\operatorname{diam}(E) = 1$ and $d(x, y) < 1$, for all $x, y \in X$.

Corollary 5.2. *If X is a non-compact space satisfying the above property, then there exists an extreme, finitely additive measure in $\mathscr{M}_\sigma^a(X) \setminus \mathscr{M}_\sigma(X)$ (i.e., not necessarily countably additive measure) which is the unique maximizing measure for some potential $f \in C_b(X, \mathbb{R})$.*

Proof. For $n \geq 0$, let $x^{(n)} = (a_n, a_n, \ldots) \in X$ and consider the associated sequence of Dirac delta measures $(\delta_{x^{(n)}})_{n \geq 1}$. By compactness of $\mathscr{M}_1^a(X)$, this sequence of measures, viewed as a topological net, has a convergent subnet $(\delta_{x^{(\alpha)}})_{\alpha \in D}$. Let $\mu = \lim_{d \in D} \delta_{x^{(\alpha)}}$. We claim that μ is not a countably additive measure. Indeed, take $B_n = X \setminus \{x \in X : d(x, x^{(0)}) < d(x^{(0)}, x^{(n)})\}$. Note that the hypothesis considered on E imply $B_n \downarrow \emptyset$. Suppose by contradiction that μ is a countably additive measure. Since for each $n \geq 1$, the set B_n is closed, follows from Portmanteau theorem (Theorem 6.1 item (c) of [Par05])

$$\mu(B_n) \geq \limsup_{\alpha \in D} \delta_{x^{(\alpha)}}(B_n) = 1.$$

Consequently, μ is not a countably additive measure which is a contradiction.

A straightforward computation shows that any such cluster point μ is a shift-invariant measure. It remains to show that μ is an extreme point of $\mathscr{M}_1^a(X)$. This fact is a consequence of the equivalence $i) \Leftrightarrow v)$ of Theorem 5.1. Indeed, for each $\alpha \in D$ the measure $\delta_{x^{(\alpha)}}$ is an extreme point of $\mathscr{M}_1^a(X)$ as $\langle \delta_{x^{(\alpha)}}, f_1 f_2 \rangle = \langle \delta_{x^{(\alpha)}}, f_1 \rangle \langle \delta_{x^{(\alpha)}}, f_2 \rangle$. In order to conclude that μ satisfies a similar relation it is enough to observe that the above equality is stable under weak-$*$ limits so we have $\langle \mu, f_1 f_2 \rangle = \langle \mu, f_1 \rangle \langle \mu, f_2 \rangle$. By using again the equivalence $i) \Leftrightarrow v)$ of Theorem 5.1 it follows that μ is an extreme point.

Let $\xi : C_b(X, \mathbb{R})^{**} \to \mathbb{R}$ be the linear functional obtained in item iii) of Theorem 5.1 to μ. Recall that ξ is of the form $\xi(\nu) = \langle \nu, g \rangle$ for some g in $C_b(X, \mathbb{R})$, see [Bre11, Proposition 3.14]. Finally, by taking the potential $f = -g$ and considering the functional $F \in C_b(X, \mathbb{R})^*$ defined by $F(\mu) = \langle \mu, f \rangle$ the result follows. ∎

6. Applications

6.1. **Finite Entropy Ground-States and Maximizing Measures.**

In this section we consider the following ergodic optimization problem. We fix a potential $f \in C_b(X,\mathbb{R})$ and consider the problem of finding an element of $\mathscr{M}_\sigma^a(X)$ with finite entropy which attains the supremum

$$m(f) = \sup_{\substack{\nu\in\mathscr{M}_\sigma^a(X)\\ \mathtt{h}^{\mathtt{v}}(\nu)>-\infty}} \int_X f d\nu.$$

An invariant measure μ having finite entropy is referred to as a *maximizing measure* for the potential f if it attains the supremum in the above variational problem, that is,

$$m(f) = \sup_{\substack{\nu\in\mathscr{M}_\sigma^a(X)\\ \mathtt{h}^{\mathtt{v}}(\nu)>-\infty}} \int f d\nu = \int f d\mu.$$

The above supremum is always finite since $f \in C_b(X,\mathbb{R})$ but the existence of a maximizing measure is a non-trivial problem because the subset of functionals in $\mathscr{M}_\sigma^a(X)$ with finite entropy is non-compact.

Consider a fixed bounded Hölder potential f and a real parameter $\beta > 0$. We denote by $\mu_{\beta f}$ the equilibrium state constructed above associated to the potential βf. We now show that any cluster point μ_∞ of the family $(\mu_{\beta f})_{\beta>0}$, such that $\mathtt{h}^{\mathtt{v}}(\mu_\infty) > -\infty$ is a maximizing measure for f. It is standard to call μ_∞ a *Gibbs State at zero temperature for the potential* f or simply a *ground state* for f.

Theorem 6.1. *Let f be a bounded continuous potential, $\beta > 0$ and $\mu_{\beta f} \in \mathrm{Eq}(\beta f)$. Suppose there is at least one cluster point μ_∞ of $(\mu_{\beta f})_{\beta>0}$ having finite entropy. Then μ_∞ is a maximizing measure for the potential f.*

Proof. Let μ_∞ be an arbitrary cluster point of the family $(\mu_{\beta f})_{\beta>0}$, such that $\mathtt{h}^{\mathtt{v}}(\mu_\infty) > -\infty$. Note that for all $\beta > 0$ we have that $\mathtt{h}^{\mathtt{v}}(\mu_{\beta f}) > -\infty$ and $\mu_{\beta f} \in \mathscr{M}_\sigma^a(X)$. Therefore,

$$\int_X f\, d\mu_\infty = \lim_{\beta\to\infty} \int_X f\, d\mu_{\beta f} \leq m(f).$$

On the other hand, for any $\nu \in \mathscr{M}_\sigma^a(X)$, we get from the variational principle that $\langle \beta f, \mu_{\beta f}\rangle + \mathtt{h}^{\mathtt{v}}(\mu_{\beta f}) \geq \langle \beta f, \nu\rangle + \mathtt{h}^{\mathtt{v}}(\nu)$, and that the inequality is non-trivial if $\mathtt{h}^{\mathtt{v}}(\nu) > -\infty$. In this case,

$$\int_X f\, d\mu_{\beta f} + \frac{\mathtt{h}^{\mathtt{v}}(\mu_{\beta f})}{\beta} \geq \int_X f\, d\nu + \frac{\mathtt{h}^{\mathtt{v}}(\nu)}{\beta}$$

and consequently, by the non positivity of $\mathtt{h}^{\mathtt{v}}$,

$$\int_X f\, d\mu_\infty = \lim_{\beta\to\infty} \int f d\, \mu_{\beta f} \geq \lim_{\beta\to\infty} \int_X f\, d\nu + \frac{\mathtt{h}^{\mathtt{v}}(\nu)}{\beta} = \int_X f\, d\nu.$$

Since this inequality holds for any $\nu \in \mathscr{M}_\sigma^a(X)$ having finite entropy, the result follows. $\square$

Remark 6.2. We remark that it is not possible to conclude from the previous proof that the cluster point μ_∞ considered above is a countably additive measure.

If we do not require that $\mathtt{h}^{\mathtt{v}}(\mu_\infty) > -\infty$, then the above argument still gives us the inequality

$$\sup_{\substack{\nu\in\mathscr{M}_\sigma^a(X)\\ \mathtt{h}^{\mathtt{v}}(\nu)>-\infty}} \int f d\nu \le \int f d\mu_\infty,$$

which, in principle, could be strict.

6.2. **Markov Chains on Standard Borel Spaces.** In this section we show how to apply the results obtained here to discrete time Markov Chains taking values in a metric space E. We then show how to construct and prove some stability results in [MT09] within the framework of Thermodynamic Formalism.

Roughly speaking, a discrete-time Markov chain Φ on a metric space E is a countable collection $\Phi \equiv \{\Phi_0, \Phi_1, \ldots\}$ of random variables, with Φ_i taking values in E so that its future trajectories depend on its present and its past only through the current value. A concrete construction of a discrete time Markov chain, can be made by specifying a measurable space $(X, \mathscr{F})$, where each element of Φ is defined, an initial probability distribution $p : \mathscr{B}(E) \to [0,1]$, and a transition probability kernel $P : E \times \mathscr{B}(E) \to [0,1]$ such that

i) for each fixed $A \in \mathscr{B}(E)$ the map $a \longmapsto P(a, A)$ is a $\mathscr{B}(E)$-measurable function,
ii) for each fixed $a \in E$ the map $A \longmapsto P(a, A)$ is a Borel probability measure on E.

Definition 6.3. A stochastic process Φ defined on $(X, \mathscr{F}, \mathbb{P}_\mu) = (E^{\mathbb{N}}, \mathscr{B}(E^{\mathbb{N}}), \mathbb{P}_\mu)$ and taking values on E is called a time-homogeneous Markov Chain, with transition probability kernel P and initial distribution μ if its finite dimensional distributions satisfy, for each $n \ge 1$,

$$\begin{aligned}&\mathbb{P}_\mu(\Phi_0 \in A_0, \ldots, \Phi_n \in A_n)\\ &\qquad = \int_{A_0} \cdots \int_{A_{n-1}} P(y_{n-1}, A_n) dP(y_{n-2}, y_{n-1}) \ldots dP(y_0, y_1) d\mu(y_0).\end{aligned}$$

Definition 6.4 (Invariant Measures)**.** A sigma-finite measure π on $\mathscr{B}(E)$ with the property

$$\pi(A) = \int_E P(x, A)\, d\pi(x)$$

will be called invariant.

The key results about the existence of invariant measures for a Markov chain are based on recurrence, see for example Theorem 10.0.1 in [MT09]. In what follows, we prove the existence of such measures for a certain class of kernels based on the results of Section 3. In order to do so, assume that $f \in \mathrm{Hol}(\alpha)$ is a summable potential with respect to some *a priori* measure p on E, that is $\|\mathscr{L}_f(1)\|_\infty < \infty$. Then, for each $x = (x_1, x_2, \ldots) \in E^{\mathbb{N}}$, the map $A \mapsto \mathscr{L}_f(1_A \circ \pi_1)(x)$, for $x \in X$ and $A \in \mathscr{B}(E)$ defines a finite measure on X. In particular, $dP(x, a) \equiv e^{f(ax)} dp(a)$, or equivalently,

$$P(x, A) = \int_E 1_A(a)\, dP(x, a) \equiv \int_E e^{f(ax)} (1_A \circ \pi_1)(ax)\, dp(a) = \mathscr{L}_f(1_A \circ \pi_1)(x)$$

defines a transition kernel which might be neither a probability measure nor constant on $\{y \in E^{\mathbb{N}} : y_1 = x_1\}$. However, it remains to check Kolmogorov's consistency conditions in order to verify that P defines a stochastic process. That is, as P induces the measure $\mathbb{P}_x$ on E^n with respect to the initial distribution δ_x for $x \in X$, given by

$$\begin{aligned}\mathbb{P}_x(\Phi_1 \in A_1, \ldots \Phi_n \in A_n) &= \mathscr{L}_f\left(1_{\pi_1^{-1}A_1}\mathscr{L}_f\left(1_{\pi_1^{-1}A_2}\cdots\mathscr{L}_f\left(1_{\pi_1^{-1}A_n}\right)\cdots\right)\right)(x)\\ &= \mathscr{L}_f^n\left(\textstyle\prod_{i=1}^n 1_{\pi_1^{-1}A_i}\circ\sigma^{n-i}\right)(x) = \mathscr{L}_f^n\left(1_{\{(y_i)\in X: y_i \in A_{n+1-i}, i=1,\ldots n\}}\right)(x),\end{aligned}$$

it is necessary and sufficient that $\mathscr{L}_f(1) = 1$, or in other words, f has to be normalized. Now let ν be the unique probability measure with $\mathscr{L}_f^*(\nu) = \nu$ as in Section 3. With respect to this initial distribution, the above implies that

$$\mathbb{P}_\nu(\Phi_1 \in A_1, \ldots, \Phi_n \in A_n) = \nu\left(\{(y_i) \in X : y_i \in A_{n-i}, i = 1, \ldots n\}\right). \tag{6}$$

As the right-hand side is well known from the construction of the natural extension of a measure-preserving dynamical system, one obtains the following relation between σ on X and the stochastic process defined by P with respect to a normalized potential f through the bilateral shift on $(E^{\mathbb{Z}}, \hat{\nu})$, where $\hat{\nu}$ is the extension of ν to $E^{\mathbb{Z}}$ through (6). In this setting, σ corresponds to the left shift whereas $(\Phi_i : i \in \mathbb{N})$ is given by $\Phi_i = y_{-i}$ for $(y_i) \in E^{\mathbb{Z}}$. Furthermore, as $\mathscr{L}_f(1) = 1$, it follows from the same argument as in the proof of Theorem 6.6 that

$$d\left(P^n(x,\cdot),\nu\right) = d\left((\mathscr{L}_f^n)^*(\delta_x),\nu\right) \le Cs^n d(\delta_x,\nu) \le Cs^n, \tag{7}$$

where d is the Wasserstein metric on the space of probability measures as defined in (2) (see Theorem 1.1.5 in [BK12]). Note that (7) is also known as *geometric ergodicity* in the literature on probability theory and that geometric ergodicity was established in [Sta17, BS16, SZ17] for non-stationary and random countable shift spaces.

Observe that (Φ_i) in general is not a Markov chain as $P(x,\cdot)$ might not only depend on the the first coordinate of x. However, by assuming that $P(x,\cdot) = P(x_1,\cdot)$, or equivalently, that f only depends on the first two coordinates, one easily obtains that $\pi \equiv \nu \circ \pi_1^{-1}$ is P-stationary as

$$\begin{aligned}\int_E P(x_1, A)\, d\pi(x_1) &= \int_E P(x_1, A)\, d[\nu\circ\pi_1^{-1}](x_1)\\ &= \int_E \mathscr{L}_f(1_A\circ\pi_1)(x_1, x_2, \ldots)\, d[\nu\circ\pi_1^{-1}](x_1)\\ &= \int_{E^{\mathbb{N}}} \mathscr{L}_f(1_A\circ\pi_1)\, d\nu = \int_{E^{\mathbb{N}}} 1_A\circ\pi_1\, d[\mathscr{L}_f^*\nu] = \int_{E^{\mathbb{N}}} 1_A\circ\pi_1\, d\nu\\ &= \int_E 1_A\, d[\nu\circ\pi_1^{-1}] = \pi(A).\end{aligned}$$

Note that the argument depends on the assumption that f only depends on the first coordinates, as if this would not be the case, the identity in line 3 would no longer be satisfied.

6.3. **Asymptotic Stability of Markov Operators.** In this section, we turn our attention to the closely related problem of asymptotic stability of Markov operators on standard Borel spaces and indicate how some of the stability problems considered in [Sza00] can be approached by the results in Section 3.

Let $\mathscr{M}_{\text{fin}}(X)$ be the set of all finite nonnegative Borel measures on X. An operator $P : \mathscr{M}_{\text{fin}}(X) \to \mathscr{M}_{\text{fin}}(X)$ is called a *Markov Operator* if it satisfies the following two conditions:

(i) *positive linearity:* $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$, for all $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathscr{M}_{\text{fin}}(X)$,

(ii) *preservation of the norm:* $P\mu(X) = \mu(X)$ for $\mu \in \mathscr{M}_{\text{fin}}(X)$.

A Markov operator is called a *Feller operator* if there is a linear operator $U : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$, the pre-dual to P, such that

$$\langle \mu, Uf \rangle = \langle P\mu, f \rangle \text{ for } f \in C_b(X, \mathbb{R}), \mu \in \mathscr{M}_{\text{fin}}(X).$$

Finally, a measure in $\mathscr{M}_{\text{fin}}(X)$ is called *stationary* if $P\mu = \mu$, and P is called *asymptotically stable* if there exists a stationary distribution ν such that

$$\lim_{n\to\infty} d(P^n\mu, \nu) = 0, \quad \text{for all } \mu \in \mathscr{M}_1(X),$$

where, as above, d refers to the Wasserstein metric.

Example 6.5. Let be $(E, \mathscr{E})$ a standard Borel space and $X = E^{\mathbb{N}}$ the product space endowed with the product metric $d_X(x, y) = \sum_{n=1}^{\infty} 1/2^n \min\{d_E(x_n, y_n), 1\}$. It is easy to see that (X, d_X) is a Polish space. If f is a bounded α-Hölder continuous normalized potential, then the restriction to $\mathscr{M}_{\text{fin}}(X)$ of the Banach transpose of the Ruelle operator $\mathscr{L}_f^*$ is a Markov operator and its associated Feller operator is $\mathscr{L}_f : C_b(X, \mathbb{R}) \to C_b(X, \mathbb{R})$.

Theorem 6.6. *Under the assumptions of the above example, the Markov operator $P = \mathscr{L}_f^*|_{\mathscr{M}_{\text{fin}}(X)}$ is asymptotically stable. Moreover, there exist $C > 0$ and $s \in (0, 1)$ such that, where ν refers to the unique stationary probability measure and d to the Wasserstein metric defined in* (2)*,*

$$d(P^n(\mu), \nu) \leq Cs^n \textit{ for all } \mu \in \mathcal{M}_1.$$

Proof. As the potential is normalized, $\mathscr{L}_f(1) = 1$ and, in particular, $\mathbb{P}_n^m = \mathscr{L}_f^m$ and $(\mathbb{P}_n^m)^* = P^m$. Therefore, it follows from Theorem 2.1 in [BS16] that there exists $t \in (0, 1)$ and $m \in \mathbb{N}$ such that $d(\mathbb{P}_n^m)^*(\mu), \mathbb{P}_n^m)^*(\tilde{\mu}) \leq td(\mu, \tilde{\mu})$. Moreover, as the $\overline{d}$-diameter of X is 1, it follows that $d(\mu, \tilde{\mu}) \leq 1$. Hence, for each $n \in \mathbb{N}$,

$$d(P^n(\mu), P^n(\tilde{\mu}) \leq t^{-1} t^{\frac{n}{m}} d(\mu, \tilde{\mu}).$$

In particular, P has a unique fixed point ν and, for $\tilde{\mu} \equiv \nu$, it follows that $d(P^n(\mu), \nu) = d(P^n(\mu), P^n(\nu)) \leq t^{n/m-1}$. □

6.4. **Infinite Interacting Random Paths.** We consider the following random path process. At each discrete time $t = n \in \mathbb{N}$, a random point $q_n \in \mathbb{R}^d$ is chosen accordingly to the d-dimensional standard Gaussian measure

$$G_d(A) = \frac{1}{(2\pi)^{n/2}} \int_A \exp\left(-\frac{1}{2}\|v\|_2^2\right) d\lambda^n(v).$$

This sequence of random points induces a random path process on $\mathbb{R}^d \times [1, +\infty)$, given by the linear interpolation among these points, that is,

$$\gamma(t) = (1 - (t - (n-1)))q_n + (t - (n-1))q_{n+1}, \quad \text{if } t \in [n, n+1]. \tag{8}$$

This construction induces a bijection $\Gamma : (\mathbb{R}^d)^{\mathbb{N}} \to \Upsilon$, where Υ is the set of all "polygonal" paths of the form (8).

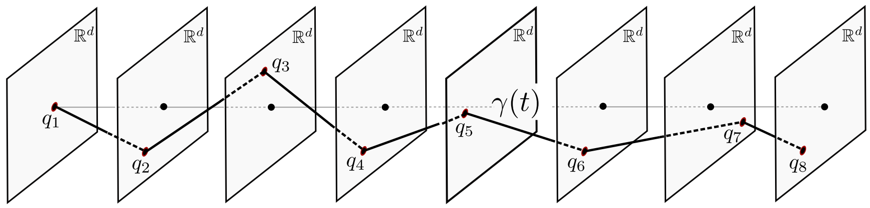

FIGURE 2. An example of a random path $\gamma(t)$ constructed from $q_1, q_2, \ldots$

Let p be the probability measure obtained by the pushforward of the infinite product measure $\prod_{i\in\mathbb{N}} G_d$ to Γ. The space Υ of such all such paths has natural structure of a standard Borel space inherited by $(\mathbb{R}^d)^{\mathbb{N}}$. In the language of the previous sections $E = \Upsilon$ and the *a priori* measure p is the push-forward of $\prod_{i\in\mathbb{N}} G_d$.

Let $f : \Upsilon^{\mathbb{N}} \to \mathbb{R}$ be a Hölder bounded potential. A point in $\Upsilon^{\mathbb{N}}$ will be denoted by $(\gamma_1, \gamma_2, \ldots)$. Note that each coordinate γ_n of a such point is actually a path in $\mathbb{R}^d \times [1, +\infty)$. Now we consider the Ruelle operator

$$\mathscr{L}_f(\varphi)(\gamma_1, \gamma_2, \ldots) = \int_{\Upsilon} \exp(f(\gamma, \gamma_1, \gamma_2, \ldots))\ \varphi(\gamma, \gamma_1, \gamma_2, \ldots)\ dp(\gamma).$$

Since we are assuming that f is a bounded Hölder continuous Theorem 4.9 implies the existence of an equilibrium measure μ_f which is also a countably additive Borel probability measure. This equilibrium measure μ_f describes what will be the law of this infinite interacting random path process in $\mathbb{R}^d \times [1, +\infty)$. The interesting feature of this approach is to allow the construction of an infinite interacting path process measure, having infinite-body interactions, since f can be chosen as a function depending on infinitely many coordinates.

Interesting examples are obtained by the following class of potentials

$$f(\gamma_1, \gamma_2, \ldots) = -\sum_{n=1}^{\infty} J(n) \frac{d_{\mathbb{H}}^{\alpha}(\gamma_1, \gamma_n)}{1 + d_{\mathbb{H}}^{\alpha}(\gamma_1, \gamma_n)}$$

where $J(n) \geq 0$, and goes to zero sufficiently fast, $0 < \alpha < 1$ and $d_{\mathbb{H}}$ stands for the Hausdorff distance. For each inverse temperature $\beta > 0$ we consider the equilibrium measure $\mu_{\beta f}$.

Conjecture 6.7. *At very low temperatures ($\beta \gg 1$) the typical configuration should be an infinite collection of paths which are closed to each other and also close to the origin (this last information comes from the dependence of μ_f on the a priori measure p). On the other hand, at very high temperatures ($0 < \beta \ll 1$) a typical configuration for μ_β should be similar to an infinite collection of independent "diffusive" paths.*

The results of the previous section also allow us to construct a Markov process that can be used to describe the time evolution of this infinite interacting random path process. Given a bounded Hölder potential f we consider a normalized potential $\bar{f}$ cohomologous to f and the following Markov pre-generator $T : C(\Upsilon^{\mathbb{N}}, \mathbb{R}) \to C(\Upsilon^{\mathbb{N}}, \mathbb{R})$ given by

$$T : \mathscr{L}_{\bar{f}} - I.$$

Clearly, this is actually a Markov generator since $\mathscr{L}_{\bar{f}}$ is bounded and everywhere defined operator. Therefore, we can apply the Hille-Yosida Theorem to construct a Markov semigroup $\{S(t) : t \geq 0\}$ given by

$$S(t)(\varphi) = \lim_{n\to\infty}(I - (t/n)T)^{-n}\varphi, \quad \forall \varphi \in C_b(\Upsilon^{\mathbb{N}}, \mathbb{R})$$

which is a diffusion in infinite dimension obtained from a potential which is not necessarily of finite-body type interaction.

Analogous considerations apply to the potential βf so the semigroup associated to this potential should be ergodic as long as $J(n)$ decays to zero exponentially fast and β is sufficiently small. Therefore for any choice of ν (countably additive probability measure), we have $S(t)^*(\nu) \rightharpoonup \mu_{\beta f}$. This observation actually follows from the famous $(M - \varepsilon)$ theorem, see [Lig05].

Conjecture 6.8. *As long as the Ruelle operator has the spectral gap property and the potential f has continuous partial derivatives, intuitively, one would expect that the scaling limit (in the sense of Donsker theorem) of the infinite-dimensional Markov process associated to this semigroup is a formal solution of the infinitely dimensional stochastic differential equation*

$$dX_t^n = dB_t^n - \langle e_n, \nabla f(\sigma^n(X_t^1, X_t^2, \ldots))\rangle\, dt$$

This stochastic differential equation has its origin in the works of Lang [Lan77a, Lan77b], where the potential f has either one or two-body interactions, satisfies some symmetry and smoothness condition. This equations are also studied using ideas of DLR-Gibbsian equilibrium states in [Fri87, Osa13, Shi79, Tan96].

7. Concluding Remarks

Compact alphabets. As mentioned early, if X is compact, then it follows from the Alexandroff Theorem [DS58, III.5.13] that $rba(X)$ is equal to the set of all signed and finite Borel regular countably additive measures. Therefore, in this case the Thermodynamic Formalism developed here is an extension of the classical one for finite ([Bal00, PP90, Rue68, Wal75, Wal78]) and compact alphabets ([BCL$^+$11, CS16, LMMS15, SdSS14]).

Shift-invariant subspaces. If $Y \subset X$ is a complete and shift-invariant subset, then the definition of pressure and entropy can be introduced analogously as we did for the full shift. Moreover, since our main results regarding the existence of equilibrium states are build upon the general theory of convex analysis, they generalize immediately for such subshifts.

Spectral radius. By using similar argument as in [CvER17], we can prove the following result. For any $f \in C_b(X, \mathbb{R})$, there exists at least one finitely additive probability measure ν_f such that

$$\mathscr{L}_f^* \nu_f = \lambda \nu_f,$$

where $0 < \lambda \leq \rho(\mathscr{L}_f)$. At this moment we do not know what are the necessary and sufficient conditions to ensure that ν_f is countably additive. It also seems that there λ may not be the spectral radius of the Ruelle operator acting on $C_b(X, \mathbb{R})$.

Uniqueness. As far as we know, the first paper proving the uniqueness of equilibrium states for Hölder potentials in an uncountable alphabet setting is [ACR18]. The techniques employed there are no longer applicable here, because they are strongly dependent on the denseness of the Hölder potentials in the space $C_b(X,\mathbb{R})$, which may not be true if X is not compact. As mentioned before, the Gâteaux differentiability of the pressure would imply this result, but to the best of our knowledge none of the known techniques can be adapted to work in the generality considered here.

Stone-Čech compactification. Due to Knowles correspondence theory developed in [Kno67], there is no technical advantage in reconstructing our theory by regarding X as a subset of its Stone-Čech compactification βX. To be more precise: the question whether an equilibrium state μ_f, for a general potential $f \in C_b(X,\mathbb{R})$, is a countably additive measure is simply translated to a question on the support of a corresponding measure. For example, as an application of Theorem 2.1 of [Kno67], it follows that the Yosida-Hewitt decomposition of the equilibrium state μ_f has no purely finitely additive part if and only if $\overline{\mu_f}(Z) = 0$ for every zero-set Z in βX disjoint from X, see [Kno67] for more details and the definition of $\overline{\mu_f}$.

Phase transitions. If we have phase transition (in the sense of multiple equilibrium states at the same temperature) for a normalized potential βf, then the semigroup $\{S(t) : t \geq 0\}$ generated by the operator $T = (\mathscr{L}_{\beta f} - I)$ is not ergodic in the sense of [Lig05]. We believe that distinct cluster points in the weak-$*$-topology of $S(t)^*(\nu)$, when t tends to infinity, for suitable choices of ν, will generate distinct solutions for the infinitely dimensional stochastic differential equation $dX_t^n = dB_t^n - \langle e_n, \nabla f(\sigma^n(X_t^1, X_t^2, \ldots))\rangle\, dt$. Although we do not have a rigorous argument that supports this claim, it seems to be at least consistent with what is known about both problems for Hölder potentials. Furthermore, a rigorous proof of such relation would have the potential of creating a beautiful bridge between Thermodynamic Formalism and the theory of infinite-dimensional diffusions.

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