Phase Transitions in One-dimensional Translation Invariant Systems: a Ruelle Operator Approach.

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Abstract

We consider a family of potentials f, derived from the Hofbauer potentials, on the symbolic space $\Omega = \{0, 1\}^{\mathbb{N}}$ and the shift mapping σ acting on it. A Ruelle operator framework is employed to show there is a phase transition when the temperature varies in the following senses: the pressure is not analytic, there are multiple eigenprobabilities for the dual of the Ruelle operator, the DLR-Gibbs measure is not unique and finally the Thermodynamic Limit is not unique. Additionally, we explicitly calculate the critical points for these phase transitions. Some examples which are not of Hofbauer type are also considered. The non-uniqueness of the Thermodynamic Limit is proved by considering a version of a Renewal Equation. We also show that the correlations decay polynomially and compute the decay ratio.

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Contents

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1 Introduction

In this work we are interested in the study of the phase transition phenomenon, as temperature varies, for some systems defined on the one-dimensional lattice $\mathbb{N}$. Our study focus mainly on the models which are generalizations of the so called Hofbauer models on $\Omega = \{0,1\}^{\mathbb{N}}$ (see [26] and [54]). The Hofbauer models, besides being interesting mathematical objects, are also related to the B. Fedelhorf and M. Fisher models (see [17] and [18]) which appear in the Physics literature (see also [56] and [57]). They are also studied in [36], [19], [27] and [34], and are associated to a family of continuous potentials. They are also singled out for being fixed points of a certain renormalization mapping (see [4], [9] and [10]).

The analysis of Phase Transitions in Thermodynamic Formalism is a problem which can be understood in several different settings (see [35] [47] [46] [45] [38] [37] [49] [13]). Loosely speaking, in most cases a phase transition is defined by an abrupt qualitative change of the model in terms of some of its parameters (in general, the inverse of the temperature) in a neighborhood of a special value, called critical point or phase transition point.

We will present here several results on phase transitions for the one dimensional lattice $\mathbb{N}$. Concepts and results which are common in Statistical Mechanics are explored here in the setting of Thermodynamic Formalism. We would like to mention the very interesting work [48] where this unifying point of view is explored and nice results are carefully presented. We refer the reader to the references [41], [2] and [52] for basic results in Thermodynamic Formalism and Ergodic Theory and [44], [32], [8], [50], [1], [15] and [24], [28] for basic results in Statistical Mechanics. In the companion paper [12] the authors presented in detail several concepts and results which are used here.

To give precise statements of the main results of this paper we introduce some notations and definitions. We consider the Bernoulli space or sometimes called symbolic space $\Omega = \{0,1\}^{\mathbb{N}}$ and the full left shift $\sigma : \Omega \to \Omega$ acting on this symbolic space.

The main focus in the paper are potentials defined on the set $\Omega = \{0,1\}^{\mathbb{N}}$ instead of the usual space $\Omega = \{0,1\}^{\mathbb{Z}}$. This is the setting where one can apply the Ruelle operator formalism (see [44]). The final result applies to $\Omega = \{0,1\}^{\mathbb{Z}}$ as we will describe in full detail in the last section. It is just a technical issue of using coboundary functions to go from one setting to the other.

A potential is a continuous function $f : \Omega \to \mathbb{R}$ which describes the interaction of spins in the lattice $\mathbb{N}$. We denote by $\mathcal{M}_1(\sigma)$ the set of invariant probabilities measures (over the Borel sigma algebra of Ω) under σ.

Definition 1 (Pressure). *For a continuous potential $f : \Omega \to \mathbb{R}$ the Pressure of f is given by*

$$P(f) = \sup_{\mu \in \mathcal{M}_1(\sigma)} \left\{ h(\mu) + \int_{\Omega} f \, d\mu \right\},$$

where $h(\mu)$ denotes the Shannon-Kolmogorov entropy of μ (see [52] for definition).

Definition 2 (Equilibrium State)**.** *A probability measure $\mu \in \mathcal{M}_1(\sigma)$ is called an equilibrium state for f if*

$$h(\mu) + \int_\Omega f\, d\mu = P(f).$$

Remark. If f is continuous there always exists at least one equilibrium state (see [53]). For any potential in the class $\mathrm{W}(X, T)$ or $\mathrm{Bow}(X, T)$ the equilibrium state is unique (see [55]). When f is assumed to be only a continuous function there are examples where we have more than one equilibrium state for f. The existence of more than one equilibrium state is a possible meaning for phase transition. An example is given in [26] and here we present some other examples. As we mention before there are several definitions of phase transition. They are not necessarily equivalents. On this paper we investigate five types of phase transition (listed below) and exhibit a continuous potential f where these different notions coincides on a single critical value $\beta_c > 0$.

1. The function $p(\beta) = P(\beta\, f)$ is not analytic at $\beta = \beta_c$
2. There are more than one equilibrium state, that is, at least two probability measures maximizing $h(\mu) + \beta_c \int_\Omega f\, d\mu$.
3. The dual of Ruelle operator (to be defined later) has more than one eigenprobability for $\beta_c f$
4. There exist more than one DLR (to be defined later) probabilities measures for the potential $\beta_c f$.
5. There is more than one Thermodynamic Limit (to be defined later).

We refer the reader to [12] where the equivalences among the definitions $3, 4$ and 5 are proved for Bowen potentials. As far as we know there is no general theory on Phase Transitions and examples are handled in a case by case basis.

Decay of correlation of exponential type (for a large class of observable functions) occurs for the equilibrium probability of a Hölder potential. By the other hand, in some cases where there is phase transition (not Hölder), for the equilibrium probability (at the transition temperature) one gets polynomial decay of correlation. We show in Section 9 that this is indeed the case for the example we analyze in the paper. In the proof we use Renewal Theory.

There are interesting questions related to what happens near this critical point and regarding the properties of the model at it. For example, the problem of maximizing probabilities and selection or non-selection at zero temperature in distinct models were analyzed in [5], [11], [51], [33], [22] [7] and [39]. In some cases, there is more than one selected ground state. For the potentials considered here (the Double Hofbauer potentials) we analyze questions about selection or non-selection at a positive critical temperature for both symmetric and asymmetric cases. The term “phase transition” shall hereby be solely employed when the critical temperature is strictly positive.

Regarding to the first notion of phase transition given a continuous potential f a natural question is whether the Pressure $P(\beta\, f)$ is differentiable or even analytic as a function of β, the inverse temperature. In such generality this question is very hard but if f is a Hölder potential then the mapping $\beta \mapsto P(\beta f)$ is real analytic function for any $\beta > 0$ (see [41] and [2]).

In the sequel we introduce the basic concepts we used in the phase transition notions listed above.

Definition 3. *Given a continuous function $f : \Omega \to \mathbb{R}$, consider the Ruelle operator (or transfer) $\mathcal{L}_f : C(\Omega) \to C(\Omega)$ (for the potential f) defined in such way that for any continuous function $\psi : \Omega \to \mathbb{R}$ we have $\mathcal{L}_f(\psi) = \varphi$, where*

$$\varphi(x) = \mathcal{L}_f(\psi)(x) = \sum_{y\in\Omega;\, \sigma(y)=x} e^{f(y)}\, \psi(y).$$

Remark. When f is a Hölder function then $\mathcal{L}_f$ sends the space of Hölder functions to itself.

This operator (which is also called transfer operator) is a very helpful tool in the analysis of equilibrium states in Thermodynamic Formalism. One important issue is the existence or not of an strictly positive continuous eigenfunction for such operator.

Definition 4. *The dual operator $\mathcal{L}_f^*$ acts on the space of probability measures. It sends a probability measure μ to a probability measure $\mathcal{L}_f^*(\mu) = \nu$ defined in the following way: the probability measure ν is unique probability measure satisfying*

$$\int_\Omega \psi d\, \mathcal{L}_f^*(\mu) = \int_\Omega \psi\, d\nu = \int_\Omega \mathcal{L}_f(\psi) d\mu$$

for any continuous function ψ.

Definition 5 (Gibbs Measures)**.** *Let $f : \Omega \to \mathbb{R}$ be a continuous function. We call a probability measure ν a Gibbs probability for f if there exists a positive $\lambda > 0$ such that $\mathcal{L}_f^*(\nu) = \lambda\, \nu$. We denote the set of such probabilities by $\mathcal{G}^*(f)$.*

In general, for any continuous function f the set $\mathcal{G}^*(f) \neq \emptyset$. We should remark that a probability measures on $\mathcal{G}^*(f)$ is not necessarily translation invariant, even if f is Lipchitz or Hölder.

For a Hölder potential f there exist a value $\lambda > 0$ (the spectral radius) which is a common eigenvalue for both Ruelle operator and its dual (see [41]). The eigenprobability ν associated to the maximal λ is unique. This probability ν (which is unique) is a Gibbs state according to the above definition.

A natural question is: for a Holder continuous potential f is it possible to have different eigenprobabilities $\nu_1, \nu_2 \in \mathcal{G}^*(f)$ for $\mathcal{L}_f^*$, associated to different eigenvalues λ_1 and λ_2? This is not possible due to properties of the involution kernel (see [39]). These two eigenprobabilities would determine via the involution kernel two positive eigenfunctions for another Ruelle operator $\mathcal{L}_{f^*}$, with the eigenvalues λ_1 and λ_2 (see [25]), where f^* is the dual potential for f. This is not possible (see for instance [41] or Proposition 1 in [39]). Anyway, eigendistributions for $\mathcal{L}_f^*$ may exist (see [25]).

This eigenvalue λ is the spectral radius of the operator $\mathcal{L}_f$. If $\mathcal{L}_f(\varphi) = \lambda\varphi$ and $\mathcal{L}_f^*(\nu) = \lambda\nu$, then up to normalization (to get a probability measure) the probability measure $\mu = \varphi\,\nu$ is the equilibrium state for f.

Because of this last mentioned fact the Gibbs probability ν helps to identify the equilibrium probability μ. If φ is constant equal to 1 then the Gibbs probability measure is the equilibrium probability.

These are non trivial results and the detailed proofs can be found in [41]. Several properties of the equilibrium state μ (mixing, exponential decay of correlations, central limit theorem, etc...) are obtained using this formalism in the Hölder case (see [41]). The bottom line is: several nice properties of the Gibbs state follow from the use of Ruelle operator properties. All these are extended to the equilibrium state via this formalism.

Under the square summability of the variation for normalized potentials there exist just one eigenprobability (see [40] and [29]) and therefore there is no phase transition in this case in this sense.

When there exists a positive continuous eigenfunction for the Ruelle operator (of a continuous potential f) it is unique (the proof in [41] works for a continuous potential f). We remark that for a general continuous potential may not exist a positive continuous eigenfunction. We present some examples here. On the other hand, eigenprobabilities always exists for a continuous potential since the space Ω is compact (see Theorem 10). If for a continuous potential f there exists an eigenfunction φ and an eigenprobability ν, then $\mu = \varphi\,\nu$ defines an equilibrium state for f (see Section 2 in [41]).

For a Hölder potential f we do not have phase transintion in any above described notions. One heuristic reason is in the Hölder case the influence over the state in fixed site n in the lattice, by any other site decays exponentially fast with respect to the distance between them.

We will consider in this paper some continuous potentials defined by P. Walters in [54] on the Bernoulli space $\{0,1\}^{\mathbb{N}}$. The values of a potential f in this class are defined just by the first strings of zeroes and ones. To be more precise consider four sequence of real numbers a_n, b_n, c_n, d_n and constants a, b, c, d. The potential f in this class satisfies

$$f(0^n 1z) = a_n, \quad f(0^\infty) = a, \quad f(10^n 1z) = d_n, \quad f(10^\infty) = d,$$
$$f(01^n 0z) = b_n, \quad f(01^\infty) = b, \quad f(1^n 0z) = c_n, \quad f(1^\infty) = c,$$

and is assumed that $a_n \to a$, $b_n \to b$, $c_n \to c$ and $d_n \to d$.

The existence of positive **continuous** eigenfunction for f is guaranteed (see Theorem 3.1 in [54]) as long as the following inequality is satisfied

$$1 < \frac{1}{e^{2\max(a,c)}} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{j+1}} \frac{e^{a_2+\ldots+a_{j+1}}}{e^{j\max(a,c)}} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{j+1}} \frac{e^{c_2+\ldots+c_{j+1}}}{e^{j\max(a,c)}} \right]. \tag{1}$$

In this case (see Theorem 3.5 in [54]) the eigenvalue λ for the Ruelle operator satisfies:

$$1 = \frac{1}{\lambda^2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{j+1}} \frac{e^{a_2+\ldots+a_{j+1}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{j+1}} \frac{e^{c_2+\ldots+c_{j+1}}}{\lambda^j} \right]. \tag{2}$$

There are continuous potentials f of the above form such that the condition (1) is not satified. Such potentials provide examples where the potential f is continuous but the Ruelle operator associated to it do not have positive continuous eigenfunction. When the eigenfunction do exists an explict formula for it is given in [54] (see page 1341). We point out that all of the above formulas are analytical expressions and even in the case where the r.h.s of (1) is equal to 1 an explicit eigenfunction which is not continuous can be obtained (because, for instance is ∞ just in the points 0^∞ and 1^∞ but finite elsewhere). This extended sense of eigenfunction will be considered here in **the critical point** in some of our examples.

In [6] it is analyzed the zero temperature limit for such family of potentials.

We will define in the next section the double Hofbauer model which is obtained from a certain potential $g : \{0,1\}^{\mathbb{N}} = \Omega \to \mathbb{R}$, depending of two fixed parameters γ and δ.

The potential view of renormalization is presented in [4]. The double Hofbauer potential is a fixed point for a renormalization operator. In this sense it is a model of special interest. We will explain briefly this point.

We define $H : \Omega = \{0,1\}^{\mathbb{N}} \to \Omega$ by: for $c_1 \geq 2$

$$H((\underbrace{0,...,0}_{c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,...)) = (\underbrace{0,...,0}_{2c_1},\underbrace{1,...,1}_{c_2}\underbrace{0,...,0}_{c_3},1,\ldots),$$

and

$$H((\underbrace{1,\ldots,1}_{c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots)) = (\underbrace{1,\ldots,1}_{2c_1},\underbrace{0,\ldots,0}_{c_2}\underbrace{1,\ldots,1}_{c_3},1,\ldots).$$

We define the renormalization operator $\mathcal{R}$ in the following way: given the potential $V_1 : \Omega \to \mathbb{R}$ we get $V_2 = \mathcal{R}(V_1)$ where for x of the form $(\underbrace{0,...,0}_{c_1}\ 1...)$, or $(\underbrace{1,...,1}_{c_1}\ 1...)$, with $c_1 \geq 2$:

$$V_2(x) = V_1(\sigma(\,(H(x)))\,) + V_1(H(x)),$$

and for x of the form (01...) or (10...) we set

$$V_2(x) = V_1(x).$$

This defines $V_2 = \mathcal{R}(V_1)$.

It is easy to see that for γ and δ fixed the corresponding double Hofbauer potential g is fixed for $\mathcal{R}$.

In [4] it is explained why this is the natural renormalization operator to be considered in the one-dimensional setting (which is inspired by the similar concept in Statistical Mechanics). It is more common in dynamics to consider the renormalization of the transformation dynamics (the M. Feigenbaum point of view) which is different from the reasoning described above.

2 The Double Hofbauer Model

Before present the model we need to introduce some notations. We define two infinite collections of cylinder sets given by

$$L_n = \overline{\underbrace{000...0}_{n}1} \quad \text{and} \quad R_n = \overline{\underbrace{111...1}_{n}0}, \quad \text{for all } n \geq 1.$$

Note that these cylinders are disjoint and $\cup_{n\geq 1}(L_n \cup R_n) = \Omega \setminus \{0^\infty, 1^\infty\}$. To define the model we also need to fix two real numbers $\gamma > 1$ and $\delta > 1$, satisfying $\delta < \gamma$. Using these parameters we can define a continuous potential $g_{\gamma,\delta} : \Omega \to \mathbb{R}$, which is simply denote by g, in the following way: for any $x \in \Omega$

$$g(x) = \begin{cases} -\gamma \log \frac{n}{n-1}, & \text{if } x \in L_n, \text{ for some } n \geq 2; \\ -\delta \log \frac{n}{n-1}, & \text{if } x \in R_n, \text{ for some } n \geq 2; \\ -\log \zeta(\gamma), & \text{if } x \in L_1; \\ -\log \zeta(\delta), & \text{if } x \in R_1; \\ 0, & \text{if } x \in \{1^\infty, 0^\infty\}, \end{cases}$$

where $\zeta(s) = \sum_{n\geq 1} 1/n^s$. By using the canonical identification of a point of the symbolic space Ω with a point on the interval $[0, 1]$ we can have a sketch of the graph of the Double Hofbauer potential

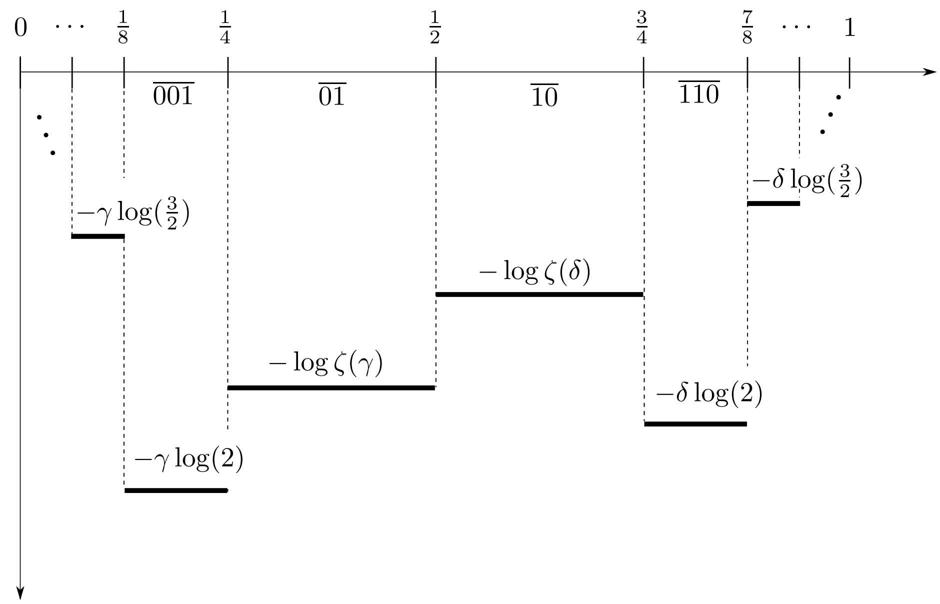

Figure 1: The Double Hofbauer potential represented on the interval $[0, 1]$.

These potentials are particular cases of a more general class considered by F.Hofbauer [26] and P. Walters [54]. They are not Hölder pontetials and have not summable variation (see [54] or [48]). In fact, following the notation of the work [54] (page 1325) our potential are obtained by considering $a_n = -\gamma\log\frac{n}{n-1}$, $a = 0$, $c_n = -\delta\log\frac{n}{n-1}$, $c = 0$, $b_n = -\log\zeta(\gamma)$, $d_n = -\log\zeta(\delta)$, $b = -\log\zeta(\gamma)$ and $d = -\log\zeta(\delta)$, $n \in \mathbb{N}$. For these choice of the sequences a_n, b_n, c_n, d_n and the constants a, b, c, d Walters proved in [54] that these potentials does not belongs to $\mathrm{W}(X, T)$ neither $\mathrm{Bow}(X, T)$. This fact follows from a simple application of the Theorem 1.1 page 1326 in [54].

When $\delta \neq \gamma$ there is a competition of two regimes. This system has a much more complex behaviour that the one presented in the Fisher-Fedelhorf model described in [17].

When $\gamma = \delta$ we say that the potential g defined above is symmetric. To avoid confusion we use the terminology Hofbauer model (indexed by γ which differs from the Double Hofbauer model which is indexed by γ and δ) for the family of potentials considered in [36], [4] and [19]. There are, of course, some similarities between the two models.

For a while we will not assume that the potential is symmetric. When we need such hypothesis we will make it clear.

3 Phase Transition I. Non-differentiability of the Pressure

Note that the delta Dirac δ_{0^∞} and δ_{1^∞} are both equilibrium states for g. On page 1341 in [54] it is presented the explicit expression for the eigenfunction φ_β associated to the eigenvalue $\lambda(\beta)$ for the Ruelle operator of the potential βg.

One important issue is to show that the pressure $p(\beta)$ of the potential $\beta\, g$ is equal to $\log \lambda(\beta)$, where $\lambda(\beta)$ is the main eigenvalue of Ruelle operator for $\beta\, g$. This is the claim of Theorem 10. As we will see the Theorem 10 says that

$$\sup_{\mu\in\mathcal{M}_1(\sigma)} \left\{ h(\mu) + \beta \int_\Omega g\, d\mu \right\} = P(\beta\, g) \equiv p(\beta) = \log \lambda(\beta),$$

where $\lambda(\beta)$ is the maximal eigenvalue of $\mathcal{L}_{\beta\, g}$. Moreover we can show that $P(1\, g) = 0$ (see Theorems 12 and 13) and $P(\beta\, g) > 0$ for $\beta < 1$.

Remark. The function $\beta \mapsto P(\beta\, g)$ is non-increasing and therefore for $\beta > 1$ we have that $P(\beta\, g) \leq 0$. Since $h(\mu) + \beta \int_\Omega g\, d\mu = 0$ for $\mu = \delta_{0^\infty}$ and $\mu = \delta_{1^\infty}$ we have that $P(\beta\, g) = 0$ for $\beta > 1$.

We are interested in to determine equilibrium states μ_β for the family of potentials $\beta\, g$, when β approaches 1 from below. If $0 < \beta < 1$ we have that $p(\beta) \equiv P(\beta\, g) > 0$, because this function is monotone increasing and $P(g) = 0$ (see Theorem 13). These observations give rise to natural questions as: is there a selection in the limit when $\beta \to 1$? In other words, does μ_β converges to δ_{0^∞} or to δ_{1^∞} when $\beta \to 1$? In the negative case is still the weak limit of μ_β a convex combination of these two probability measures ? In what follows we show some computations that will help to answer these questions.

Using the general result described by Corollary 3.5 in [54] in our particular case we have that the eigenvalue $\lambda(\beta) = \exp(P(\beta\, g)) \equiv \exp(p(\beta))$ satisfies the following identity

$$1 = \frac{\sum_{n=1}^{\infty} \frac{n^{-\gamma\, \beta}}{\lambda(\beta)^n}}{\zeta(\gamma)^\beta} \; \frac{\sum_{n=1}^{\infty} \frac{n^{-\delta\, \beta}}{\lambda(\beta)^n}}{\zeta(\delta)^\beta}. \tag{3}$$

For different values of the parameter γ the above family provides examples where we have phase transition of type 1 and 2 as defined on the page 3. In all these examples the critical inverse temperature β_c can be explicitly obtained and its value is $\beta_c = 1$. To be more precise if $2 > \gamma > \delta > 1$ there exists just two ergodic equilibrium probabilities at $\beta = 1$: the Dirac measure concentrated on 0^∞ and the Dirac measure concentrated on 1^∞. In the case $\gamma, \delta > 2$ there exists three ergodic equilibrium probabilities at $\beta = 1$: the Dirac measure concentrated on 0^∞, the Dirac measure concentrated on 1^∞ and a probability measure μ_1 which gives positive probability to open sets (which is described latter).

Question: Since $\beta = 1$ is a critical temperature for the Double Hofbauer model it is interesting to know what is the asymptotic behavior of $p(\beta) = P(\beta\, g)$ when $\beta \to 1$ with $\beta < 1$. Of course, the other lateral limit is trivial because pressure vanish for $\beta > 1$.

Let $p_1(\beta) = \log(\lambda_1(\beta))$ denote the pressure for the Hofbauer model associated to $\gamma > 1$ and $p_2(\beta) = \log(\lambda_2(\beta))$ be the pressure for the Hofbauer model associated to $\delta > 1$ according to [36]. If $\gamma, \delta > 1$ the asymptotics expansions of $p_1(\beta)$ and $p_2(\beta)$, when $\beta \to 1$ from below, were determined in [36] (Theorem A) and for the reader's convenience we give the statement of the theorem below:

Theorem 6. *For the Hofbauer model with parameter γ we have:*

a) $\zeta(\gamma)^\beta = \sum_{n=1}^\infty \frac{e^{-n\, p_1(\beta)}}{n^{\gamma\beta}}$, *for* $\beta < 1$, *for any* $\gamma > 1$,

b) If $1 < \gamma < 2$, then, when $\beta \leq 1$, $\beta \to 1$, we get that

$$p_1(\beta) = \left(\frac{\zeta(\gamma)\, \log \zeta(\gamma) - \gamma\, \zeta'(\gamma)}{-\Gamma(1-\gamma)} \right)^{\frac{1}{\gamma-1}} (1-\beta)^{\frac{1}{(\gamma-1)}} + \text{ high order terms,}$$

c) If $2 < \gamma < 3$, then, when $\beta \leq 1$, $\beta \to 1$, there is constant A so that

$$p_1(\beta) = \frac{\zeta(\gamma)\, \log \zeta(\gamma) - \gamma\, \zeta'(\gamma)}{\gamma\, \zeta'(\gamma-1)} (1-\beta) + A(1-\beta)^{\gamma-1}(1+o(1)).$$

In this case the entropy of the probability measure μ (the equilibrium state for the Hofbauer model) is

$$\frac{\zeta(\gamma)\, \log \zeta(\gamma) - \gamma\, \zeta'(\gamma)}{\gamma\, \zeta'(\gamma-1)}.$$

The case $\gamma > 3$ can be also analyzed but the formulas are more complex.

d) when $\beta \to 1$, $3 \leq m < \gamma < m+1$, we get the expansion

$$p_1(\beta) = A_1(1-\beta) + A_2(1-\beta)^2 + \cdots + A_{m-1}(1-\beta)^{m-1} + (C + o(1))(1-\beta)^{\gamma-1}.$$

for some constants $A_1, A_2, \ldots, A_{m-1}$ and C.

Remark. Obviously, $p_2(\beta)$ has similar properties.

Theorem 7. *If $1 < \delta < \gamma < 2$, then, $p'(\beta) = \frac{d}{d\beta} \log \lambda(\beta) \to 0$, when $\beta \to 1$, $\beta < 1$. In the case $\gamma > \delta > 2$, we have*

$$\lim_{\beta \to 1^-} p(\beta) = \frac{1}{2} \left(\lim_{\beta \to 1^-} p_1(\beta) + \lim_{\beta \to 1^-} p_2(\beta) \right).$$

Since $p(\beta) = 0$ for $\beta > 1$ there is a lack of analyticity of the pressure $p(\beta)$ at $\beta = 1$.

Proof. It is known from [36] that

$$1 = \lambda_1(\beta)^{-1}\,\frac{1+\sum_{n=2}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda_1(\beta)^{n-1}}}{\zeta(\gamma)^{\beta}} = \frac{\sum_{n=1}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda_1(\beta)^{n}}}{\zeta(\gamma)^{\beta}}, \tag{4}$$

and

$$1 = \lambda_2(\beta)^{-1}\frac{1+\sum_{n=2}^{\infty}\frac{n^{-\delta\,\beta}}{\lambda_2(\beta)^{n-1}}}{\zeta(\delta)^{\beta}} = \frac{\sum_{n=1}^{\infty}\frac{n^{-\delta\,\beta}}{\lambda_2(\beta)^{n}}}{\zeta(\delta)^{\beta}}. \tag{5}$$

If we assume that

$$\limsup_{\beta\to 1}\frac{\lambda(\beta)}{\lambda_1(\beta)} > 1 \qquad \text{and} \qquad \limsup_{\beta\to 1}\frac{\lambda(\beta)}{\lambda_2(\beta)} > 1, \tag{6}$$

then by (4) and (5), respectively, we have that

$$\limsup_{\beta\to 1}\frac{\sum_{n=1}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda(\beta)^{n}}}{\zeta(\gamma)^{\beta}} < 1. \qquad \text{and} \qquad \limsup_{\beta\to 1}\frac{\sum_{n=1}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda(\beta)^{n}}}{\zeta(\gamma)^{\beta}} < 1.$$

But this would contradict the equation (3). Therefore, it is not possible that both inequalities in (6) holds.

The functions $p_1(\beta)$ and $p_2(\beta)$ are convex, monotonous decreasing and differentiable in β, for $\gamma, \delta > 1$ and $t < 1$. It is also known that for $1 < \delta < \gamma < 2$, it is true that $p_1'(\beta) \to 0$ and $p_2'(\beta) \to 0$, when $\beta \to 1$, $\beta < 1$.

From Theorem 6 item b) and the L'Hospital Rule follows that the limit

$$\lim_{\beta\to 1^-}\frac{\lambda_1(\beta)}{\lambda_2(\beta)} = \lim_{\beta\to 1^-}\frac{c_1+\frac{1}{\gamma-1}\,\log(1-\beta)}{c_2+\frac{1}{\delta-1}\log(1-\beta)} = \frac{\delta-1}{\gamma-1} < 1, \tag{7}$$

Therefore, for β close to 1 we have that

$$\lambda_2(\beta) > \lambda_1(\beta). \tag{8}$$

Since $\lambda(\beta), \lambda_1(\beta)$, and $\lambda_2(\beta)$ are all convex as functions of β it is not possible that $\lambda(\beta) \leq \lambda_1(\beta)$, for $\beta < 1$ close to 1 (otherwise would contradict (3)). Therefore, we get that $\lambda_2(\beta) \geq \lambda(\beta) \geq \lambda_1(\beta)$. It follows from Abel's Theorem that

$$\frac{e^{2\,p(\beta)}}{e^{p_1(\beta)}\,e^{p_2(\beta)}} = \frac{\lambda(\beta)^2}{\lambda_1(\beta)\,\lambda_2(\beta)} = \frac{1+\sum_{n=2}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda(\beta)^{n-1}}}{1+\sum_{n=2}^{\infty}\frac{n^{-\gamma\,\beta}}{\lambda_1(\beta)^{n-1}}}\;\frac{1+\sum_{n=2}^{\infty}\frac{n^{-\delta\,\beta}}{\lambda(\beta)^{n-1}}}{1+\sum_{n=2}^{\infty}\frac{n^{-\delta\,\beta}}{\lambda_2(\beta)^{n-1}}} \to 1, \tag{9}$$

when, $\beta \to 1^-$, where $p(\beta) = P(\beta\, g) \equiv P(\beta\, g_{\lambda,\delta})$.

Since $\lambda_1(\beta) > 1$ it is not possible that $\lambda(\beta) \geq \lambda_2(\beta) > \lambda_1(\beta)$. Indeed, if it was true then the two quotients on the right side of (9) would be smaller than one and this is a contradiction. For similar reason we can not have $\lambda_2(\beta) > \lambda_1(\beta) \geq \lambda(\beta)$. Therefore, we get that $\lambda_2(\beta) \geq \lambda(\beta) \geq \lambda_1(\beta)$. From these observations follows that $p'(\beta) \to 0$, when $\beta \to 1$, $\beta < 1$. Note that when $\beta \to 1$ from below the right hand side of (9) goes to 1. Therefore,

$$\lim_{\beta \to 1^-} p(\beta) = \frac{1}{2} \left(\lim_{\beta \to 1^-} p_1(\beta) + \lim_{\beta \to 1^-} p_2(\beta) \right).$$

The above equality shows the existence of phase transition, in the case $\gamma, \delta > 2$, in the sense of lack of differentiability of the pressure. □

Theorem 8. *Consider the Double Hofbauer model. If* $2 < \delta < \gamma < 3$*, then,*

$$\lim_{\beta \to 1^-} p(\beta) = \frac{1}{2} \left(\lim_{\beta \to 1^-} p_1(\beta) + \lim_{\beta \to 1^-} p_2(\beta) \right).$$

Moreover,

$$\lim_{\beta \to 1^-} p'(\beta) = \frac{1}{2} \left(\frac{\zeta(\gamma) \log \zeta(\gamma) - \gamma \zeta'(\gamma)}{\gamma \zeta'(\gamma - 1)} + \frac{\zeta(\delta) \log \zeta(\delta) - \delta \zeta'(\gamma)}{\delta \zeta'(\delta - 1)} \right). \qquad (10)$$

Since $p(\beta) = 0$ *for* $\beta > 1$ *there is a lack of differentiability of the pressure at* $\beta = 1$*.*

Proof. The proof is analogous to the previous one, but here we have to use 9 and item c) of Theorem 6 and also to note that the function

$$\gamma \mapsto \frac{\zeta(\gamma) \log \zeta(\gamma) - \gamma \zeta'(\gamma)}{\gamma \zeta'(\gamma - 1)} < 0$$

is monotonous decreasing for $\gamma > 2$. □

4 The Main Eigenfunction

The eigenfunction of the Ruelle operator (when it exists) help us to understand important properties of the equilibrium state of a given potential. Continuous eigenfunctions of the Double Hofbauer model do exists for any $0 < \beta < 1, \gamma > 1$ and $\delta > 1$. Indeed, given real number $a > 0$ let $b(\beta)$ be defined such that

$$b(\beta) = \frac{a}{\lambda(\beta)} \left(1 + \sum_{j=2}^{\infty} \frac{j^{-\gamma \beta}}{\lambda(\beta)^{j-1}} \right) \zeta(\gamma)^{-\beta}.$$

Note that when $\beta \to 1$ we have that $b(\beta) \to a$. From the general result described in [54] applied to our particular case we get that the eigenfunction φ_β, for $\beta < 1$ and $n \geq 1$ is given by

$$\varphi_\beta(0^n 1...) = a \left(1 + n^{\gamma\,\beta} \sum_{j=2}^{\infty} \frac{(j+n-1)^{-\gamma\,\beta}}{\lambda(\beta)^{j-1}} \right), \tag{11}$$

$$\varphi_\beta(1^q 0...) = b(\beta) \left(1 + q^{\delta\,\beta} \sum_{n=2}^{\infty} \frac{(n+q-1)^{-\delta\,\beta}}{\lambda(\beta)^{n-1}} \right), \tag{12}$$

$$\varphi_\beta(0^\infty) = a \qquad \text{and} \qquad \varphi_\beta(1^\infty) = b(\beta).$$

We remark that all the above series are absolutely convergent because $\lambda(\beta) > 1$ and from the definitions of $\lambda(\beta)$ and $b(\beta)$ follows that $\varphi_\beta(10...) = a\,(\lambda(\beta)-1)\zeta(\delta)$.

For $\beta = 1$ all the above is fine, up to $\varphi_1(0^\infty) = \infty$ and $\varphi_1(1^\infty) = \infty$. In this case φ_1 is positive but it has infinite values just in these two points. Straightforward calculations shown that

$$\varphi_1(0^n 1..) \sim \frac{n}{\gamma - 1} \qquad \text{and} \qquad \varphi_1(1^n 0..) \sim \frac{n}{\delta - 1}. \tag{13}$$

From where it follows that $\varphi_1(0^\infty) = \infty$ and $\varphi_1(1^\infty) = \infty$. We point out that this will not be a big problem.

Note that if $\gamma > \delta$, then $\zeta(\gamma) < \zeta(\delta)$, and $\varphi_\beta(1^\infty) = b(\beta) \to \varphi_1(1^\infty) = \varphi_1(0^\infty)$, when $\beta \to 1^-$. In the symmetric case when $1 < \gamma = \delta < 2$ it follows from Theorem 1 page 141 in [36] that

$$P(\beta\, g) \sim c\,(1-\beta)^{\frac{1}{\gamma-1}} + \text{ high order terms}, \quad \text{when } \beta \to 1^-.$$

In the non symmetric case it is not clear how to obtain the asymptotic expansion of $P(\beta\, g)$ near the critical point.

We also observe that for $\beta = 1$ the pressure vanish and $\mathcal{L}_g(\varphi_1) = \varphi_1$, therefore for any $x \in \Omega$ we have that

$$\sum_{\sigma(y)=x} e^{g(y)}\, \varphi_1(y) = \varphi_1(x),$$

even for $x = 0^\infty$ and $x = 1^\infty$, because both sides of the above equality are equal to $+\infty$. By extending the equality below in the obvious way we obtain for all $x \in \Omega$

$$\sum_{\sigma(y)=x} e^{g(y) + \log \varphi_1(y) - \log \varphi_1(\sigma(y))} = 1.$$

The function $J \equiv J_g$ defined by $\log J = g + \log \varphi_1 - \log(\varphi_1 \circ \sigma)$ is called the Jacobian associated to g. Using the above observations we can define an operator $\mathcal{G}$ which sends any continuous function ψ to $\mathcal{G}(\psi) = \phi$, where ϕ is a function defined on $\Omega \setminus \{0^\infty, 1^\infty\}$ by

$$\phi(x) = \mathcal{G}(\psi)(x) = \sum_{\sigma(y)=x} J_g(y)\, \psi(y).$$

Since $\mathcal{G}(1) = 1$ the dual operator $\mathcal{G}^*$ acts on the space of probabilities measures mapping a probability measure μ on a probability measure $\mathcal{G}^*(\mu) = \nu$ so that for any continuous function ψ the following equality holds

$$\int_\Omega \psi d\, \mathcal{G}^*(\mu) = \int_\Omega \psi\, d\nu = \int_\Omega \mathcal{G}(\psi) d\mu.$$

When $\gamma, \delta > 2$, there exists a probability μ_1 (positive in open sets) which is fixed by $\mathcal{G}^*$. We can show that in this case this probability measure μ_1 is an equilibrium state for the potential g (see [26], [36] and [19]). More detailed description of the probability measure μ_1 is given in the next section.

5 The Eigenprobability

By using the Caratheodory Extension Theorem we can define a finite measure ν_1 on the Borelians of Ω such that for any natural number $q \geq 1$, we have

$$\nu_1\left(\overline{0^q1}\right) = q^{-\gamma} \qquad \text{and} \qquad \nu_1\left(\overline{1^q0}\right) = q^{-\delta}. \tag{14}$$

We chose below the values of $\nu_1(\overline{0})$ and $\nu_1(\overline{1})$ so that the measure ν_1 satisfies $\mathcal{L}_g^*(\nu_1) = \nu_1$. Note that this fixed point equation is equivalent to say that for any function of the form $I_{\overline{0^q1}}$ we have

$$q^{-\gamma} = \int_\Omega I_{\overline{0^q1}}\, d\nu_1 = \int_\Omega \mathcal{L}_g(\, I_{\overline{0^q1}}\,) d\nu_1,$$

and, moreover for any function of the form $I_{\overline{1^q0}}$ we have

$$q^{-\delta} = \int_\Omega I_{\overline{1^q0}}\, d\nu_1 = \int_\Omega \mathcal{L}_g(\, I_{\overline{1^q0}}\,) d\nu_1.$$

Let us compute the last integral above. First by the definition of the Ruelle Operator we have

$$\mathcal{L}_g(\, I_{\overline{0^q1}}\,)(x) = e^{g(0x)} I_{\overline{0^q1}}(0x) + e^{g(1x)} I_{\overline{0^q1}}(1x) = e^{g(0x)} I_{\overline{0^q1}}(0x).$$

The last expression is nonzero if and only if $x \in L_{q-1}$. Therefore,

$$\int_\Omega \mathcal{L}_g(\, I_{\overline{0^q1}}\,) d\nu_1 = \int_\Omega e^{g(0x)} I_{\overline{0^q1}}(0x) d\nu_1(x) = \frac{q^{-\gamma}}{(q-1)^{-\gamma}}(q-1)^\gamma = q^{-\gamma}.$$

Analogously we can compute the integral of $I_{\overline{1^q0}}$. For the probability measure ν_1 to be a eigenprobability for g it must also satisfy

$$\nu_1(\overline{0}) = \int_\Omega I_{\overline{0}}\, d\nu_1 = \int_\Omega \mathcal{L}_g(\, I_{\overline{0}}\,) d\nu_1.$$

By using again the definition of the Ruelle Operator we have

$$\mathcal{L}_g(\, I_{\overline{0}}\,)(x) = e^{g(0x)} I_{\overline{0}}(0x) + e^{g(1x)} I_{\overline{0}}(1x) = e^{g(0x)} I_{\overline{0}}(0x).$$

The point x must be in the cylinder $\overline{1}$ or, in the cylinder $\overline{0}$, which means in some of the sets L_n, $n \geq 1$. Then,

$$\nu_1(\overline{0}) = \int_{\overline{1}} e^{g(0x)} I_{\overline{0}}(0x) d\nu_1(x) + \sum_{n=1}^{\infty} \int_{L_n} e^{g(0x)} I_{\overline{0}}(0x) d\nu_1(x).$$

Therefore, it follows from the definitions of g and the Lebesgue integral that

$$\begin{aligned}
\nu_1(\overline{0}) &= \zeta(\gamma)^{-1}\nu_1(\overline{1}) + 2^{-\gamma}\nu_1(\overline{01}) + \frac{3^{-\gamma}}{2^{-\gamma}}\nu_1(\overline{001}) + \dots \\
&= \zeta(\gamma)^{-1}\nu_1(\overline{1}) + 2^{-\gamma} + 3^{-\gamma} + \dots \\
&= \zeta(\gamma)^{-1}\nu_1(\overline{1}) + \zeta(\gamma) - 1.
\end{aligned}$$

In the same way we obtain $\nu_1(\overline{1}) = \zeta(\delta)^{-1}\nu_1(\overline{0}) + \zeta(\delta) - 1$. Solving the system we get that

$$\nu_1(\overline{0}) = \frac{\zeta(\gamma)^{-1}\zeta(\delta) - \zeta(\gamma)^{-1} + \zeta(\gamma) - 1}{1 - \zeta(\gamma)^{-1}\zeta(\delta)^{-1}}$$

and

$$\nu_1(\overline{1}) = \frac{\zeta(\delta)^{-1}\zeta(\gamma) - \zeta(\delta)^{-1} + \zeta(\delta) - 1}{1 - \zeta(\delta)^{-1}\zeta(\gamma)^{-1}}.$$

This measure ν_1 is not necessarily a probability measure. Since it is a finite measure all we have to do is multiplying it by a suitable constant in order to get an eigenprobability. From now on, we will assume that ν_1 is a probability measure. If $\gamma > \delta$ then some tedious manipulation yields that $\nu_1(\overline{1}) > \nu_1(\overline{0})$. This means that the eigenprobability ν_1 gives more mass for regions where the potential is less flat.

Piecing together all the observations on this section we have proved the following proposition.

Proposition 9. *The above defined probability* ν_1 *satisfies* $\mathcal{L}_g^*(\nu_1) = \nu_1$.

Now we need the following result:

Proposition 10. *Suppose* $g : \Omega \to \mathbb{R}$ *is the continuous function we consider above. For any* $\beta > 0$ *there exists a eigenprobability* ν_β *and eigenvalue* $\Lambda(\beta)$ *such that* $\mathcal{L}_{\beta g}^* \nu_\beta = \Lambda(\beta)\, \nu_\beta$. *Moreover,* $\Lambda(\beta) = \lambda(\beta) = \log p(\beta)$, *for all* $0 \le \beta < 1$.

Proof. Since the potential $\beta\, g$ is continuous we can define a transformation $\mathcal{T}$ in the space of probabilities measures over Ω such that $\mathcal{T}(\mu) = \rho$, where for any continuous function f we have

$$\int_\Omega f\, d\mathcal{T}(\mu) = \int_\Omega f d\rho = \frac{\displaystyle\int_\Omega \mathcal{L}_{\beta\, g}(f) d\mu}{\displaystyle\int_\Omega \mathcal{L}_{\beta\, g}(1) d\mu}.$$

By the Thichonov-Schauder Theorem there exists a fixed point ν_β for such $\mathcal{T}$ and we have $\Lambda(\beta) = \int_\Omega \mathcal{L}_{\beta\, g}(1) d\nu_\beta$. Finally, by using the same reasoning of Section 2 in [41] we get that $\lambda(\beta) = \Lambda(\beta)$.

It remains to prove that $\Lambda(\beta)$, the eigenvalue of the dual operator, satisfies $\log p(\beta) = \Lambda(\beta)$. The proof is similar to the one given in [41] to the Proposition 3.4. In [41] the potential is Lipchitz but the same approach can be adopted to the potential we are considering here. $\square$

6 Phase Transition II. Non-Uniqueness of the Equilibrium State

In this section we show the existence of at least two equilibrium probability states for the potential βg at $\beta = 1$. Keeping the notation of the previous section, for $n \ge 1$, $\beta > 1$, consider the finite measure μ_β such that

$$\mu_\beta(L_n) = \nu_t(L_n)\, \varphi_\beta(0^n 1 \ldots), \qquad \mu_\beta(R_n) = \nu_\beta(R_n)\, \varphi_\beta(1^n 0 \ldots)$$

$\mu_\beta(\overline{0}) = \sum_n \mu_\beta(L_n)$ and $\mu_\beta(\overline{1}) = \sum_n \mu_\beta(R_n)$. This defines the probability measure μ_β in a unique way. For instance, (not normalizing) we have $\mu_1(\overline{01}) = \varphi_1(01\ldots)\nu_1(\overline{01}) = \zeta(\gamma)$, and $\mu_1(\overline{001}) = \varphi_1(001\ldots)\nu_1(\overline{001}) = \varphi_1(001\ldots)\, 2^{-\gamma}$. The bottom line is

$$\mu_1([0^q 1]) \sim q^{1-\gamma} \text{ and } \mu_1([1^q 0]) \sim q^{1-\delta}. \tag{15}$$

Proposition 11. *The above defined probability* μ_β *is invariant for* σ.

Proof. Given a continuous f we have that

$$\int_\Omega f\circ\sigma\, d\mu_\beta = \int_\Omega (f\circ\sigma)\,\varphi_\beta\, d\nu_\beta = \int_\Omega \frac{1}{\lambda(\beta)}\mathcal{L}_{\beta g}[\,(f\circ\sigma)\,\varphi_\beta\,]d\nu_\beta.$$

by using the fact that for any $x\in\Omega$ we have $\mathcal{L}_{\beta g}[\,(f\circ\sigma)\,\varphi_\beta\,](x) \;=\; f(x)\mathcal{L}_{\beta g}[\,\varphi_\beta\,](x)$, follows that the r.h.s. above is equal to

$$\int_\Omega \frac{1}{\lambda(\beta)}\, f\,\mathcal{L}_{\beta g}(\varphi_\beta\,)\, d\nu_\beta = \int_\Omega \frac{1}{\lambda(\beta)}\, f\,\lambda(\beta)\,\varphi_\beta\, d\nu_\beta = \int_\Omega f\, d\mu_\beta. \qquad \square$$

For $\gamma,\delta>2$, the probability μ_1 is also invariant because is the weak limit of invariant probabilities. Remember that when $\gamma,\delta<2$ such invariant probility measures do not exist (the natural candidates would be invariant measure which maximize pressure and are different from the Delta Diracs).

For $\beta=1$ we have a small problem: the eigenfunction have the following asymptotic behavior $\varphi_1(0^n1..)\sim n/(\gamma-1)$ and $\varphi_1(1^n0..)\sim n/(\delta-1)$ (compare with expressions (4) and (8) pages 1077 and 1078 in [19]). To get a probability measure using the above procedure, we have to assume that $\gamma,\delta>2$. In the cases $1<\gamma<2$ or $1<\delta<2$ the above method breaks down and we do not get a probability measure at $\beta=1$, just a σ-finite measure. By assuming $\gamma,\delta>2$ we have that μ_1 is an equilibrium probability (which gives positive mass to open sets at $\beta=1$) and the entropy of μ_1 is positive. The lower semicontinuity of the entropy (and the fact that $p(\beta)\to 0$) implies that any weak limit ν of μ_β, when $\beta\to 1$, is an equilibrium probability for g. Therefore, for $\gamma,\delta>2$ we have that μ_1 is an invariant and equilibrium probability for g.

An interesting question is: in the case $2>\gamma>\delta>1$ what happens with the equilibrium probability μ_β, when $\beta\to 1^-$. We claim that μ_β selects δ_{1^∞}. Indeed, any probability ν which is not δ_{0^∞} or δ_{1^∞} is such that $\int_\Omega g\, d\nu<0$. Now, we consider for large and fixed n and $\beta\sim 1$ the quotient

$$\frac{\mu_\beta(\overline{1^n0})}{\mu_\beta(\overline{0^n1})}\sim\frac{n^{-\delta}\,\frac{n}{\delta-1}}{n^{-\gamma}\,\frac{n}{\gamma-1}}\to\infty.$$

The above asymptotic expansion means that measure gives much more mass to the sets closest to the point 1^∞ than those close to the point 0^∞. This shows our claim.

The measure μ_1 is a probability measure only if $\delta,\gamma>2$. In this case by using continuity arguments we can prove that μ_β converges to μ_1, when $\beta\to 1$. Therefore, μ_1 is selected. It is interesting to remark that the mass of $\mu_1(\overline{1})$ is greater than the mass $\mu_1(\overline{0})$ using similar arguments we mentioned above. This describes the influence of the flatness in the phase transition point.

The Theorem 7 looks more natural taken in account the above analysis: for $1 < \delta < \gamma < 2$ the left derivative of pressure at $\beta = 1$ is zero (the only selected equilibrium state is δ_{1^∞}). On the other hand, for $\gamma = \delta > 2$ we have that the left derivative of pressure at $\beta = 1$ is non-zero (see Theorem 1 page 141 in [36] for Hofbauer model where the lack of differentiability is obtained when we vary the parameter γ). In this case we also have that the correlations with respect to μ_1 of some cylinder functions decays polynomially fast (see for instance [36] and Theorem 2.8 in [19] for the Hofbauer model). The decay ratio is also explicitly determined as a function of γ.

Proposition 12. *If $\gamma, \delta > 2$, then the above defined probability measure μ_β, $0 < \beta < 1$ is a fixed point for $\mathcal{G}_\beta^*$, where for any ψ*

$$\mathcal{G}_\beta(\psi)(x) = \sum_{\sigma(y)=x} J_{\beta,g}(y)\,\psi(y),$$

and $\log J_{\beta,g} = \beta\, g + \log \varphi_\beta - (\varphi_\beta \circ \sigma) - \log \lambda(\beta)$. *This meaning that* $\mathcal{G}_\beta^*(\mu_\beta) = \mu_\beta$. *Moreover,* $\log(\lambda(\beta)) = P(\beta g) \equiv p(\beta)$ *is monotonous decreasing.*

Proof. The proof follows from the fact that if ν_β is such that $\mathcal{L}_{\beta\, g}^*(\nu_\beta) = \lambda(\beta)\nu_\beta$, then the σ-invariant probability $\mu_\beta = \varphi_\beta\, \nu_\beta$ is fixed for $\mathcal{L}^*_{\beta\, g + \log \varphi_\beta - \log(\varphi_\beta \circ \sigma)\, - \log \lambda(\beta)}$. Indeed, given $f : \Omega \to \mathbb{R}$ we have

$$\begin{aligned} \int_\Omega \mathcal{G}_\beta(f) d\mu_\beta &= \int_\Omega \mathcal{L}_{\beta\, g + \log \varphi_\beta - (\varphi_\beta \circ \sigma) - \log \lambda(\beta)}\,(f)\,\varphi_\beta\, d\nu_\beta \\ &= \int_\Omega \frac{1}{\lambda(\beta)}\, \mathcal{L}_{\beta,g}\,(f\,\varphi_\beta)\, \frac{\varphi_\beta}{\varphi_\beta} d\nu_\beta = \int_\Omega f \varphi_\beta\, d\nu_\beta = \int_\Omega f\, d\mu_\beta. \end{aligned}$$

Note that $\log J_{\beta,g} = \beta\, g\, +\, \log \varphi_\beta\, -\, (\varphi_\beta \circ \sigma) - \log \lambda(\beta)$ is normalized, that is, $\mathcal{L}_{\beta\, g + \log \varphi_\beta - (\varphi_\beta \circ \sigma) - \log \lambda(\beta)}(1) = 1$, so we just shown that

$$\mathcal{L}^*_{\beta\, g + \log \varphi_\beta - (\varphi_\beta \circ \sigma) - \log \lambda(\beta)}(\mu_\beta) = \mu_\beta.$$

The Theorem 3.4 in [41] can be used under our hypothesis. In particular, we concluded that the Pressure of βg is equal to $\log \lambda(\beta)$ (that is, log of the main eigenvalue). Since the supremum in the pressure definition is taken over all the shift invariant probability measures it follows that

$$\begin{aligned} 0 &\geq P(\beta\, g\, +\, \log \varphi_\beta\, -\, (\varphi_\beta \circ \sigma) - \log \lambda(\beta)) \\ &= P(\beta\, g\, - \log \lambda(\beta)) \\ &= \sup_{\mu \in \mathcal{M}_1(\sigma)} \left\{ h(\mu) + \int_\Omega [\beta g - \log(\lambda(\beta))]\, d\mu \right\} \\ &= P(\beta\, g) - \log(\lambda(\beta)). \end{aligned}$$

On the other hand, when $\mu = \delta_{0^\infty} \in \mathcal{M}_1(\sigma)$ we have that $h(\mu) + \beta \int_\Omega g d\mu = 0$ and therefore $P(\beta g) = \log(\lambda(\beta))$. This argument shown that there exists at least two equilibrium states at the critical point $\beta_c = 1$.

The last statement follows from the nonpositivity of the potential g which implies that the derivative $p'(\beta)$ is nonpositive for $\beta < 1$. $\square$

As a consequence of this proposition we have the following corollary.

Corollary 13. *The pressure of the potential g vanish, that is, $P(g) = 0$. Moreover, there are at least three equilibrium probabilities at the phase transition point $\beta = 1$ for $\delta, \gamma > 2$.*

Proof. Note that the potential $g + \log \varphi_1 - \log(\varphi_1 \circ \sigma)$ is normalized, that is $\mathcal{L}_{g+\log\varphi_1 - \log(\varphi_1\circ\sigma)}(1) = 1$. Moreover, $\mathcal{L}^*_{\beta g}(\nu_1) = \nu_1$. If we consider $\mu_1 = \varphi_1 \nu_1$, then using the same reasoning of last proposition we get that

$$\mathcal{L}^*_{g+\log\varphi_1 - \log(\varphi_1\circ\sigma)}(\mu_1) = \mu_1.$$

Although $\varphi_1(0^\infty)$ and $\varphi_1(0^\infty)$ are not defined the above argument can be applied because ν_1 and μ_1 has no atoms. By invoking the Theorem 3.4 of [41] again it follows that $P(g) \leq 0$. Indeed, for $\mu = \delta_{0^\infty}$, we have that $h(\mu) + \int_\Omega g d\mu = 0$ and therefore $P(g) = 0$. We also have that in any case δ_{0^∞} and δ_{1^∞} are equilibrium states. $\square$

Remark 14. *Among the equilibrium probabilities obtained above one of them assign positive values to cylinders sets which is the one we got from the Ruelle Operator. Therefore, there are at least three ergodic equilibrium states; of course, convex combinations of them are also equilibrium states. In this case, as we mention before (by continuity arguments) we have that μ_β converges to μ_1, when $\beta \to 1$. In this case there is selection of the limit probability in the phase transition point.*

7 Phase Transitions III. Non-Uniquess of the DLR Measure

In this section we move towards a more probabilistic approach to obtain the Gibbs measures. The exposition is based on the Section 2.1 of [48]. We refer the reader to [12] for definitions and results on DLR probabilities and its relation with Thermodynamic Limit probabilities.

Conditional Expectation: basic facts and notation

Let $\mathcal{B}$ denote the Borel sigma-algebra on $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{X}_n = \sigma^{-n}(\mathcal{B})$, that is, the σ-algebra generated by the random variables $X_n, X_{n+1}, \ldots$ on the Bernoulli space, where $X_n(x) = x_n$ for all $x \in \Omega$. Fixed a probability measure m defined over Ω and given a cylinder set $\overline{a_0 a_1 \ldots a_{n-1}}$, where $a_j \in \{0,1\}$, we define

$$\alpha_{\overline{a_0 a_1 \ldots a_{n-1}}}(x) = \mathbb{E}_m[I_{\overline{a_0 a_1 \ldots a_{n-1}}} \,|\, \mathcal{X}_n],$$

where $\mathbb{E}_m[f|\mathcal{X}_n]$ is the conditional expectation of f with respect to m given the σ-algebra $\mathcal{X}_n$. From a elementary property of the conditional expectation for any fixed $b_n, b_{n+1}, .., b_r$ we have that

$$\int_{X_n=b_n,\ldots,X_r=b_r} I_{\overline{a_0 a_1 \ldots a_{n-1}}}(x)\, dm(x) = \int_{X_n=b_n,\ldots,X_r=b_r} \alpha_{\overline{a_0 a_1 \ldots a_{n-1}}}(x) d\, m(x).$$

In other words

$$m\left(\overline{a_0 a_1 \ldots a_{n-1} b_n \ldots b_r}\right) = \int_{X_n=b_n,\ldots,X_r=b_r} \alpha_{\overline{a_0 a_1 \ldots a_{n-1}}}(x) d\, m(x).$$

The measurable functions with respect to $\mathcal{X}_n$ are the functions of the form $\varphi(\sigma^n(x))$ where φ is Borel measurable. So we can characterize $\alpha_{\overline{a_0 a_1 \ldots a_{n-1}}}$ by the following property: for any $\mathcal{B}$-measurable (or continuous) $\varphi : \Omega \to \mathbb{R}$

$$\int_\Omega \varphi(\sigma^n(x)) I_{\overline{a_0 a_1 \ldots a_{n-1}}}(x)\, dm(x) = \int_\Omega \varphi(\sigma^n(x)) \alpha_{\overline{a_0 a_1 \ldots a_{n-1}}}(x) d\, m(x).$$

Definition 15. *Given a potential ϕ we say that a probability measure m is a DLR probability for ϕ if for all $n \in \mathbb{N}$ and any cylinder set $\overline{x_0 x_1 \ldots x_{n-1}}$, we have m-almost every $z = (z_0, z_1, z_2, \ldots)$ that*

$$\mathbb{E}_m(I_{\overline{x_0 x_1 \ldots x_{n-1}}} \,|\, \mathcal{X}_n)(z) = \frac{e^{\phi(z)+\phi(\sigma(z))+\ldots+\phi(\sigma^{n-1}(z))}}{\sum_{\sigma^n(z)=\sigma^n(y)} e^{\phi(y)+\phi(\sigma(y))+\ldots+\phi(\sigma^{n-1}(y))}}.$$

The set of all DLR probabilities for ϕ is denoted by $\mathcal{G}^{DLR}(\phi)$. In general this set is not unique, but for a very large class of potentials $\mathcal{G}^{DLR}(\beta\phi)$ is unique for β large enough, by the Dobrushin Uniqueness Theorem. So a possible sense of phase transition is the existence of a inverse temperature β so that $\mathcal{G}^{DLR}(\beta\phi)$ posses more than one element.

In the sequel we shown among other things that the equilibrium probability μ_1 is a DLR probability for the potential g. We refer the reader to [12] for more results about DLR probabilities.

Definition 16. *A continuous positive function $J : \Omega \to \mathbb{R}$ such that for any $x \in \Omega$ we have $\sum_{\sigma(y)=x} J(y) = 1$ is called a Jacobian.*

Here we consider the case where $\phi = \log J$ where J is a Jacobian (the general case is analyzed in [12]). In this case the Ruelle operator $\mathcal{L}_{\log J}$ (for the potential $\log J$) is defined as usual for any continuous function ψ by

$$\mathcal{L}_{\log J}(\psi)(x) = \sum_{\sigma(y)=x} J(y)\,\psi(y).$$

By the definition of a Jacobian we have that $\mathcal{L}_{\log J}(1) = 1$. Remember that the dual operator $\mathcal{L}^*_{\log J}$ acts on the space of probability measures. Its action on μ give us a probability measure $\mathcal{L}^*_{\log J}(\mu) = \nu$ such that for any continuous function ψ we have

$$\int_\Omega \psi d\,\mathcal{L}^*_{\log J}(\mu) = \int_\Omega \psi\, d\nu = \int_\Omega \mathcal{L}_{\log J}(\psi) d\mu.$$

Definition 17. *A probability m is called a $g-$measure if it is a fixed point for $\mathcal{L}^*_{\log J}$.*

Lemma 18. *For any Jacobian J the operator $\mathcal{L}^*_{\log J}$ has a fixed point which is invariant probability measure for σ.*

Proof. Since $\mathcal{L}_{\log J}(1) = 1$ then $\mathcal{L}^*_{\log J}$ takes probability measures to probability measures. So existence of a fixed point for $\mathcal{L}^*_{\log J}$ is a straightforward application of the Tychonov-Schauder Theorem. For the shift invariance of the fixed probability, see below the Lemma 23. □

If the Jacobian J is in the Hölder class the probability measure m provided by the above lemma is unique, that is, the operator $\mathcal{L}^*_{\log J}$ has only one fixed point. If J is not in the Hölder class this is not always true.

In [3], [43] and [21] are presented examples where the Jacobian J is **continuous and strictly positive** and such that there are at least two fixed points for $\mathcal{L}^*_{\log J}$. In these cases $\mathcal{G}^*(\log J)$ does not have cardinality one and so we have phase transition in this sense. In [21] is presented a criteria for the existence of more than one g-measure. We will show (see next theorem) how to use this criteria to exhibit a class of examples, where we have phase transition in the sense of existence of more than one DLR probability.

In the examples presented here, where we have more than one Gibbs probability (see Sections 6 and 8) the Jacobian can be zero in some finite subset of Ω.

For the potentials g, with $\gamma, \delta > 2$, which were considered in the previous sections it is also true that the $m = \mu_1$ is the unique fixed point for the corresponding $\mathcal{L}^*_{\log J_g}$. Therefore, $\mathcal{G}^*(\log J_g)$ has cardinality one.

Proposition 19. *Suppose that J is positive and continuous. If $\mathcal{L}^*_{\log J}(m) = m$ it follows that m is DLR probability for the potential $\phi = \log J$. In other words, $\mathcal{G}^*(\log J) \subset \mathcal{G}^{DLR}(\log J)$.*

Remark 20. *By using the criteria obtained in [21] and the above proposition it is possible to shown the existence of more than one DLR probability for the Double Hofbauer model.*

In what follows we give the proof of the Proposition 19. The proof will be divided into three lemmas which the statements and proofs are given below.

Lemma 21. *If $\mathcal{L}^*_{\log J}(m) = m$, then for any continuous f and g we have*

$$\int_\Omega \mathcal{L}_{\log J}(f)\, g\, dm = \int_\Omega f\,(g \circ \sigma)\, dm.$$

Proof. Since $\mathcal{L}_{\log J}(f\,(\,g \circ \sigma)) = g\,\mathcal{L}_{\log J}(f)$, it follows from the definition of the dual operator that

$$\begin{aligned}\int_\Omega f\,(g \circ \sigma)\, dm = \int_\Omega f\,(g \circ \sigma)\, d\mathcal{L}^*_{\log J} m &= \int_\Omega \mathcal{L}_{\log J}\,(f\,(g \circ \sigma)\,)\, dm \\ &= \int_\Omega \mathcal{L}_{\log J}(f)\, g\, dm.\end{aligned}$$

□

Remark 22. *In the Hilbert space $L^2(\Omega, \mathcal{B}, m)$ the dual of $\mathcal{L}_{\log J}$ is the Koopman operator $g \to \mathcal{K}(g) = g \circ \sigma$.*

Lemma 23. *If $\mathcal{L}^*_{\log J}(m) = m$ then m is invariant for σ.*

Proof. Indeed, given any continuous function g we have

$$\begin{aligned}\int_\Omega g \circ \sigma\, dm = \int_\Omega (g \circ \sigma)\, d\mathcal{L}^*_{\log J}(m) &= \int_\Omega \mathcal{L}_{\log J}(g \circ \sigma)\, dm \\ &= \int_\Omega g\, \mathcal{L}_{\log J}(1)\, dm = \int_\Omega g\, dm.\end{aligned}$$ □

Lemma 24. *If $\mathcal{L}^*_{\log J}(m) = m$ then for any continuous function $f : \Omega \to \mathbb{R}$ we have that*

$$\mathbb{E}_m(f \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(f)(\sigma^n(x)).$$

Proof. Let $g : \Omega \to \mathbb{R}$ be an arbitrary continuous function. Since for any $n \in \mathbb{N}$ we have $\mathcal{L}^n_{\log J}(f\,(\,g \circ \sigma^n)) = g\,\mathcal{L}^n_{\log J}(f)$ it follows that

$$\begin{aligned}\int_\Omega (g \circ \sigma^n(x))\, f(x) dm(x) &= \int_\Omega \mathcal{L}^n_{\log J}\,[\,(g \circ \sigma^n(x))\, f(x)\,] dm(x)\\ &= \int_\Omega g(x)\,\mathcal{L}^n_{\log J}\,(f)(x)\, dm(x)\\ &= \int_\Omega g(\sigma^n(x))\,\mathcal{L}^n_{\log J}\,(f)(\sigma^n(x))\, dm(x),\end{aligned}$$

where in the last equality we use the fact that m is invariant for σ. From the previous lemma we get that

$$\mathbb{E}_m(I_{\overline{a_0 a_1 \ldots a_{n-1}}} \,|\, \mathcal{X}_n)(x) = \mathcal{L}^n_{\log J}(I_{\overline{a_0 a_1 \ldots a_{n-1}}})(\sigma^n(x))$$

$$= \sum_{\sigma^n(y)=\sigma^n(x)} \exp\big(\log J(y) + \log J(\sigma(y)) + \ldots + \log J(\sigma^{n-1}(y))\big)\; I_{\overline{a_0 a_1 \ldots a_{n-1}}}(y)$$

$$= \exp\big(\log J(a_0, a_1, .., a_{n-1} x) + \ldots + \log J(\sigma^{n-1}(a_0, a_1, .., a_{n-1} x))\big)$$

$$= \frac{\exp\left(\log J(a_0, a_1, .., a_{n-1} x) + \ldots + \log J(\sigma^{n-1}(a_0, a_1, .., a_{n-1} x))\right)}{\sum_{\sigma^n(y)=\sigma^n(x)} \exp\left(\log J(y) + \ldots + \log J(\sigma^{n-1}(y))\right)},$$

where in the denominator on last equality we use that $\mathcal{L}_{\log J}(1) = 1$. □

Piecing together the three previous lemmas we have proved the Proposition 19. When $\phi = \log J$ is such that $\mathcal{L}^*_{\log J}$ has two invariant probabilities then there exist two DLR probabilities for $\phi = \log J$, meaning that we have phase transition in the DLR sense.

From the Proposition 12 we get that μ_β is fixed by $\mathcal{L}^*_{\log J_{\beta\, g}}$. So we can concluded that μ_β defines a DLR probability (for more details see [12]). We observe that for the Double Hofbauer potential the hypothesis required to prove that $\mathcal{G}^*(g) \subset \mathcal{G}^{DLR}(g)$ in [12] are also satisfied.

8 Phase Transition IV. Non Uniqueness of T.L. - Renewal equation

In what follows we introduce the so called finite volume Gibbs measures with a boundary condition $y \in \Omega$ (see [48]). For a given $n \in \mathbb{N}$ consider the probability measure in $(\Omega, \mathscr{F})$ so that for any $F \in \mathscr{F}$, we have

$$\mu_n^y(F) = \frac{1}{Z_n^y} \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} 1_F(x)\exp(-(\, f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x)\,)\,)$$

where Z_n^y is a normalizing factor called partition function given by

$$Z_n^y = \sum_{\substack{x\in\Omega;\\ \sigma^n(x)=\sigma^n(y)}} \exp(-(\, f(x) + f(\sigma(x)) + ... + f(\sigma^{n-1}(x)\,)).$$

A straightforward computations shows that $\mu_n^y(\cdot)$ is a probability measure. This probability measure can be written in the Ruelle Operator formalism in the following way:

$$\mu_n^y(F) = \frac{\mathcal{L}_f^n\,(1_F)\,(\sigma^n(y))}{\mathcal{L}_f^n\,(1)\,(\sigma^n(y))} \quad \text{or} \quad \mu_n^y = \frac{1}{\mathcal{L}_f^n(1)(\sigma^n(y))}[\,(\mathcal{L}_f)^*\,]^n\,(\delta_{\sigma^n(y)}). \tag{16}$$

Definition 25. *For a fixed $y\in\Omega$ any weak limit of the subsequences $\mu_{n_k}^y$, when $k\to\infty$ is called Thermodynamic Limit with boundary conditions y. Now we consider the collection of all the Thermodynamic Limits varying $y\in\Omega$ and take the closed convex hull of this collection. This set is denoted by $\mathcal{G}^{TL}(f)$.*

For a while we assume that the main eigenvalue $\lambda = 1$ and $f = \log J$ where J is a Jacobian. In this case, $Z_n^y = 1$ for all $n\in\mathbb{N}$ and $y\in\Omega$. If f is Hölder it is known (see [41]) that for any fixed y we have $\lim_{n\to\infty}\mathcal{L}_f^n(I_{[a]})(y) = m([a])$, where m is the fixed point for the operator $\mathcal{L}_f^*$ (which is the equilibrium state for $\log J$) and $[a]$ is any cylinder set. In this way the Thermodynamic Limit probability is unique (independent of the boundary condition). In other words, if $f = \log J$ is Hölder, then for any $y\in\Omega$ we have that

$$\lim_{n\to\infty}\mu_{n,1}^y = \mu,$$

where μ is the equilibrium state for $\log J$. In this case $\mathcal{G}^{TL}(f)$ has cardinality one and there is no phase transition in the sense of the number of probability measures in $\mathcal{G}^{TL}(f)$. For a discussion about non-normalized potentials f we refer the reader to [12].

We want to analyze what happens in the case of $f = \log J_g$ (of previous section) which is not in the Hölder class. The main question is: for $\beta = 1$, the limits $\lim_{n\to\infty}\mu_n^{z_1}$ and $\lim_{n\to\infty}\mu_n^{z_2}$ can be different ? The purpose of this section is to answer this question. Before going into computational details we recall that in this case we already know (from the previous section) that the phase transition occurs in the DLR sense.

In this section we follow the results and ideas from [19]. From now we denote $\log J$ the normalized Jacobian associated to g, where g is the Double Hofbauer potential.

Definition 26. *We denote $\mathcal{G}^{TL}_{Per}$ the set of Thermodynamic Limits probabilities obtained from all periodic points y.*

We will show that the set $\mathcal{G}^{TL}_{Per}(\log J)$ is the convex hull of the three probability measures δ_{0^∞}, δ_{1^∞} and μ_1 defined on the Section 6. Of course, this implies the existence of phase transition in the Thermodynamic Limit sense.

An interesting remark is in [12] it is shown that for the potential $\log J$ as we are considering here the set $\mathcal{G}^{TL}(\log J) = \mathcal{G}^{DLR}(\log J)$. Note that $\mathcal{G}^{*}(\log J) = \{\mu_1\}$ is strictly contained in $\mathcal{G}^{TL}_{Per}(\log J)$. The set of equilibrium states for the pressure is also equals to $\mathcal{G}^{TL}_{Per}(\log J)$.

Our proof is based on some properties associated to a kind of Renewal Equation. Given a sequence $a : \mathbb{N} \to \mathbb{R}$ and a probability measure p defined on $\mathbb{N}$ we can ask whether exists or not another sequence $A : \mathbb{N} \to \mathbb{R}$ satisfying the following associated Renewal Equation: for all $q \in \mathbb{N}$

$$A(q) = [A(0)p_q + A(1)p_{q-1} + A(2)\, p_{q-2} + ... + A(q-2)p_2 + A(q-1)p_1] + a(q). \quad (17)$$

If $M = \sum_{q=1}^{\infty} q\, p_q$ then the Renewal Theorem (Cap VII Theorem 6.1 in [30] and Cap V Theorem 5.1 in [31]) claims that

$$\lim_{q\to\infty} A(q) = \frac{\sum_{q=1}^{\infty} a(q)}{M}.$$

One important feature of the Renewal Theorem is that the limit value of $A(q)$, as $q \to \infty$, is provided without knowing the explicit values of the $A(q)$.

We want to investigate the Thermodynamic Limit

$$\lim_{q\to\infty} \mu_q^y([a]) = \lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[a]})(\sigma^q(y)),$$

for different points y in the Bernoulli space Ω for an arbitrary cylinder set $[a]$.

We are interested in to find a fixed cylinder $[a]$ for which the above limit does depends on $y \in \Omega$. If such cylinder do exists one can say that the Double Hofbauer model has phase transition in the TL sense. To accomplish this we will consider the cylinder $[0]$ and periodic points y in the Bernoulli space. We will show that:

Proposition 27. *For the Double Hofbauer model*

$$\lim_{q\to\infty} \mu_q^{0^\infty}([0]) = 1 \quad \textit{and} \quad \lim_{q\to\infty} \mu_q^{1^\infty}([0]) = 0.$$

This shows that there exist more than one probability on the set of Thermodynamic Limits. This means a phase transition in this sense.

The proof of the above proposition will be presented later. Of course, the aim of this proposition is to prove the existence of phase transition for the Double

Hofbauer model in the Thermodynamic Limit sense. But much more can be said in this case and we want to have the complete picture of this problem concerning on what happens to this limit when we consider other points $y \in \Omega$. In this direction we obtained the following result.

Proposition 28. *For any periodic points y and $z \in \Omega$ (being not the fixed points) we have*

$$\lim_{q\to\infty} \mu_q^y([0]) = \lim_{q\to\infty} \mu_q^z([0]).$$

As they were stated the above propositions characterize all the possible Thermodynamic Limit values for the cylinder [0] for any periodic boundary conditions. It is possible to show more: for any cylinder set $[a]$ the analogous result is true, but its proof requires a more elaborate calculation and we will not present it here.

The proof of the Proposition 28 will follow easily from:

Proposition 29. *The limit*

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[a]})(y))$$

is the same for any point of the form $y = \underbrace{000...0}_{n}1...$ *or* $y = \underbrace{111...1}_{n}0....$

To facilitate the understanding why the Proposition 28 follows from Proposition 29 we compute the limit in a simple case. Let y be a periodic of the form $y = (011)^\infty = 011\,011\,011....$ Assuming the Proposition 29 we have that the following limits are equal

$$\lim_{n\to\infty} \mathcal{L}^{3\,n}_{\log J}(I_{[a]})(011)^\infty = \lim_{n\to\infty} \mathcal{L}^{3\,n+1}_{\log J}(I_{[a]})(110)^\infty = \lim_{n\to\infty} \mathcal{L}^{3\,n+2}_{\log J}(I_{[a]})(101)^\infty.$$

For this case the statement of the Proposition 28 holds true because of

$$\sigma^{3\,n}(011)^\infty = (011)^\infty,\ \ \sigma^{3\,n+1}(011)^\infty = (110)^\infty \text{ and } \sigma^{3\,n+2}(011)^\infty = (101)^\infty.$$

In order to prove the Proposition 29 we will take advantage of some properties of Renewal Theory. The proof is based on the geometric structure of the tree graph generated by the pre-images of the point y that one has to consider to compute the value of the Ruelle operator in the point y. This idea was used in [19] but we should remark that the situation here is more complex. The Figure 2 helps to understand how the procedure works.

Proof of the Proposition 27. The preimages of 0^∞ are 0^∞ and $1\,0^\infty$. By using that $J(10^\infty) = 0$ and $J(0^\infty) = 1$ we get that

$$\mu_1^{0^\infty}([0]) = \frac{\mathcal{L}_{\log J}\,(1_{[0]})\,(\sigma(0^\infty))}{\mathcal{L}_{\log J}\,(1)\,(\sigma(0^\infty))} = \mathcal{L}_{\log J}\,(1_{[0]})\,(\sigma(0^\infty)) = \mathcal{L}_{\log J}\,(1_{[0]})\,(0^\infty)\ = 1.$$

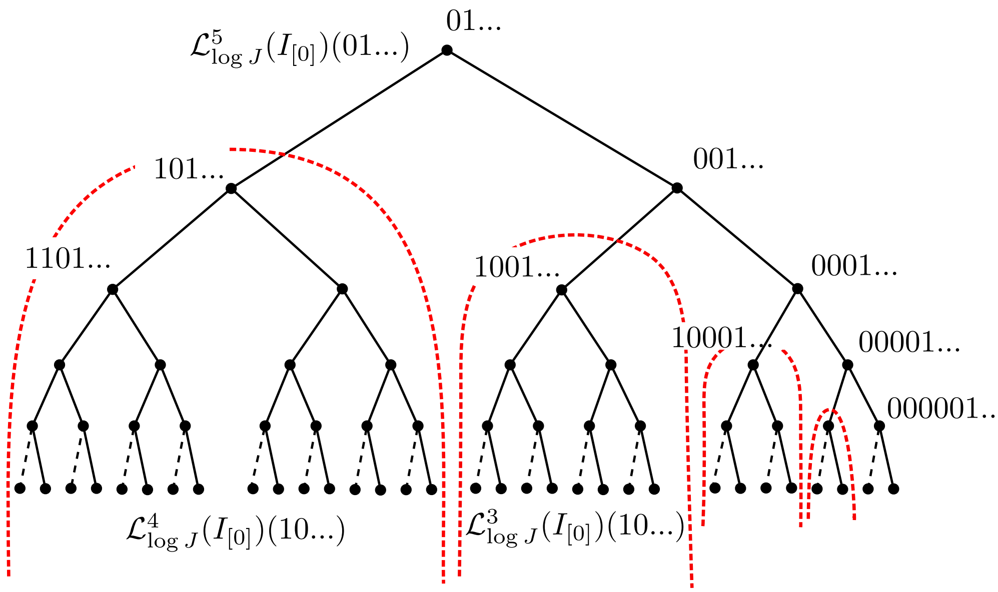

Figure 2: The tree graph representation.

We proceed by induction. By assuming that $\mathcal{L}^n_{\log J}\left(1_{[0]}\right)\left(\sigma(0^\infty)\right) = 1$ it is easy to see that

$$\mathcal{L}^{n+1}_{\log J}\left(1_{[0]}\right)\left(\sigma(0^\infty)\right) = J(0^\infty)\mathcal{L}^n_{\log J}\left(1_{[0]}\right)\left(0^\infty\right) + J(10^\infty)\mathcal{L}^n_{\log J}\left(1_{[0]}\right)\left(10^\infty\right) = 1.$$

Therefore, $\mu_n^{0^\infty}([0]) \to 1$, when $n \to \infty$. In the same manner we can see that $\mu_n^{1^\infty}([0]) \to 0$, when $n \to \infty$. $\square$

Proof of the Proposition 29. The Jacobian for the Double Hofbauer potential at the inverse temperature $\beta = 1$ is such that $\log J = g + \log \varphi_1 - \log(\varphi_1 \circ \sigma)$. For simplicity we introduce some notations and split the computation $\log J(x)$ in six cases:

a) for $q \geq 2$ and $x \in L_q$ we have

$$\begin{aligned}\log J(x) = -\gamma \log \frac{q}{q-1} &+ \log\left(1 + q^\gamma \sum_{n=2}^{\infty}(n+q-1)^{-\gamma}\right) \\ &- \log\left(1 + (q-1)^\gamma \sum_{n=2}^{\infty}(n+q-2)^{-\gamma}\right) \\ := -\gamma \log \frac{q}{q-1} &+ \log r(q) - \log r(q-1).\end{aligned}$$

b) for $q \geq 2$ and $x \in R_q$, we have

$$\begin{aligned}\log J(x) = -\gamma \log \frac{q}{q-1} + \log\left(1 + q^{\delta} \sum_{n=2}^{\infty} (n+q-1)^{-\delta}\right) \\ - \log\left(1 + (q-1)^{\delta} \sum_{n=2}^{\infty} (n+q-2)^{-\delta}\right) \\ := -\delta \log \frac{q}{q-1} + \log s(q) - \log s(q-1).\end{aligned}$$

c) for $x = 0\overbrace{111...1}^{q}0... \in L_1$ we have

$$\log J(x) = -\log\left(1 + q^{\delta} \sum_{n=2}^{\infty} (n+q-1)^{-\delta}\right) = -\log(s(q)).$$

d) for $x = 1\overbrace{000...0}^{q}1... \in R_1$ we have

$$\log J(x) = -\log\left(1 + q^{\gamma} \sum_{n=2}^{\infty} (n+q-1)^{-\gamma}\right) = -\log(r(q)).$$

e) for $x = 0^{\infty}$ or $x = 1^{\infty}$ we have $J(x) = 1$.

f) for $x = 10^{\infty}$ or $x = 01^{\infty}$ we have $J(x) = 0$.

We point out that $\log J(010..) = -\log \zeta(\gamma) = -\log(r(1))$ and $\log J(101..) = -\log \zeta(\delta) = -\log(s(1))$. Simple computation shows that $\log J(10^n 1..) \sim n^{-1}$ and $\log J(01^n 0..) \sim n^{-1}$.

We proceed with the computation of $\mathcal{L}^n_{\log J}(I_{[a]})(y)$ for $y = 01\ldots$ and $[a] = [0]$. Let us stress again that the idea is to follow [19] but here we use the scheme of smaller trees on the right side (see Figure 2). Note that for all $n \in \mathbb{N}$ the value $\mathcal{L}^n_{\log J}(I_{[0]})(y)$ is constant and independent of $y \in [01]$ (analogously for $y \in [10]$). One element in the sum determined by $\mathcal{L}^q_{\log J}(I_{[0]})(01...)$ is

$$\exp\left(\log J(\underbrace{000...0}_{q+1}1) + \log J(\underbrace{000...0}_{q}1) + \log J(\underbrace{000...0}_{q-1}1) + ... + \log J(001...)\right)$$

which simplifies to

$$\frac{r(q+1)}{r(q)}\left(\frac{q+1}{q}\right)^{-\gamma}\frac{r(q)}{r(q-1)}\left(\frac{q}{q-1}\right)^{-\gamma}...\left(\frac{2}{1}\right)^{-\gamma}\frac{r(2)}{\zeta(\gamma)} = \frac{(q+1)^{-\gamma}\, r(q+1)}{\zeta(\gamma)}.$$

For $q = 1$ this is simply given by $\mathcal{L}^1_{\log J}(I_{[0]})(01..) = 2^{-\gamma} r(2)/\zeta(\gamma)$. The general term of $\mathcal{L}^q_{\log J}(I_{[0]})(01...)$ for $1 \leq j \leq q-1$ is given by the following expression

$$\Bigg[\exp\Bigg(\log J(1\underbrace{000...0}_{q-j}1) + \log J(\underbrace{000...0}_{q-j}1) + \log J(\underbrace{000...0}_{q-j-1}1) + \ldots + \log J(0\,0\,1...)\Bigg)\Bigg]\,\mathcal{L}^j_{\log J}(I_{[0]})(10...)$$

which is equals to

$$\frac{(q-j)^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}^j_{\log J}(I_{[0]})(10...).$$

Therefore for any $q \geq 2$ we have that

$$\mathcal{L}^q_{\log J}(I_{[0]})(01\ldots) = \frac{(q-1)^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}^1_{\log J}(I_{[0]})(10\ldots) + \ldots + \frac{2^{-\gamma}}{\zeta(\gamma)}\,\mathcal{L}^{q-2}_{\log J}(I_{[0]})(10\ldots) + \frac{1}{\zeta(\gamma)}\,\mathcal{L}^{q-1}_{\log J}(I_{[0]})(10\ldots) + \frac{(q+1)^{-\gamma} r(q+1)}{\zeta(\gamma)}. \quad (18)$$

The above expression is not exactly a renewal type of equation because of the powers of operators involving on it for points of the form (01..) are described by the powers of the operator evaluated on points of the form (10...). We need some more work in order to get an true renewal equation. This is the main motivation for our next step which is the computation of $\mathcal{L}^n_{\log J}(I_{[a]})(y)$ for $y = 10..$ and $[a] = [0]$ using the scheme of smaller trees on the left side.

Proceeding as above and splitting $\mathcal{L}^q_{\log J}(I_{[0]})(10..)$ we find the following term

$$\exp\Bigg(\log J(0\underbrace{111...1}_{q}0) + \log J(\underbrace{111...1}_{q}0) + \log J(\underbrace{111...1}_{q-1}0) + ... + \log J(1\,1\,0...)\Bigg)$$

which can be reduced to the expression below by straightforward computations

$$\frac{1}{s(q)}\left(\frac{q}{q-1}\right)^{-\delta}\frac{s(q)}{s(q-1)}\left(\frac{q-1}{q-2}\right)^{-\delta}\frac{s(q-1)}{s(q-2)}\cdots\left(\frac{2}{1}\right)^{-\delta}\frac{s(2)}{s(1)} = \frac{1}{\zeta(\delta)}\,q^{-\delta}.$$

Similarly to the previous step we can see that for all $n \in \mathbb{N}$ we have that the value of $\mathcal{L}^n_{\log J}(I_{[0]})(x)$ is constant and independent of $x \in [10]$. Following the geometric picture of the Renewal Equation it is easy to see that

$$\mathcal{L}^{q}_{\log J}(I_{[0]})(10...) = \frac{1}{\zeta(\delta)}\, q^{-\delta} + \frac{1}{\zeta(\delta)}(q-1)^{-\delta}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(01...) + \ldots$$
$$+ \frac{1}{\zeta(\delta)} 3^{-\delta}\, \mathcal{L}^{q-3}_{\log J}(I_{[0]})(01..) + \frac{2^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^{q-2}_{\log J}(I_{[0]})(01..) + \frac{1}{\zeta(\delta)}\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(010..). \tag{19}$$

Similarly to (18) the above expression is not exactly a renewal type equation and to obtain a genuine renewal equation the idea is to replace (18) in (19). By doing this we obtain the identity below which we write down with several terms aiming to help the reader to identify the pattern emerging from this replacement

$$\begin{aligned}
\mathcal{L}^{q}_{\log J}(I_{[0]})(10..) = \;& \tfrac{1}{\zeta(\delta)}\, q^{-\delta} + \tfrac{1}{\zeta(\delta)}(q-1)^{-\delta}\, \tfrac{2^{-\gamma}\, r(2)}{\zeta(\gamma)} + \\
& \tfrac{1}{\zeta(\delta)}(q-2)^{-\delta}\Big[\, \tfrac{1}{\zeta(\gamma)}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) \;+\; \tfrac{3^{-\gamma}\, r(3)}{\zeta(\gamma)} \Big] + \\
& \tfrac{1}{\zeta(\delta)}(q-3)^{-\delta}\Big[\, \tfrac{2^{-\gamma}}{\zeta(\gamma)}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) \;+ \tfrac{1}{\zeta(\gamma)}\, \mathcal{L}^{2}_{\log J}(I_{[0]})(10..) + \\
& \qquad\qquad \tfrac{4^{-\gamma}\, r(4)}{\zeta(\gamma)} \Big] + \ldots \\
& \tfrac{1}{\zeta(\delta)} 3^{-\delta} \Big[\tfrac{(q-4)^{-\gamma}}{\zeta(\gamma)}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) + .. + \tfrac{1}{\zeta(\gamma)}\, \mathcal{L}^{q-4}_{\log J}(I_{[0]})(10..) + \\
& \qquad\qquad \tfrac{(q-2)^{-\gamma}\, r(q-2)}{\zeta(\gamma)} \Big] + \\
& \tfrac{1}{\zeta(\delta)} 2^{-\delta} \Big[\tfrac{(q-3)^{-\gamma}}{\zeta(\gamma)}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) + .. + \tfrac{1}{\zeta(\gamma)}\, \mathcal{L}^{q-3}_{\log J}(I_{[0]})(10..) + \\
& \qquad\qquad \tfrac{(q-1)^{-\gamma}\, r(q-1)}{\zeta(\gamma)} \Big] + \\
& \tfrac{1}{\zeta(\delta)} \Big[\tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)}\, \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) + .. + \tfrac{1}{\zeta(\gamma)} \mathcal{L}^{q-2}_{\log J}(I_{[0]})(10..) + \\
& \qquad\qquad \tfrac{q^{-\gamma}\, r(q)}{\zeta(\gamma)} \Big].
\end{aligned}$$

By rearranging the terms of the sum and make the trivial simplifications we can see that the above expression is equal to

$$\begin{aligned}
& \tfrac{1}{\zeta(\delta)}\, q^{-\delta} + \tfrac{1}{\zeta(\delta)}(q-1)^{-\delta}\, \tfrac{2^{-\gamma}\, r(2)}{\zeta(\gamma)} + \ldots + \tfrac{1}{\zeta(\delta)} 2^{-\delta}\, \tfrac{(q-1)^{-\gamma}\, r(q-1)}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)}\, \tfrac{q^{-\gamma}\, r(q)}{\zeta(\gamma)} + \\
& \mathcal{L}^{1}_{\log J}(I_{[0]})(10..) \Big[\tfrac{(q-2)^{-\delta}}{\zeta(\delta)} \tfrac{1}{\zeta(\gamma)} + \tfrac{(q-3)^{-\delta}}{\zeta(\delta)} \tfrac{2^{-\gamma}}{\zeta(\gamma)} + \ldots + \tfrac{2^{-\delta}}{\zeta(\delta)} \tfrac{(q-3)^{-\gamma}}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)} \tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)} \Big] + \\
& \mathcal{L}^{2}_{\log J}(I_{[0]})(10..) \Big[\tfrac{(q-3)^{-\delta}}{\zeta(\delta)} \tfrac{1}{\zeta(\gamma)} + \tfrac{(q-4)^{-\delta}}{\zeta(\delta)} \tfrac{2^{-\gamma}}{\zeta(\gamma)} + \ldots + \tfrac{2^{-\delta}}{\zeta(\delta)} \tfrac{(q-4)^{-\gamma}}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)} \tfrac{(q-3)^{-\gamma}}{\zeta(\gamma)} \Big] + \\
& \qquad \ldots + \mathcal{L}^{q-3}_{\log J}(I_{[0]})(10..) \Big[\tfrac{2^{-\delta}}{\zeta(\delta)} \tfrac{1}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)} \tfrac{2^{-\gamma}}{\zeta(\gamma)} \Big] + \mathcal{L}^{q-2}_{\log J}(I_{[0]})(10..) \tfrac{1}{\zeta(\delta)} \tfrac{1}{\zeta(\gamma)}.
\end{aligned}$$

Now we define $A(0)=0$, $p_1=0$ and for $q\geq 2$, $A(q)=\mathcal{L}^q_{\log J}(I_{[0]})(10..)$ and

$$p_q = \tfrac{(q-1)^{-\delta}}{\zeta(\delta)}\,\tfrac{1}{\zeta(\gamma)} + \tfrac{(q-2)^{-\delta}}{\zeta(\delta)}\,\tfrac{2^{-\gamma}}{\zeta(\gamma)} + ... + \tfrac{2^{-\delta}}{\zeta(\delta)}\,\tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)}\,\tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)} \ . \tag{20}$$

Let $a(q)$ be the sequence defined for $q\geq 1$ by

$$a(q) = \tfrac{q^{-\delta}}{\zeta(\delta)} + \tfrac{(q-1)^{-\delta}}{\zeta(\delta)}\tfrac{2^{-\gamma}\ r(2)}{\zeta(\gamma)} + ... + \tfrac{2^{-\delta}}{\zeta(\delta)}\,\tfrac{(q-1)^{-\gamma}\ r(q-1)}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)}\,\tfrac{q^{-\gamma}\ r(q)}{\zeta(\gamma)} \ , \tag{21}$$

From the definitions we have $a(1)=1/\zeta(\delta)=A(1)$ and

$$\sum_{j=2}^{\infty} p_j = \sum_{n=1}^{\infty}\frac{n^{-\gamma}}{\zeta(\gamma)}\sum_{n=1}^{\infty}\frac{n^{-\delta}}{\zeta(\delta)} = 1.$$

By bringing together all the above results we obtain the following genuine renewal equation

$$A(q) = [A(0)p_q + A(1)p_{q-1} + A(2)\,p_{q-2} + ... + A(q-2)p_2 + A(q-1)p_1] + a(q)$$

Recalling that $p_1=0$ we have

$$A(q) = [A(1)p_{q-1} + A(2)\,p_{q-2} + ... + A(q-2)p_2] + a(q). \tag{22}$$

Lemma 30. *Let p_q be the sequence above defined. Then when $q\to\infty$ we have*

$$p_q \sim q^{1-\delta-\gamma}, \tag{23}$$

Proof. To prove this asymptotic behavior it is enough to prove that the quotient $p_q/(q\ q^{-\gamma}\ q^{-\delta})$ has a limit, when $q\to\infty$. This quotient is explicitly given by

$$\frac{\tfrac{(q-1)^{-\delta}}{\zeta(\delta)}\,\tfrac{1}{\zeta(\gamma)} + \tfrac{(q-2)^{-\delta}}{\zeta(\delta)}\,\tfrac{2^{-\gamma}}{\zeta(\gamma)} + ... + \tfrac{2^{-\delta}}{\zeta(\delta)}\,\tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)} + \tfrac{1}{\zeta(\delta)}\,\tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)}}{q\ q^{-\gamma}\ q^{-\delta}}$$

which can be rewritten as

$$\tfrac{1}{\zeta(\gamma)\,\zeta(\delta)}\textstyle\sum_{j=1}^{q-1}\ \tfrac{1}{q}\ \left(\tfrac{j}{q}\right)^{\delta}\ \left(\tfrac{q-j}{q}\right)^{\gamma}.$$

Looking at this expression as Riemann sums we can guarantee that it has a limit, when $q\to\infty$, and

$$\tfrac{1}{\zeta(\gamma)\,\zeta(\delta)}\textstyle\sum_{j=1}^{q-1}\ \tfrac{1}{q}\ \left(\tfrac{j}{q}\right)^{\delta}\ \left(1-\tfrac{j}{q}\right)^{\gamma}\ \longrightarrow\ \tfrac{1}{\zeta(\gamma)\,\zeta(\delta)}\textstyle\int_0^1\ x^{\delta}(1-x)^{\gamma}\,dx$$

which finish the proof of the lemma. $\square$

By assuming that $\delta < \gamma$ and using a similar argument as above one can show that $a(q) \sim q^{2-\delta-\gamma}$. At this point we have proved that the hypothesis of the Theorem 6.1 of the reference [30] holds and this theorem allow us to estimate the limit we are interested in. It is very important to note that we are also able to use the result appearing in a remark below the Theorem 6.1 of [30] (even though $p_1 = 0$). Let us denote $M = \sum_{q=1}^{\infty} q\, p_q$. If $\gamma > 2$ then M is finite. The Renewal Theorem assures that

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[0]})(10..) \,=\, \lim_{q\to\infty} A(q) = \frac{\sum_{q=1}^{\infty} a(q)}{M} \tag{24}$$

One can show that $\sum_{q=1}^{\infty} a(q) = 1 \,+\, \sum_{j=2}^{\infty} \frac{r(j)\, j^{-\gamma}}{\zeta(\gamma)}$.

Now we proceed to another big step in this proof. In this step we need to obtain a similar renewal equation for $\mathcal{L}^q_{\log J}(I_{[0]})(01..)$. In the previous step we have replaced (18) in (19). Now, we need instead to replace (19) in (18). Starting as before we write

$$\begin{aligned}
\mathcal{L}^q_{\log J}(I_{[0]})(01..) = \;& \tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} + \tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)} \Big[\tfrac{2^{-\delta}}{\zeta(\delta)} + \tfrac{1}{\zeta(\delta)}\, \mathcal{L}^1_{\log J}(I_{[0]})(01..) \Big] + \\
& \tfrac{(q-3)^{-\gamma}}{\zeta(\gamma)} \Big[\tfrac{3^{-\delta}}{\zeta(\delta)} + \tfrac{2^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^1_{\log J}(I_{[0]})(01..) \,+\, \tfrac{1}{\zeta(\delta)}\, \mathcal{L}^2_{\log J}(I_{[0]})(01..) \Big] + \\
& \ldots + \tfrac{2^{-\gamma}}{\zeta(\gamma)} \Big[\tfrac{(q-2)^{-\delta}}{\zeta(\delta)} \,+\, \tfrac{(q-3)^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^1_{\log J}(I_{[0]})(01..) + \ldots + \\
& \qquad\qquad \tfrac{2^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^{q-4}_{\log J}(I_{[0]})(01..) \,+\, \tfrac{1}{\zeta(\delta)}\, \mathcal{L}^{q-3}_{\log J}(I_{[0]})(010..) \Big] + \\
& + \tfrac{1}{\zeta(\gamma)} \Big[\tfrac{(q-1)^{-\delta}}{\zeta(\delta)} \,+\, \tfrac{(q-2)^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^1_{\log J}(I_{[0]})(01..) + \ldots + \\
& \qquad\qquad \tfrac{2^{-\delta}}{\zeta(\delta)}\, \mathcal{L}^{q-3}_{\log J}(I_{[0]})(01..) \,+\, \tfrac{1}{\zeta(\delta)}\, \mathcal{L}^{q-2}_{\log J}(I_{[0]})(010..) \Big] + \\
& + \tfrac{(q+1)^{-\gamma}\; r(q+1)}{\zeta(\gamma)}.
\end{aligned}$$

By performing obvious simplifications, the above expression becomes

$$\begin{aligned}
& \tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} + \tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{2^{-\delta}}{\zeta(\delta)} + ... + \tfrac{1}{\zeta(\gamma)}\, \tfrac{(q-1)^{-\delta}}{\zeta(\delta)} + \tfrac{(q+1)^{-\gamma}\; r(q+1)}{\zeta(\gamma)} + \\
& \quad \mathcal{L}^1_{\log J}(I_{[0]})(01..) \Big[\tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} + \tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{2^{-\delta}}{\zeta(\delta)} + ... + \tfrac{1}{\zeta(\gamma)}\, \tfrac{(q-2)^{-\delta}}{\zeta(\delta)} \Big] + \\
& \quad \mathcal{L}^2_{\log J}(I_{[0]})(01..) \Big[\tfrac{(q-3)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} + \tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{2^{-\delta}}{\zeta(\delta)} + ... + \tfrac{1}{\zeta(\gamma)}\, \tfrac{(q-3)^{-\delta}}{\zeta(\delta)} \Big] + \\
& \quad ... + \mathcal{L}^{q-3}_{\log J}(I_{[0]})(01..) \left\{ \tfrac{2^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} \,+\, \tfrac{1}{\zeta(\gamma)}\, \tfrac{2^{-\delta}}{\zeta(\delta)} \right\} + \mathcal{L}^{q-2}_{\log J}(I_{[0]})(01..)\, \tfrac{1}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)}.
\end{aligned}$$

Analogously we define $B(0) = 0$, $p_1 = 0$ and for $q \geq 2$, $B(q) = \mathcal{L}^q_{\log J}(I_{[0]})(01..)$. We also define the following sequence

$$b(q) = \tfrac{(q-1)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{1}{\zeta(\delta)} + \tfrac{(q-2)^{-\gamma}}{\zeta(\gamma)}\, \tfrac{2^{-\delta}}{\zeta(\delta)} + ... + \tfrac{1}{\zeta(\gamma)}\, \tfrac{(q-1)^{-\delta}}{\zeta(\delta)} + \tfrac{(q+1)^{-\gamma}\; r(q+1)}{\zeta(\gamma)}. \tag{25}$$

Note that the first term of this sequence satisfies $b(1) = 2^{-\gamma}\, r(2)/\zeta(\gamma) = B(1)$. By using the identities established above we have proved the following renewal equation:

$$B(q) = [B(0)p_q + B(1)p_{q-1} + B(2)\, p_{q-2} + ... + B(q-2)p_2 + B(q-1)p_1] + b(q),$$

where p_j is same sequence we consider in the previous step. Since $B(0) = 0$ we have in fact

$$B(q) = [B(1)p_{q-1} + B(2)\, p_{q-2} + ... + B(q-2)p_2] + b(q). \tag{26}$$

Again one can show that $b(q) \sim q^{2-\delta-\gamma}$. By applying the Renewal Theorem we obtain

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[0]})(01..) \;=\; \lim_{q\to\infty} B(q) = \frac{\sum_{q=1}^{\infty} b(q)}{M}. \tag{27}$$

Since the following equality holds

$$\sum_{q=1}^{\infty} b(q) = 1 \,+\, \sum_{j=2}^{\infty} \frac{r(j)\, j^{-\gamma}}{\zeta(\gamma)}$$

we can define a constant K so that

$$K := \frac{\sum_{q=1}^{\infty} b(q)}{M} = \frac{\sum_{q=1}^{\infty} a(q)}{M}.$$

The above described procedure allows to obtain other Thermodynamic Limits. For instance, if $y = 110...$ we have

$$\begin{aligned}\mathcal{L}^q_{\log J}(I_{[0]})(10..) &= J(010..)\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(010..) + J(110..)\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(110..)\\ &= \frac{1}{\zeta(\delta)}\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(010..) + \frac{2^{-\delta}\, s(2)}{\zeta(\delta)} \mathcal{L}^{q-1}_{\log J}(I_{[0]})(110..).\end{aligned}$$

Taking the limit in q we get

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[0]})(110..) = H_1 \qquad \text{where} \qquad H_1 = \left(K - \frac{1}{\zeta(\delta)} K\right) \frac{\zeta(\delta)}{2^{-\delta}\, s(2)}$$

By performing simple algebraic manipulations we can see that $H_1 = K$. Let us apply the method again but now for $y = 001...$ From the equation

$$\begin{aligned}\mathcal{L}^q_{\log J}(I_{[0]})(01..) &= J(110..)\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(101..) + J(0010..)\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(001..)\\ &= \frac{1}{\zeta(\delta)}\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(010..) + \frac{2^{-\delta}\, s(2)}{\zeta(\delta)}\, \mathcal{L}^{q-1}_{\log J}(I_{[0]})(110..),\end{aligned}$$

we get again, by taking the limits in q, that

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[0]})(001..) = K.$$

After prove similar result for $y = 1110..$ we it is immediate to extend the method for periodic points by a formal induction. For this particular case, we have

$$\begin{aligned}\mathcal{L}^q_{\log J}(I_{[0]})(110..) &= J(0110..)\,\mathcal{L}^{q-1}_{\log J}(I_{[0]})(011..) + J(1110..)\,\mathcal{L}^{q-1}_{\log J}(I_{[0]})(1110..)\\ &= \frac{1}{s(2)}\,\mathcal{L}^{q-1}_{\log J}(I_{[0]})(0110..) + \frac{3^{-\delta}\,s(3)}{2^{-\delta}\,s(2)}\mathcal{L}^{q-1}_{\log J}(I_{[0]})(1110..),\end{aligned}$$

and we get in the same way that

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[0]})(1110..) = K.$$

This finish the proof of Proposition 29.

It is worth pointing out that for any $y \in \Omega$ we have that

$$\mathcal{L}^q_{\log J}(I_{[0]})(y) + \mathcal{L}^q_{\log J}(I_{[1]})(y) = \mathcal{L}^q_{\log J}(1)(y) = 1.$$

Therefore for any point $y \in \Omega$ for which the Proposition 29 applies we have

$$\lim_{q\to\infty} \mathcal{L}^q_{\log J}(I_{[1]})(y) = 1 - K.$$

9 Polynomial Decay of Correlations

In this section we want to estimate the decay of correlation of the observable $I_{[0]}$ for the equilibrium probability at the critical inverse temperature.

That is, we will show that the integral below decays in a polynomial way, with respect to q, and also determine its precise asymptotic behavior

$$\int_\Omega (I_{[0]}\circ\sigma^q)\,[\,I_{[0]}\ -\mu_1[0]\,]\,d\mu_1 \ \sim\ q^{2-\delta}.$$

The technique is similar to the one employed in [19] and here we will proofs in full details. Other known results on polynomial decay of correlations are presented in [36], [23] and [42].

To deduce the polynomial decay we will need several preliminary estimations. We begin with the proof of the following asymptotic relation

$$\mu_1[0] - \mathcal{L}^q_{\log J}(I_{[0]})(01..) \sim q^{2-\delta}.$$

Define $V_q = \mu_1[0] - B(q) = \mu_1[0] - \mathcal{L}^q_{\log J}(I_{[0]})(01..)$. We want to obtain the behavior of $V(q)$, when $q \to \infty$. From the renewal equation (26) we get another renewal equation: for $q \geq 3$

$$V_q = \sum_{j=1}^{q-2} V_j p_{q-j} + \left[\mu_1[0] \sum_{j=q}^{\infty} p_j - b(q) \right]. \tag{28}$$

To simplify the notation for $q \geq 3$. we denote by K_q the last term on the above equality, that is, $K_q = \mu_1[0] \sum_{j=q}^{\infty} p_j - b(q)$. Now consider the following formal power series

$$f(z) = \sum_{j=2}^{\infty} p_j \, z^j, \quad V(z) = \sum_{j=1}^{\infty} V_j \, z^j, \quad \text{and} \quad K(z) = \sum_{j=3}^{\infty} K_j \, z^j.$$

From the renewal equation (28) we get that $V(z)\, f(z) + K(z) = V(z) - V_1 z - V_2 z^2$. Therefore

$$V(z) = \frac{K(z) + V_1 z + V_2 z^2}{1 - f(z)} = \frac{K(z) + V_1 z + V_2 z^2}{1 - z} \, \frac{1 - z}{1 - f(z)}.$$

For δ and γ large we have that $f(z)$ is differentiable on $z = 1$ and the derivative is not zero. From the previous estimations it is simple to see that $K_n \sim n^{1-\gamma}$. Up to a bounded multiplicative constant we get

$$V(z) \sim \frac{K(z) + V_1 z + V_2 z^2}{1 - z}.$$

from where we obtain (asymptotically)

$$\begin{aligned}
(1 - z)\, V(z) &= \sum_{j=1}^{\infty} V_j \, z^j \; - z \sum_{j=1}^{\infty} V_j \, z^j \\
&= V_1 z + V_2 z^2 + K_3 z^3 + K_4 z^4 + K_5 z^5 + ... \\
&= \sum_{j=3}^{\infty} K_j \, z^j + V_1 z + V_2 z^2 \\
&= K(z) + V_1(z) + V_2 z^2.
\end{aligned}$$

Note that

$$\frac{1}{(1 - z)} \left(K(z) + V_1 z + v_2 z^2 \right) = (1 + z + z^2 + z^3 + ...) \; (V_1 z + V_2 z^2 + K_3 z^3 + K_4 z^4 + ...) =$$

$$V_1\, z \,+\, (V_1+V_2)\, z^2 + (V_1+V_2+K_3) z^3 + (V_1+V_2+K_3+K_4) z^4 ....$$

In this way we get the following recurrence relation: $V_1 = 0$ and $V_n = K_3 + K_4 + .. + K_n + V_2$. Once $K_n \sim n^{1-\gamma}$ we get that $V_n \sim n^{2-\gamma}$, where we have assumed for last estimate to holds that $\delta < \gamma$. This argument complete the proof of

$$\mu_1[0] - \mathcal{L}^q_{\log J}(I_{[0]})(01..) \sim n^{2-\gamma}. \tag{29}$$

By using the renewal equation (22) and a similar estimation procedure one can prove that

$$\mu_1[0] - \mathcal{L}^q_{\log J}(I_{[0]})(10..) \sim n^{3-\delta-\gamma}. \tag{30}$$

Now for each $s \geq 2$ we need to evaluate the difference

$$\mu_1[0] - \mathcal{L}^q_{\log J}(I_{[0]})(\underbrace{00...0}_{s}\,1..).$$

Let $B^s_q = \mathcal{L}^q_{\log J}(I_{[0]})(\overbrace{00...0}^{s}\,1..)$, $A_t = \mathcal{L}^t_{\log J}(I_{[0]})(10...)$ and for $j \geq 1$ $p^s_j = (s+j-1)^{-\gamma} r(s)^{-1} s^{\gamma}$. Using similar arguments presented above (see figure 3 page 1092 [19]) one can show that for any $s, q \geq 2$

$$\begin{aligned}
B^s_q &= J(1\,\underbrace{00...0}_{s}\,1..)\,A_{q-1} + J(1\,\underbrace{00...0}_{s+1}\,1..)\,A_{q-2} + ... + J(1\,\underbrace{00...0}_{s+q-2}\,1..)\,A_1 \\
&\qquad + \frac{(s+q)^{-\gamma}\, r(s+q)}{s^{-\gamma}\, r(s)} \\
&= \frac{1}{r(s)}\, A_{q-1} + \frac{(s+1)^{-\gamma}}{r(s)\, s^{-\gamma}}\, A_{q-2} + ... + \frac{(s+q-2)^{-\gamma}}{r(s)\, s^{-\gamma}}\, A_1 + \frac{(s+q)^{-\gamma}\, r(s+q)}{s^{-\gamma}\, r(s)} \\
&= p^s_1\, A_{q-1} + p^s_2\, A_{q-2} + ... + p^s_{q-1}\, A_1 + \alpha(q,s).
\end{aligned} \tag{31}$$

Note that $\sum_{j=1}^{\infty} p^s_j = 1$. Let us now consider for $s \geq 2$ and $n \geq 1$, the following sequence $V^s_n = \mu_1[0] - B^s_q$. Recalling the definitions given above if $n = 1$ we have $V^s_1 = \mu_1[0] - \frac{(s+1)^{-\gamma}\, r(s+1)}{s^{-\gamma}\, r(s)}$. We also introduce, for $n \geq 1$, the sequences $K_n = \mu_1[0] - A_n$, and $U^s_n = \mu_1[0]\,(p^s_n + p^s_{n+1} + ...) - \alpha(n,s)$. Note that $\mu_1[0]\,(p^s_n + p^s_{n+1} + ...) \sim \frac{s^{1-\gamma}}{\gamma-1}$. From the equation (31) we deduce that

$$V^s_q = K_{q-1}\, p^s_1 + K_{q-2}\, p^s_2 ... +\, K_1\, p^s_{q-1} - U^s_q.$$

The last term behavior is know to be $U^s_q \sim \mu_1[0]\, \frac{1}{\gamma-1}\, \frac{(q+s)^{-\gamma+1}}{s^{-\gamma}\, r(s)} - \frac{(s+q)^{-\gamma}\, r(s+q)}{s^{-\gamma}\, r(s)}$, and we know from (30) how to control K_q, that is, $K_q \sim q^{3-\delta-\gamma}$. Since $p^s_{q-1} = \frac{(s+q-2)^{-\gamma}}{r(s)\, s^{-\gamma}}$

we get that for fixed s the dominant term as a function of q in the right hand side of the above inequality is $U_q^s \sim \frac{(q+s)^{1-\gamma}}{s^{-\gamma} r(s)}$ because

$$K_{q-1}\, p_1^s + K_{q-2}\, p_2^s ... +\, K_1\, p_{q-1}^s \sim \frac{q^{4-2\gamma-\delta}}{s^{1-\gamma}}. \tag{32}$$

The above analysis is similar to the one in Lemma A4 page 1102 in [19].
Putting all these estimates together we finally obtain the estimation

$$\mu_1[0] - \mathcal{L}_{\log J}^q (I_{[0]})(\underbrace{00...0}_{s}\,1..) \sim \frac{(q+s)^{1-\gamma}}{s^{-\gamma} r(s)}. \tag{33}$$

As mentioned before the Ruelle operator $\mathcal{L}_{\log J}$ is the dual (in the $\mathcal{L}^2(\Omega, \mathcal{B}, \mu_1)$ sense) of the Koopman operator $\mathcal{K}(\varphi) = \varphi \circ \sigma$, using this duality we get that for fixed q that

$$\begin{aligned}
\int_\Omega (I_{[0]} \circ \sigma^q)\, [\, I_{[0]}\; - \mu_1[0]\,]\, d\mu_1 &= \int I_{[0]}\, \mathcal{L}_{\log J}^q [\, I_{[0]}\; - \mu_1[0]\,]\, d\mu_1 \\
&= \int_\Omega I_{[0]}(x) \left[\mathcal{L}_{\log J}^q\, I_{[0]}(x)\; - \mu_1[0]\right] d\mu_1(x) \\
= \sum_{j=1}^{\infty} \int_\Omega I_{[0]}(\underbrace{00...0}_{j}\,1..) &\left[\mathcal{L}_{\log J}^q\, I_{[0]}(\underbrace{00...0}_{j}\,1..)\; - \mu_1[0]\right] d\mu_1 \\
&= \sum_{j=1}^{\infty} \mu_1([\underbrace{00...0}_{j}\,1]\, \left[\mathcal{L}_{\log J}^q\, I_{[0]}(\underbrace{00...0}_{j}\,1..)\; - \mu_1[0]\right] \\
&= \sum_{j=1}^{\infty} j^{1-\gamma}\, \frac{(q+j)^{1-\delta}}{j^{-\gamma}\, r(q)} \sim q^{2-\delta},
\end{aligned}$$

where in the last equality we used (33) and (15). We point out that the decay of correlations of other observables can also be obtained by variations of the above method.

10 Potentials on the lattices $\mathbb{N}$ and $\mathbb{Z}$

In this section we describe the setting where the Ruelle operator provides results for the one-dimensional lattice $\mathbb{Z}$. This is a classical topic and we present it here just for completeness (see [44], [32], [2], [12] and [41]).

The elements of the symbolic space $\Omega = \{0,1\}^{\mathbb{N}}$ as in the previous section are be denoted by $(x_0, x_1, x_2, ..)$, while the elements in two-sided lattice $\hat{\Omega} = \{0,1\}^{\mathbb{Z}}$ are denoted by $(...x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...)$. The action of the left shift $\hat{\sigma}$ on $\hat{\Omega} = \{0,1\}^{\mathbb{Z}}$ is described by $\sigma(..., x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...) = (..., x_{-2}, x_{-1}, x_0 \,|\, x_1, x_2, ...)$.

Consider a potential $\hat{f} : \hat{\Omega} = \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ and suppose that we are interested in finding an equilibrium state $\hat{\mu}$ for the potential $\hat{f}$, that is, a probability measure on $\hat{\Omega}$ satisfying

$$P(\hat{f}) = \sup_{\nu \in \mathcal{M}(\hat{\sigma})} \left\{ h(\nu) + \int_{\hat{\Omega}} \hat{f}\, d\nu \right\} = h(\hat{\mu}) + \int_{\hat{\Omega}} \hat{f}\, d\hat{\mu}.$$

Since there is no reason to consider the site $0 \in \mathbb{Z}$ a special one in the lattice, then it is natural to looking for shift-invariant probability measures among those solving the variational problem.

The crucial observation that relates one sided and two-sided lattices is the following: the equilibrium probabilities for $\hat{f}$ and $g : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, satisfying $\hat{f} = g + h - h \circ \hat{\sigma}$ for some continuous $h : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ are the same. When such h do exists the above equation is called a coboundary equation. The main point is that under some mild regularity assumptions on the potential f, one can get a special g which depends just on future coordinates, that is, for any pair $x = (..., x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...)$ and $y = (..., y_{-2}, y_{-1} \,|\, x_0, x_1, x_2, ...) \in \hat{\Omega}$, we have $g(x) = g(y)$. By abusing the notation we frequently write $g(x) = g(x_0, x_1, x_2, ...)$, which allow us to think about g is a function on $\{0,1\}^{\mathbb{N}}$.

For the potential g one can apply the formalism of the Ruelle Operator $\mathcal{L}_g$ to study the properties of μ (a probability measure on $\{0,1\}^{\mathbb{N}}$), which is the equilibrium state for $g : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$. In those cases it is possible to show that the equilibrium state for $\hat{f}$ is the probability measure $\hat{\mu}$ (a probability on $\{0,1\}^{\mathbb{Z}}$), which is given by the natural extension of μ (see [2] and [41]). Let us elaborate on that: what we call the natural extension of the shift-invariant probability measure μ on $\{0,1\}^{\mathbb{N}}$ is the probability measure $\hat{\mu}$ on $\{0,1\}^{\mathbb{Z}}$ defined in the following way: for any given cylinder set on the space $\hat{\Omega}$ of the form $[a_k, a_{k+1}, ..., a_{-2}, a_{-1} \,|\, a_0, a_1, a_2, ..a_{k+n}]$, where $a_j \in \{0,1\}$ and $j \in \{k, k+1, .., k+n\} \subset \mathbb{Z}$, we define

$$\hat{\mu}\Big([a_k, a_{k+1}, ..., a_{-1} \,|\, a_0, a_1, ..., a_{k+n}]\Big) = \mu\Big([a_k, a_{k+1}, ..., a_{-1}, a_0, a_1, ..a_{k+n}]\Big),$$

where $[a_k, a_{k+1}, ..., a_{-1}, a_0, a_1, ..., a_{k+n}]$ is now a cylinder on Ω. Notice that if μ is shift-invariant, then $\hat{\mu}$ is shift-invariant.

Given the potential $\hat{f} : \hat{\Omega} \to \mathbb{R}$, it is natural to denote $\mathcal{G}^*(\hat{f})$ as the set of natural extensions $\hat{\nu}$ of the probabilities ν which are eigenprobabilities of the Ruelle operator $\mathcal{L}_g$ (where g was the associated coboundary).

All the above statements holds true if $\hat{f}$ is in Hölder (and then we get that g is also Hölder) or Walters class (see [58]). In some cases where $\hat{f}$ (or g) is not Hölder, part of the above formalism also works. This is indeed the case of all potentials we considered here.

We shall remind that in order to use the Ruelle operator formalism we have to work with potentials f which are defined on the symbolic space Ω, i.e., $f : \Omega = \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$.

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