Graphical Representations for Ising and Potts Models in General External Fields

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Abstract

This work is concerned with the theory of graphical representation for the Ising and Potts models over general lattices with non-translation invariant external field. We explicitly describe in terms of the random-cluster representation the distribution function and, consequently, the expected value of a single spin for the Ising and q-state Potts models with general external fields. We also consider the Gibbs states for the Edwards-Sokal representation of the Potts model with non-translation invariant magnetic field and prove a version of the FKG inequality for the so called general random-cluster model (GRC model) with free and wired boundary conditions in the non-translation invariant case.

Adding the amenability hypothesis on the lattice, we obtain the uniqueness of the infinite connected component and the almost sure quasilocality of the Gibbs measures for the GRC model with such general magnetic fields. As a final application of the theory developed, we show the uniqueness of the Gibbs measures for the ferromagnetic Ising model with a positive power-law decay magnetic field with small enough power, as conjectured in [8].

Contents

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1 Introduction

Graphical representations are extremely useful tools for the study of phase transition in Equilibrium Statistical Mechanics. Fortuin and Kasteleyn [22], marked the beginning of four decades of intense activity that produced a rather complete theory for translation invariant systems. These representations were successfully employed to obtain non-perturbative and deep results for Ising and Potts models on the hypercubic lattice using percolation-type methods, namely the discontinuity of the magnetization at the phase transition point for the one-dimensional Ising and Potts models with $1/r^2$ interactions [3], the knowledge of the asymptotic behavior of the eigenvalues of the covariance matrix of the Potts model [10], the Aizenman-Higuchi Theorem on the Choquet decomposition of the two-dimensional Ising and Potts models [1, 15, 16, 25, 32] and the proof that the self-dual point on the square lattice $p_{sd}(q) = \sqrt{q}/(1+\sqrt{q})$ is the critical point for percolation in the random-cluster model ($q \geqslant 1$) [5], see also the review [40]. For a detailed introduction to the random-cluster model we refer the reader to [18, 24, 27, 30].

The relationship between graphical representations and phase transitions in Ising/Potts-type models is typically considered with respect to the random-cluster model (RC model) and in view of the Edwards-Sokal coupling [20].

Most papers employing such representations use spin models with null or translation invariant magnetic field, whereas we shall analyze graphical representations of the Ising and Potts models under arbitrary and non-translation invariant external fields, which is a significantly more complicated task for several reasons: when general boundary conditions are considered, the FKG property is harder to prove - as previously noticed by [7], this property does not even hold for certain boundary conditions. In the absence of the magnetic field, phase transitions in the spin system can be directly detected by the random-cluster representation, but now this relationship is subtle since in some cases the phase transitions (in the percolation sense) in the random-cluster model does not correspond to a transition in the corresponding spin model. Such difficulties also appear in the analyses of Dobrushin-like states [26], large q order-disorder at the transition temperature [14] and the effect of "weak boundary conditions" in the q-state Potts model [11].

Here the absence of symmetry brings questions regarding the color(s) of the infinite connected component(s), which need not be addressed in the case of null magnetic field, for instance. Furthermore, non-translation invariance causes many technical issues when using basic results from the classical theory of spin models and Ergodic Theory. To avoid confusion, on this paper the terms phase transition and critical inverse temperature shall hereby be solely employed to express changing in the number of the Gibbs measures when the temperature varies.

This paper is motivated by some recent works on ferromagnetic Ising model in non-uniform external fields [4, 8, 9, 33, 36, 37]. Here, we are interested in developing the theory of graphical representation for non-translation invariant models whilst aiming for the problem of classifying which are the positive magnetic fields such that the ferromagnetic Ising model on the square lattice passes through a first order phase transition, in terms of its power law decay exponent. The formal Hamiltonian of this model is given by

$$H(\sigma) = -J \sum_{\{i,j\}} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \tag{1}$$

where the first sum ranges over the pairs of nearest neighbors. In this model, if the magnetic field $\boldsymbol{h} = (h_i : i \in \mathbb{Z}^d)$ satisfies $\liminf h_i > 0$, it has been proved [9] that for any positive temperature the set of the Gibbs measures is a singleton, therefore for essentially bounded-from-below positive magnetic fields the conclusion is similar to the one obtained by Lee and Yang [34]. In [8], the authors considered a positive, decreasing magnetic field and employed the Isoperimetric inequality and a Peierls-type argument to show that if the magnetic field is given by $h_i = h^*/|i|^\alpha$, where h^* is a positive constant, then the model presents first order phase transition in every dimension $d \geqslant 2$, for any fixed exponent $\alpha > 1$. On the other hand, if $\alpha < 1$, they proved by means of a contour expansion that the uniqueness of the Gibbs measures holds at very low and by other methods at very high temperatures, and conjectured that the set of Gibbs measures at any positive temperature should be a singleton. The authors in [8] justified why the extension of their results to any positive temperature is not obvious by resorting to most of the known techniques, but we prove as an application of the theory to be developed that the conjecture holds true. This is done by extending some results of the seminal work [7] to the non-translation invariant setting.

The paper is organized in three parts: the first part presents the relevant background material, including notation and the basic definitions of the models to be treated in subsequent parts. The second part is comprised of the theory on general finite graphs with free boundary conditions, the main results of which are the extension of the Edwards-Sokal coupling for general external fields and the explicit computation (in terms of the RC model) of the distribution function of a single spin of the Ising model with general external field and its expected values. These results are also generalized to the q-state Potts model in general external fields. The third part is concerned with the Potts, Edwards-Sokal and General random-cluster models in the non-translation invariant external fields setting with general boundary conditions. It is inspired by the reference [7], but extends their results to non-translation invariant magnetic fields - a task that was occasionally non-trivial. In some cases, their results were essentially proved for very general fields and our work was simply to point out the necessary technical modifications. Fundamental results such as the FKG inequality required non-trivial adaptations and for this reason we presented its detailed proof for both free and wired boundary conditions in the so called general random-cluster model (GRC model) with non-translation invariant external field. Even with null external field, the random cluster measures lacked the key property of the quasilocality of the Gibbs measures, although it is possible to have the said property almost surely by assuming the uniqueness of the infinite connected component. For a null magnetic field on the hypercubic lattice, this fact was first proved in [38], however the geometry of the graph in this type of question is very important because for some non-amenable graphs such as regular trees even almost sure quasilocality fails, see [21, 29]. For the random-cluster measures with translation invariant magnetic field, defined over amenable graphs, almost quasilocality was shown in [7] for those measures having almost surely at most one infinite connected component. These results were recovered here for GRC models with non-translation invariant magnetic fields. The proofs of both the uniqueness of the infinite connected component and of the quasilocality of the Gibbs measures are given and new ideas are introduced to circumvent the lack of translation invariance.

The conjecture stating the uniqueness of the Gibbs measures for the Ising model with power-law-decay magnetic field ($\alpha < 1$) is proved in the last section of the third part. As a corollary of one of the main results of this part (Theorem 11), we have obtained a characterization of the critical [1] inverse temperature $\beta_c(\boldsymbol{J}, \boldsymbol{h})$ of the ferromagnetic Ising model given by (1) where $h_i = h^*/|i|^\alpha$, with $\alpha > 1$ on the hypercubic lattice. Few facts are known about this inverse critical temperature. For example, in the positive external field case of the two-dimensional model (1) with the coupling constant $\boldsymbol{J} \equiv 1$ and $\sum_{i\in\mathbb{Z}^2} h_i < \infty$, it follows from the Onsager exact solution and a general result [23] about summable perturbations of the Gibbs measures that $\beta_c(\boldsymbol{J}, \boldsymbol{h}) = \log(1 + \sqrt{2})$. From [9] it follows that $\beta_c(\boldsymbol{J}, \boldsymbol{h}) = +\infty$ as long as $\liminf h_i > 0$ in any dimension. The last section contains the proof that $\beta_c(\boldsymbol{J}, \boldsymbol{h})$ is also trivial, i.e., $\beta_c(\boldsymbol{J}, \boldsymbol{h}) = +\infty$ when $h_i = h^*/|i|^\alpha$, with $\alpha < 1$. The most interesting cases are those where we do have phase transition and the magnetic field is given by $h_i = h^*/|i|^\alpha$, with $1 < \alpha < 2$ (not summable on entire lattice). For such cases, to the best of our knowledge, the only known fact about this critical point is that $\log(1 + \sqrt{2}) \leqslant \beta_c(1, \boldsymbol{h})$, which is derived from the correlation inequalities. It is not known whether the Lieb-Simon inequality [39, 35], the Aizenman-Barsky-Fernández Theorem [2] and other characterizations of the critical point (for example, [19])can be extended for the case $h_i = h^*/|i|^\alpha$, with $1 < \alpha < 2$.

Part I
Basic definitions and models

2 Background in graph theory

We say that a graph $G = (V, E)$ is a countable graph if its vertex set V is countable. As usual, a **path** γ on G is an alternating sequence of vertices and edges $\gamma = (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$, such that $v_i \neq v_j$ for all $0 \leqslant i, j \leqslant n-1$, $v_n \in V \setminus \{v_1, v_2, \ldots, v_{n-1}\}$ and $e_j = \{v_{j-1}, v_j\}$ for $1 \leqslant j \leqslant n$. In case $v_0 = v_n$ we say that γ is a closed path or a circuit. The vertices v_0 and v_n of γ are called initial and final vertices, respectively. We say that $x, y \in V$ are **connected** if $x = y$ or there is a path γ on G so that $x = v_0$ and $y = v_n$, denoted $x \overset{}{\underset{G}{\longleftrightarrow}} y$, and whenever it is clear from context, we

[1] in case $\beta_c(\boldsymbol{J}, \boldsymbol{h}) = +\infty$ we mean that the set of the Gibbs measures is a singleton for all $0 < \beta < +\infty$.

shall remove the subscript G from the notation. The length of a path $\gamma = (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$ is defined as $|\gamma| = n$.

A graph G is said to be a **connected graph** if any two vertices $i, j \in V$ are connected, otherwise we say that G is disconnected. The connected component of $x \in V$ is the vertex set $C_x \equiv \{y \in V : y \leftrightarrow x\}$. The **distance** $d_G(x, y)$ between $x, y \in \mathbb{V}$ is defined by $d_G(x, y) = 0$ if $x = y$; $d_G(x, y) = +\infty$ if $x \notin C_y$ and $d_G(x, y) = \inf\{|\gamma| : \gamma \text{ is a path connecting } x \text{ to } y\}$, if $x \in C_y$.

A graph $H = (\tilde{V}, \tilde{E})$ is a subgraph of $G = (V, E)$ if $\tilde{V} \subset V$ and $\tilde{E} \subset E$. A subgraph H of G is an **induced subgraph** if it has the same vertex set as G and a random subgraph of G is an induced subgraph such that the edges are chosen randomly.

Any infinite countable connected graph $\mathbb{L} \equiv (\mathbb{V}, \mathbb{E})$ will be called a lattice and from now on a finite subgraph of $\mathbb{L}$ will be denoted by $G = (V, E)$. The vertex set V will sometimes be called *the volume* V.

There are several definitions for the boundary of a vertex set V contained in $\mathbb{L}$. In this work, the boundary is defined as follows.

Definition 1 (Boundary of V)**.** *The boundary of an arbitrary vertex set V in $\mathbb{L}$ is defined by $\partial V \equiv \{i \in \mathbb{V} \setminus V : d_{\mathbb{L}}(i, V) = 1\}$, where $d_{\mathbb{L}}$ is the distance on the lattice $\mathbb{L}$. See Figure 3.*

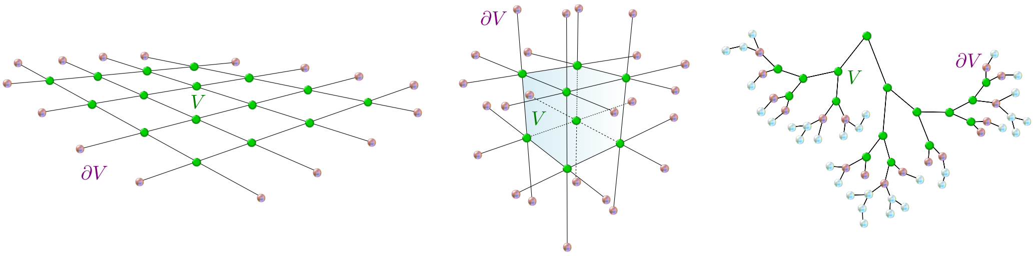


Figure 1: *Examples of boundary of V in three different lattices. The boundary of V in each case is the vertex set colored pink.*

3 The Ising model on countable graphs

Let $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ be an arbitrary lattice and Σ the standard configuration space of the Ising model, i.e.,

$$\Sigma \equiv \big\{\sigma = (\sigma_i : i \in \mathbb{V}) : \sigma_i \in \{-1, +1\},\ \forall i \in \mathbb{V}\big\} = \{-1, +1\}^{\mathbb{V}}.$$

This configuration space has a standard metric, for which the distance between any pair of configurations $\sigma, \omega \in \Omega$ is given by

$$d(\sigma, \omega) = \frac{1}{2^R}, \text{ and } R = \inf\left\{ r > 0 : \begin{array}{c} \sigma_i = \omega_i,\ \forall i \in B(o, r) \text{ and} \\ \exists\, j \in \partial B(o, r) \text{ such that } \sigma_j \neq \omega_j \end{array} \right\},$$

where $B(o,r)$ is the open ball in $\mathbb{L}$ of center $o \in \mathbb{V}$ (fixed) and radius $r > 0$. Since the metric d induces the product topology on Σ, it follows from Tychonoff's Theorem that (Σ, d) is a compact metric space. As a measure space, we always consider Σ endowed with the Borel σ-algebra $\mathscr{B}(\Sigma)$, which is generated by the open sets on (Σ, d).

The Hamiltonian of the Ising model on a finite volume $V \subset \mathbb{L}$ with boundary condition $\mu \in \Sigma$ is given by

$$\mathscr{H}^{\mu,\mathrm{Ising}}_{\boldsymbol{h},V}(\sigma) \equiv -\sum_{\substack{i,j \in V \\ \{i,j\} \in \mathbb{E}}} J_{ij}\,\sigma_i\sigma_j - \sum_{i \in V} h_i\,\sigma_i - \sum_{\substack{i \in V,\ j \in \partial V \\ \{i,j\} \in \mathbb{E}}} J_{ij}\,\sigma_i\mu_j, \tag{2}$$

where the coupling constant $\boldsymbol{J} \equiv (J_{ij} : \{i,j\} \in \mathbb{E}) \in [0, +\infty)^{\mathbb{E}}$ satisfies the regularity condition $\sum_{j \in \mathbb{V}} J_{ij} < +\infty$, $\forall i \in \mathbb{V}$ and the magnetic field is $\boldsymbol{h} \equiv (h_i : i \in \mathbb{V}) \in \mathbb{R}^{\mathbb{V}}$.

Gibbs measures. For any fixed finite volume V and boundary condition μ, we define the (finite) set of configurations compatible with μ outside V as being the set of configurations $\Sigma^{\mu}_{V} \equiv \{\sigma \in \Sigma : \sigma_i = \mu_i \text{ for } i \in \mathbb{V} \setminus V\}$. The Gibbs measures of the Ising model on the finite volume V with boundary condition μ at the inverse temperature $\beta > 0$ is the probability measure $\lambda^{\mu}_{\beta,\boldsymbol{h},V} : \mathscr{B}(\Sigma) \to \mathbb{R}$ given by

$$\lambda^{\mu}_{\beta,\boldsymbol{h},V}(\{\sigma\}) = \begin{cases} \frac{1}{\mathscr{Z}^{\mu,\mathrm{Ising}}_{\beta,\boldsymbol{h},V}} \exp\big(-\beta \mathscr{H}^{\mu,\mathrm{Ising}}_{\boldsymbol{h},V}(\sigma)\big), & \text{if } \sigma \in \Sigma^{\mu}_{V} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathscr{Z}^{\mu,\mathrm{Ising}}_{\beta,\boldsymbol{h},V}$ is a normalizing constant called the **partition function** given by

$$\mathscr{Z}^{\mu,\mathrm{Ising}}_{\beta,\boldsymbol{h},V} = \sum_{\sigma \in \Sigma^{\mu}_{V}} \exp\big(-\beta \mathscr{H}^{\mu,\mathrm{Ising}}_{\boldsymbol{h},V}(\sigma)\big).$$

We denote by $\mathscr{G}^{\mathrm{Ising}}_{\beta}(\boldsymbol{J}, \boldsymbol{h})$ the set of infinite-volume Gibbs measures which is given by the closure of the convex hull of the set of all the weak limits $\lim_{V_n \uparrow \mathbb{V}} \lambda^{\mu}_{\beta,\boldsymbol{h},V}$, where $V_n \subset V_{n+1}$ and μ runs over all possible sequences of boundary conditions.

The Ising model with free boundary condition. The Gibbs measure of the Ising model on a finite subgraph $G \subset \mathbb{L}$ with free boundary condition is given by

$$\lambda_{\beta,\boldsymbol{h},V}(\{\sigma\}) = \frac{1}{\mathscr{Z}^{\mathrm{Ising}}_{\beta,\boldsymbol{h},V}} \exp\big(-\beta \mathscr{H}^{\mathrm{Ising}}_{\boldsymbol{h},V}(\sigma)\big),$$

where $\mathscr{Z}^{\mathrm{Ising}}_{\beta,\boldsymbol{h},V}$ is the partition function and the Hamiltonian is given by

$$\mathscr{H}^{\mathrm{Ising}}_{\boldsymbol{h},V} = -\sum_{\{i,j\}\in E} J_{ij}\,\sigma_i\sigma_j - \sum_{i\in V} h_i\,\sigma_i.$$

The expected value of a random variable $f : \Sigma \to \mathbb{R}$, with respect to $\lambda^{\mu}_{\beta,\boldsymbol{h},V}$ is given by

$$\lambda^{\mu}_{\beta,\boldsymbol{h},V}(f) \equiv \sum_{\sigma\in\Sigma^{\mu}_{V}} f(\sigma)\lambda^{\mu}_{\beta,\boldsymbol{h},V}(\{\sigma\}).$$

4 The Potts model with inhomogeneous magnetic field

Let $q \in \mathbb{Z}^{+}$ be a fixed positive integer. The state space of the q-state Potts model on the lattice $\mathbb{L}$ is defined as

$$\Sigma_q \equiv \big\{\hat{\sigma} = (\hat{\sigma}_i : i \in \mathbb{V}) : \hat{\sigma}_i \in \{1,2,\ldots,q\},\ \forall i \in \mathbb{V}\big\} = \{1,2,\ldots,q\}^{\mathbb{V}}.$$

To define a q-state Potts model with magnetic field, we fix a family of coupling constants $\boldsymbol{J} \equiv (J_{ij} : \{i,j\} \in \mathbb{E}) \in [0,\infty)^{\mathbb{E}}$ and magnetic fields $\hat{\boldsymbol{h}} \equiv (h_{i,p} : i \in \mathbb{V}; p = 1,\ldots,q) \in \mathbb{R}^{\mathbb{V}} \times \cdots \times \mathbb{R}^{\mathbb{V}}$. The Hamiltonian on a finite volume G with boundary condition $\hat{\mu} \in \Sigma_q$ is given by

$$\mathscr{H}^{\hat{\mu},\mathrm{Potts}}_{\hat{\boldsymbol{h}},q,V}(\hat{\sigma}) \equiv -\sum_{\substack{i,j\in V\\ \{i,j\}\in\mathbb{E}}} J_{ij}\delta_{\hat{\sigma}_i,\hat{\sigma}_j} - \sum_{p=1}^{q}\sum_{i\in V}\frac{h_{i,p}}{q}\delta_{\hat{\sigma}_i,p} - \sum_{\substack{i\in V,\ j\in\partial V\\ \{i,j\}\in\mathbb{E}}} J_{ij}\delta_{\hat{\sigma}_i,\hat{\mu}_j}, \tag{3}$$

where $\delta_{\hat{\sigma}_i,\hat{\sigma}_j}$ is the Kronecker delta function.

The Gibbs measure of Potts model on a finite volume G with boundary condition $\hat{\mu}$ is defined analogously to the Ising model. We consider the set of all compatible configurations with the boundary condition $\hat{\mu}$, i.e., $\Sigma^{\hat{\mu}}_{q,V} \equiv \{\hat{\sigma} \in \Sigma_q : \hat{\sigma}_i = \hat{\mu}_i \text{ for } i \in \mathbb{V}\setminus V\}$ and define the Gibbs measure on the volume G with boundary condition $\hat{\mu}$ as the probability measure $\pi^{\hat{\mu}}_{\beta,\hat{\boldsymbol{h}},q,V}$ on $\big(\Sigma_q, \mathscr{B}(\Sigma_q)\big)$ such that

$$\pi^{\hat{\mu}}_{\beta,\hat{\boldsymbol{h}},q,V}(\hat{\sigma}) = \begin{cases} \frac{1}{\mathscr{Z}^{\hat{\mu},\mathrm{Potts}}_{\beta,\hat{\boldsymbol{h}},q,V}} \exp\big(-\beta\mathscr{H}^{\hat{\mu},\mathrm{Potts}}_{\hat{\boldsymbol{h}},q,V}(\hat{\sigma})\big), & \text{if } \hat{\sigma} \in \Sigma^{\hat{\mu}}_{q,V} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathscr{Z}^{\hat{\mu},\mathrm{Potts}}_{\beta,\hat{\boldsymbol{h}},q,V}$ is the partition function. The free boundary condition case can be treated similarly to the previous section. The expected value of a random variable $f : \Sigma_q \to \mathbb{R}$ in this model is denoted by $\pi^{\hat{\mu}}_{\beta,\hat{\boldsymbol{h}},q,V}(f)$. The set of infinite-volume Gibbs measures is defined similarly to the previous section and denoted by $\mathscr{G}^{\mathrm{Potts}}_{\beta}(\boldsymbol{J},\hat{\boldsymbol{h}})$.

Remark 1. *In general, we use $\hat{\boldsymbol{h}}$ to denote the magnetic field. In the special case where $q = 2$ and the magnetic field satisfies $h_{i,1} = -h_{i,2} \equiv h_i$ we drop the hat from notation $\hat{\boldsymbol{h}}$ and write the Hamiltonian, for example in the free boundary condition case, as follows*

$$\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat{\sigma}) \equiv -\sum_{\{i,j\}\in E} J_{ij}\delta_{\hat{\sigma}_i,\hat{\sigma}_j} - \sum_{i\in V} \frac{h_i}{2}(\delta_{\hat{\sigma}_{i},1} - \delta_{\hat{\sigma}_{i},2}). \tag{4}$$

Proposition 1. *Fix a finite graph $G = (V, E)$ and assume that the magnetic field of the 2-state Potts model satisfies $h_{i,1} = -h_{i,2} \equiv h_i$ for all $i \in V$. If $\hat{\sigma} \in \{1, 2\}^V$ denotes the configuration obtained from $\sigma \in \{-1, 1\}^V$ using the spins identification $-1 \leftrightarrow 2$ and $1 \leftrightarrow 1$, then we have for any $\beta > 0$ that*

$$\lambda^{\mu}_{\beta,\boldsymbol{h},V}(\{\sigma\}) = \pi^{\hat{\mu}}_{2\beta,\boldsymbol{h},2,V}(\{\hat{\sigma}\}).$$

5 The random-cluster model with external field

This section is devoted to the $q = 2$ inhomogeneous random-cluster models on a finite graph $G = (V, E)$. The general random-cluster model in external field will be introduced in the Section 9, more precisely by the expression (9). The state space over which these models are defined is the cartesian product $\{0, 1\}^E$. A generic element of this space will be denoted by ω and called an edge configuration. We say that an edge e is open in the configuration ω if $\omega_e = 1$, and we otherwise say e is closed. Given $\omega \in \{0, 1\}^E$, its set of open edges is denoted by $\eta(\omega) = \{e \in E : \omega_e = 1\}$. We say that a path $\gamma := (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$ on the graph G is an open path on ω if all of its edges belong to $\eta(\omega)$, i.e., $\omega_{e_i} = 1,\ \forall i = 1, \ldots, n$.

Two distinct vertices $x, y \in V$ are said to be connected in ω if there exists an open path $\gamma := (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$ on this edge configuration such that $v_0 = x$ and $v_n = y$. If $x, y \in V$ are connected on ω, we write $x \leftrightarrow y$. A subgraph H of G is connected on ω if any pair of vertices of H can be connected through a open path entirely contained in H. The open connected component of a vertex $x \in V$ is defined by $C_x(\omega) \equiv \{y \in V : x \leftrightarrow y \text{ in } \omega\} \cup \{x\}$. The set $C_x(\omega)$ is called the open connected component of x in the configuration ω.

To define the probability measure of the random-cluster model with external field we fix two families $\boldsymbol{p} \equiv (p_{ij} \in [0,1] : \{i,j\} \in E) \in [0,1]^E$ and $\boldsymbol{h} \equiv (h_i : i \in V) \in \mathbb{R}^V$. For convenience we will assume that the family $\boldsymbol{p}$ is given by a family of coupling constants $\boldsymbol{J} = (J_{ij} \in [0,+\infty] : \{i,j\} \in E)$ and the inverse temperature $\beta > 0$ so that $p_{ij} = 1 - \exp(-2\beta J_{ij})$. Following [17], the probability measure of the random-cluster model with external field $\boldsymbol{h}$ on the finite volume G is defined for each $\omega \in \{0,1\}^E$ by

$$\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega) = \frac{1}{\mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\boldsymbol{h},G}} B_{\boldsymbol{J}}(\omega) \prod_{\alpha=1}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha(\omega))\big),$$

where $\boldsymbol{h}(K_\alpha(\omega)) \equiv \beta \sum_{i \in K_\alpha(\omega)} h_i$, with the sets $K_1(\omega), \ldots, K_{k(\omega,G)}(\omega)$ being composed by the connected components of $(V, \eta(\omega))$, $B_{\boldsymbol{J}}(\omega)$ representing the Bernoulli factors

$$B_{\boldsymbol{J}}(\omega) \equiv \prod_{\{i,j\}:\omega_{ij}=1} p_{ij} \prod_{\{i,j\}:\omega_{ij}=0} (1 - p_{ij}) \tag{5}$$

and $\mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\boldsymbol{h},G}$ being the partition function

$$\mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\boldsymbol{h},G} = \sum_{\omega \in \{0,1\}^E} B_{\boldsymbol{J}}(\omega) \prod_{\alpha=1}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha(\omega))\big).$$

From now on, in order to ease the notation, we shall omit the ω-dependence from the components $K_1(\omega), \ldots, K_{k(\omega,G)}(\omega)$ and simply write $K_1, \ldots, K_{k(\omega,G)}$.

Part II
Free boundary conditions

6 The Edwards-Sokal Coupling

In this section we present the Edwards-Sokal model on a finite graph $G = (V, E)$. The configuration space of this model is given by the cartesian product $\{-1,+1\}^V \times \{0,1\}^E$. A pair of configurations $\sigma \in \{-1,+1\}^V$ and $\omega \in \{0,1\}^E$ are deemed consistent if $\omega_{ij} = 1 \Rightarrow \sigma_i = \sigma_j$, $\forall\ \{i,j\} \in E$. The indicator function of the consistency of a pair $(\sigma, \omega) \in \{-1,+1\}^V \times \{0,1\}^E$, is denoted by

$$\Delta(\sigma, \omega) \equiv \mathbb{1}_{\{(\xi,\eta) \in \{-1,+1\}^V \times \{0,1\}^E :\ \text{if } \eta_{ij}=1 \text{ then } \xi_i = \xi_j\}}(\sigma, \omega).$$

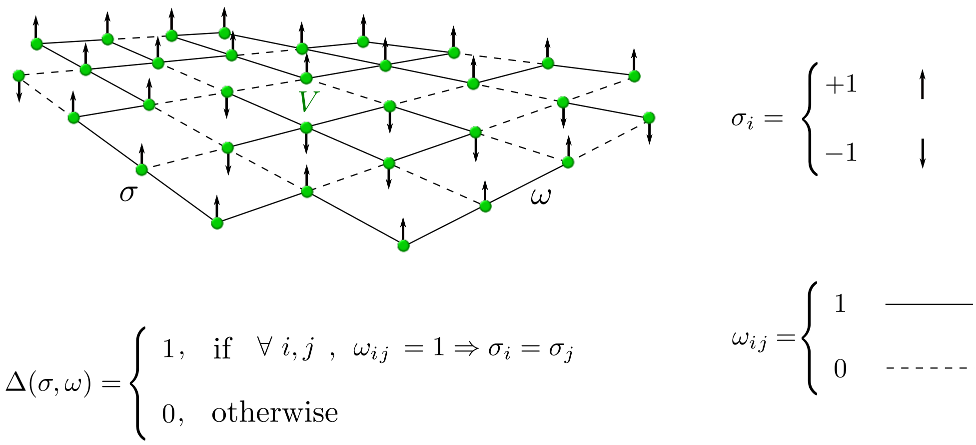


Figure 2: *An example of spin-edge compatible configuration.*

Similarly to the previous section, we fix $\beta > 0$, coupling constants $\boldsymbol{J}$ and magnetic field $\boldsymbol{h}$, and put $p_{ij} \equiv 1-\exp(-2\beta J_{ij})$. In the Edwards-Sokal model, the probability of a configuration (σ, ω) on a finite volume $G = (V, E) \subset \mathbb{L}$ is defined by

$$\nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) \equiv \frac{1}{\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}} B_{\boldsymbol{J}}(\omega)\Delta(\sigma,\omega) \times \exp\big(\beta \sum_{i\in V} h_i\,(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\,\big),$$

where $B_{\boldsymbol{J}}$ represents the Bernoulli factors introduced in (5), $\delta_{\sigma_i,\sigma_j}$ is Kronecker's delta function and $\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}$ is the partition function.

Lemma 1. *Let $G = (V, E)$ be a finite graph and consider the 2-state Potts model on G with free boundary condition and Hamiltonian given by (4). Suppose that $p_{ij} \equiv 1-\exp(-2\beta J_{ij})$ and $\hat{\sigma}$ is obtained from σ as in the Proposition 1. Then*

$$\exp\big(-2\beta(\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat{\sigma}) + \textstyle\sum_{\{i,j\}\in E} J_{ij})\big)$$
$$=$$
$$\sum_{\omega}\Big(\prod_{\{i,j\}:\omega_{ij}=1} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}:\omega_{ij}=0}(1-p_{ij})\Big) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).$$

Proof. Using the relation between $\hat{\sigma}$ and σ, we first obtain the following equality

$$\exp\big(-2\beta(\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat{\sigma}) + \textstyle\sum_{\{i,j\}\in E} J_{ij})\big)$$
$$=$$
$$\exp\big(2\beta(\textstyle\sum_{\{i,j\}\in E} J_{ij}(\delta_{\sigma_i,\sigma_j} - 1) + \sum_{i\in V} h_i \frac{1}{2}(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1}))\big).$$

By using that $p_{ij} = 1 - \exp(-2\beta J_{ij})$ and the elementary properties of the exponential, a straightforward computation shows that the above expression equals

$$\prod_{\{i,j\}\in E} \big(p_{ij}\delta_{\sigma_i,\sigma_j} + (1-p_{ij})\big) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).$$

By expanding the product in the above expression we get

$$\sum_{E'\subset E}\Big(\prod_{\{i,j\}\in E'} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}\in E\setminus E'} (1-p_{ij})\Big) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).$$

Since the collection of all the induced subgraphs of G is in bijection with $\{0,1\}^E$ we can rewrite the last expression as follows

$$\sum_{\omega}\Big(\prod_{\{i,j\}:\omega_{ij}=1} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}:\omega_{ij}=0} (1-p_{ij})\Big) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big)$$

which completes the proof. □

Lemma 2. *Under the hypothesis of Lemma 1 we can show that there exists a constant $C \equiv C(\beta, G) > 0$ so that*

$$\mathscr{Z}^{\mathrm{Potts}}_{2\beta,\boldsymbol{h},2,V} = C\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}.$$

Proof. Observe that

$$\begin{aligned}
\mathscr{Z}^{\mathrm{Potts}}_{2\beta,\boldsymbol{h},2,V} &= \sum_{\hat\sigma\in\{1,2\}^V} \exp\big(-2\beta\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat\sigma)\big) \\
&= \frac{1}{C}\sum_{\hat\sigma\in\{1,2\}^V} \exp\big(-2\beta(\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat\sigma) + \textstyle\sum_{\{i,j\}\in E} J_{ij})\big),
\end{aligned}$$

where $C \equiv \exp(2\beta\sum_{\{i,j\}\in E} J_{ij}) > 0$. From Lemma 1 it follows that the right-hand-side above is equal to

$$C\sum_{\sigma,\omega}\Big(\prod_{\{i,j\}:\omega_{ij}=1} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}:\omega_{ij}=0} (1-p_{ij})\Big)\times\exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big)$$

$$=$$

$$C\sum_{\sigma,\omega}\Big(\prod_{\{i,j\}:\omega_{ij}=1} p_{ij} \prod_{\{i,j\}:\omega_{ij}=0} (1-p_{ij})\Big)\Delta(\sigma,\omega)\times\exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big)$$

$$=$$

$$C\sum_{\sigma,\omega} B_{\boldsymbol{J}}(\omega)\Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big) = C\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}. \quad \square$$

Lemma 3. *Let G be a finite graph and fix an edge configuration $\omega \in \{0,1\}^E$. If $\boldsymbol{h}(K_\alpha) \equiv \beta\sum_{i\in K_\alpha} h_i$, where $K_1, \ldots, K_{k(\omega,G)}$ denote the connected components of $(V, \eta(\omega))$ we have*

$$\sum_{\sigma\in\{-1,+1\}^V} \Delta(\sigma,\omega) \times \exp\big(\beta \sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big) = \prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right).$$

Proof. For a fixed ω, if $K_1, \ldots, K_{k(\omega,G)}$ denotes the decomposition of V on its connected components, we get

$$\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1}) = \sum_{\alpha=1}^{k(\omega,G)} \sum_{i\in K_\alpha} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1}).$$

For each spin configuration $\sigma \in \{-1,1\}^V$ so that the pair $(\sigma,\omega) \in \{-1,1\}^V \times \{0,1\}^E$ satisfies $\Delta(\sigma,\omega) = 1$, we have that the value of all the spins in the same component has the same sign, see Figure 2. From the elementary properties of the exponential function we obtain the following equality

$$\Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big)$$
$$=$$
$$\Delta(\sigma,\omega) \times \prod_{\alpha=1}^{k(\omega,G)} \exp\big(\beta \textstyle\sum_{i\in K_\alpha} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).$$

Since $V = \sqcup_{\alpha=1}^{k(\omega,G)} K_\alpha$, we have a natural bijection between the following spaces:

$$\{-1,+1\}^V \quad \text{and} \quad \prod_{\alpha=1}^{k(\omega,G)} \{-1,1\}^{K_\alpha}.$$

For sake of simplicity, we denote a generic element of the cartesian product $\prod_{\alpha=1}^{k(\omega,G)}\{-1,1\}^{K_\alpha}$ by $(\sigma_{K_1}, \ldots, \sigma_{K_{k(\omega,G)}})$, where $\sigma_{K_j} \equiv (\sigma_i : i \in K_j)$, $\forall j = 1, \ldots, k(\omega,G)$. In this way we can simply write $\sigma = (\sigma_{K_1}, \ldots, \sigma_{K_{k(\omega,G)}})$. By abusing the notation, we write

$$\Delta(\sigma,\omega) = \prod_{\alpha=1}^{k(\omega,G)} \Delta(\sigma_{K_\alpha},\omega).$$

Using the previous observations and $h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1}) = h_i\sigma_i$ we obtain

$$\sum_{\sigma} \Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big)$$

$$= \sum_{(\sigma_{K_1},\ldots,\sigma_{K_{k(\omega,G)}})} \prod_{\alpha=1}^{k(\omega,G)} \Delta(\sigma_{K\alpha},\omega)\exp\big(\beta \textstyle\sum_{i\in K_\alpha} h_i\sigma_i\big)$$

$$= \prod_{\alpha=1}^{k(\omega,G)} \sum_{\sigma_{K_\alpha}} \Delta(\sigma_{K\alpha},\omega)\exp\big(\beta \textstyle\sum_{i\in K_\alpha} h_i\sigma_i\big).$$

Because of the consistency condition, for each fixed α, the sum appearing on the right-hand-side above has exactly two non zero terms where the spins in K_α take the values ± 1. Therefore the above expression reduces to

$$\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\big(\beta \sum_{i\in K_\alpha} h_i\big),$$

thus the lemma is proved. □

Lemma 4. *For any finite graph $G = (V, E)$, $\boldsymbol{p} = (p_{ij} : i, j \in \mathbb{V})$, $\boldsymbol{h} = (h_i : i \in V)$ and $\beta > 0$ we have*

$$\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G} = \mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\boldsymbol{h},G}.$$

Proof. The proof of this lemma is trivial given the above results. It is enough to change the sum order of the partition function of the Edwards-Sokal model, apply Lemma 3 and finally use the definition of the partition function of the random-cluster model as we show below

$$\begin{aligned}\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G} &= \sum_{\omega} B_{\boldsymbol{J}}(\omega) \sum_{\sigma} \Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big) \\ &= \sum_{\omega} B_{\boldsymbol{J}}(\omega) \prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\mathbf{h}(K_\alpha)\right) = \mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\boldsymbol{h},G}.\end{aligned}$$

□

In the sequel we prove the main result of this section. The technicalities of the proof were worked out in the previous lemmas and now the remaining task is to simply piece them together.

Theorem 1 (The marginals of $\nu_{\boldsymbol{p},\boldsymbol{h},G}$)**.** *Let $G = (V, E)$ be a finite graph, $\beta > 0$, $\boldsymbol{p} = (p_{ij} : i, j \in \mathbb{V})$ as above and $\boldsymbol{h} = (h_i : i \in V)$ an external field. Then*

(1) $$\sum_{\omega\in\{0,1\}^E} \nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) = \lambda_{\beta,\boldsymbol{h},V}(\sigma) \qquad \textit{(spin-marginal of } \nu_{\boldsymbol{p},\boldsymbol{h},G})$$

(2) $$\sum_{\sigma\in\{-1,+1\}^V} \nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) = \phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega). \qquad \textit{(edge-marginal of } \nu_{\boldsymbol{p},\boldsymbol{h},G})$$

Proof. We first prove *(1)*. Using the definition of Bernoulli factors $B_{\boldsymbol{J}}$, and Lemma 2 with $C \equiv \exp(2\beta \sum_{\{i,j\}\in E} J_{ij}) > 0$, we obtain

$$\begin{aligned}
&\sum_{\omega} \nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) \\
&\quad = \frac{1}{\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}} \sum_{\omega} B_{\boldsymbol{J}}(\omega)\Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big) \\
&\quad = \frac{C}{\mathscr{Z}^{\mathrm{Potts}}_{2\beta,\boldsymbol{h},2,V}} \sum_{\omega} \big(\textstyle\prod_{\{i,j\}:\omega_{ij}=1} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}:\omega_{ij}=0}(1-p_{ij})\big)\times \\
&\qquad\qquad \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).
\end{aligned}$$

By applying Lemma 1, it follows that the rhs above is equal to

$$\frac{C}{\mathscr{Z}^{\mathrm{Potts}}_{2\beta,\boldsymbol{h},V}} \exp\big(-2\beta(\mathscr{H}^{\mathrm{Potts}}_{\boldsymbol{h},2,V}(\hat{\sigma}) + \textstyle\sum_{\{i,j\}\in E} J_{ij})\big) = \pi_{2\beta,\boldsymbol{h},2,V}(\hat{\sigma}) = \lambda_{\beta,\boldsymbol{h},V}(\sigma).$$

To prove *(2)* it is enough to use Lemmas 3 and 4 as follows

$$\begin{aligned}
\sum_{\sigma} \nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) &= \frac{B_{\boldsymbol{J}}(\omega)}{\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\boldsymbol{h},G}} \sum_{\sigma} \Delta(\sigma,\omega) \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big) \\
&= \frac{1}{\mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\mathbf{h},G}} B_{\boldsymbol{J}}(\omega) \prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right) = \phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).
\end{aligned}$$

□

Corollary 1 (Conditional measure of $\nu_{\beta,\boldsymbol{h},G}$)**.** *Let $\omega \in \{0,1\}^E$ be a fixed edge configuration. For each $\sigma \in \{-1,+1\}^V$ we have that*

$$\nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma|\omega) = \frac{\Delta(\sigma,\omega)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)} \times \exp\big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\big).$$

Proof. From Proposition 1 and Lemma 1 we have, for any random variable $g : \{-1, 1\}^V \to \mathbb{R}$, that

$$\lambda_{\beta,\boldsymbol{h},V}(g) = \sum_{\hat{\sigma}\in\{1,2\}^V} g(\hat{\sigma})\pi_{2\beta,\boldsymbol{h},2,V}(\hat{\sigma})$$

$$= \frac{C}{\mathscr{Z}^{\text{Potts}}_{2\beta,\boldsymbol{h},2,V}} \sum_{\sigma\in\{-1,+1\}^V} g(\sigma) \sum_{\omega\in\{0,1\}^E} \Big(\prod_{\{i,j\}:\omega_{ij}=1} p_{ij}\delta_{\sigma_i,\sigma_j} \prod_{\{i,j\}:\omega_{ij}=0} (1-p_{ij})\Big)\times$$

$$\times \exp\Big(\beta \sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\Big),$$

where $C \equiv \exp(2\beta \sum_{\{i,j\}\in E} J_{ij}) > 0$. By changing the order of the sums in the last expression we get

$$\frac{C}{\mathscr{Z}^{\text{Potts}}_{2\beta,\boldsymbol{h},2,V}} \sum_{\omega\in\{0,1\}^E} B_{\boldsymbol{J}}(\omega) \sum_{\sigma\in\{-1,+1\}^E} g(\sigma)\,\Delta(\sigma,\omega)\exp\Big(\beta \textstyle\sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\Big).$$

According to Lemmas 2 and 4, we have $C^{-1}\mathscr{Z}^{\text{Potts}}_{2\beta,\boldsymbol{h},2,V} = \mathscr{Z}^{\text{ES}}_{\boldsymbol{p},\boldsymbol{h},G} = \mathscr{Z}^{\text{RC}}_{\boldsymbol{p},\boldsymbol{h},G}$. By introducing the product appearing in the definition of the random-cluster model, we can see that the above expression is equal to

$$\sum_{\omega} \frac{B_{\boldsymbol{J}}(\omega)\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}{\mathscr{Z}^{\text{RC}}_{\boldsymbol{p},\boldsymbol{h},G}}\times$$

$$\times \sum_{\sigma} g(\sigma)\frac{\Delta(\sigma,\omega)\times\exp\Big(\beta \sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\Big)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}$$

$$= \sum_{\omega,\sigma} g(\sigma)\frac{\Delta(\sigma,\omega)\times\exp\Big(\beta \sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\Big)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

Therefore

$$\lambda_{\beta,\boldsymbol{h},V}(g) = \sum_{\omega}\left[\sum_{\sigma} g(\sigma)\frac{\Delta(\sigma,\omega)\exp\Big(\beta \sum_{i\in V} h_i(\delta_{\sigma_i,1} - \delta_{\sigma_i,-1})\Big)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

On the other hand, we get from Theorem 1 that

$$\lambda_{\beta,\boldsymbol{h},V}(g) = \sum_{\omega,\sigma} g(\sigma)\nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma,\omega) = \sum_{\omega}\Big[\sum_{\sigma} g(\sigma)\nu_{\boldsymbol{p},\boldsymbol{h},G}(\sigma|\omega)\Big]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

The proof is completed upon comparison of the two previous expressions. □

7 Two-point function

The **two-point function** of the q-state Potts model is defined by

$$\tau_{\beta,\hat{\boldsymbol{h}},q,V}(x,y) \equiv \pi_{\beta,\hat{\boldsymbol{h}},q,V}(\hat{\sigma}_x = \hat{\sigma}_y) - \frac{1}{q}.$$

The term $1/q$ represents the probability that two independent spins uniformly chosen have the same value. In the random-cluster model, the **connectivity function** plays the role of the two-point function of the Potts model. This function is precisely the probability, with respect to $\phi_{\boldsymbol{p},\boldsymbol{h},G}$, that x and y are in the same connected component, notation $\phi_{\boldsymbol{p},\boldsymbol{h},G}(x \leftrightarrow y)$.

Lemma 5. *Let $G = (V,E)$ be a finite graph and $x, y \in V$ two distinct vertices. Fix an edge configuration $\omega \in \{0,1\}^E$. If $x \not\leftrightarrow y$ in ω, then*

$$\sum_{\sigma\in\{-1,+1\}^V} \mathbb{1}_{\{\sigma_x=\sigma_y\}}\Delta(\sigma,\omega)\exp\big(\beta \textstyle\sum_{i\in V} h_i\sigma_i\big)$$
$$= 2\cosh\left(\boldsymbol{h}(K_t)+\boldsymbol{h}(K_u)\right)\prod_{\substack{\alpha=1\\ \alpha\neq t,u}}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha)\big),$$

where $K_t \equiv K_t(\omega)$ and $K_u \equiv K_u(\omega)$ are two disjoint connected components containing the vertices x and y, respectively.

Proof. The basic ideas used to prove this lemma are the same we employed to prove Lemma 3, which we once more present for sake of completeness. Let $V = \sqcup_{\alpha=1}^{k(\omega,G)} K_\alpha$ be a decomposition in terms of the connected components of the graph $(V, \eta(\omega))$. We recall that $\{-1,+1\}^V \cong \prod_{\alpha=1}^{k(\omega,G)}\{-1,1\}^{K_\alpha}$ and its elements are denoted by $(\sigma_{K_1},\ldots,\sigma_{K_{k(\omega,G)}})$, with $\sigma_{K_j} \equiv (\sigma_i : i \in K_j),\ \forall j = 1,\ldots,k(\omega,G)$. We also use the natural identification $\sigma = (\sigma_{K_1},\ldots,\sigma_{K_{k(\omega,G)}})$.

Suppose that $x \not\leftrightarrow y$ in ω. Denote K_t and K_u the components containing the vertices x and y, respectively. Taking into account the decomposition of V mentioned above, we have

$$\sum_{\sigma} \mathbb{1}_{\{\sigma_x=\sigma_y\}}\Delta(\sigma,\omega)\exp\big(\beta \textstyle\sum_{i\in V} h_i\sigma_i\big)$$
$$= \sum_{\sigma} \mathbb{1}_{\{\sigma_x=\sigma_y=+1\}}\Delta(\sigma,\omega)\prod_{\alpha=1}^{k(\omega,G)} \exp\big(\beta \textstyle\sum_{i\in K_\alpha} h_i\sigma_i\big)$$
$$+ \sum_{\sigma} \mathbb{1}_{\{\sigma_x=\sigma_y=-1\}}\Delta(\sigma,\omega)\prod_{\alpha=1}^{k(\omega,G)} \exp\big(\beta \textstyle\sum_{i\in K_\alpha} h_i\sigma_i\big).$$

As previously observed, $\Delta(\sigma,\omega)=\prod_{\alpha=1}^{k(\omega,G)}\Delta(\sigma_{K_\alpha},\omega)$, so from a simple computation we get that the expression above is equal to

$$2\cosh\big(\boldsymbol{h}(K_t)+\boldsymbol{h}(K_u)\big)\prod_{\substack{\alpha=1\\ \alpha\neq t,u}}^{k(\omega,G)}\sum_{\sigma_{K_\alpha}}\Delta(\sigma_{K_\alpha},\omega)\exp\big(\beta\sum_{i\in K_\alpha}h_i\sigma_i\big). \tag{6}$$

Because of the consistency condition, the sums above over σ_{K_α} have actually two non-null terms. In each of such term the value of the spins is constant and therefore the product simplifies to

$$\prod_{\substack{\alpha=1\\ \alpha\neq t,u}}^{k(\omega,G)}\sum_{\sigma_{K_\alpha}}\Delta(\sigma_{K_\alpha},\omega)\exp\big(\beta\textstyle\sum_{i\in K_\alpha}h_i\sigma_i\big)$$
$$=\prod_{\substack{\alpha=1\\ \alpha\neq t,u}}^{k(\omega,G)}\big(\exp(\boldsymbol{h}(K_\alpha))+\exp(-\boldsymbol{h}(K_\alpha))\big)=\prod_{\substack{\alpha=1\\ \alpha\neq t,u}}^{k(\omega,G)}2\cosh\big(\boldsymbol{h}(K_\alpha)\big).$$

Finally, by replacing the last expression in (6), we end the proof. □

Lemma 5 is vital for the most important result of this section, which is the next theorem. We state below the theorem for the 2-state Potts model, but in fact the theorem is valid for much more general Potts models. The general case is treated in the last section.

Theorem 2 (Correlation-connectivity)**.** *Let $G=(V,E)$ be a finite graph and x,y two distinct vertices in V. Then*

$$\tau_{2\beta,\boldsymbol{h},2,V}(x,y)=\frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}(x\leftrightarrow y)+\frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\mathbb{1}_{\{x\not\leftrightarrow y\}}\cdot\tanh\left(\boldsymbol{h}(K_t)\right)\cdot\tanh\left(\boldsymbol{h}(K_u)\right)\big),$$

where $K_t\equiv K_t(\omega)$ e $K_u\equiv K_u(\omega)$ are two disjoint connected components containing the vertices x and y, respectively.

Proof. By using the definition of the two-point function and Theorem 1 we get

$$\begin{aligned}
\tau_{2\beta,\boldsymbol{h},2,V}(x,y) &= \pi_{2\beta,\boldsymbol{h},2,V}(\hat{\sigma}_x=\hat{\sigma}_y)-\frac{1}{2}\\
&= \sum_{\hat{\sigma}\in\{1,2\}^V}\Big(\mathbb{1}_{\{\hat{\sigma}_x=\hat{\sigma}_y\}}-\frac{1}{2}\Big)\pi_{2\beta,\boldsymbol{h},2,V}(\hat{\sigma})\\
&= \sum_{(\sigma,\omega)\in\{-1,+1\}^V\times\{0,1\}^E}\Big(\mathbb{1}_{\{\sigma_x=\sigma_y\}}-\frac{1}{2}\Big)\nu_{\beta,\boldsymbol{h},G}(\sigma,\omega)\\
&= \sum_{\omega\in\{0,1\}^E}\Big[\sum_{\sigma\in\{-1,+1\}^V}\Big(\mathbb{1}_{\{\sigma_x=\sigma_y\}}-\frac{1}{2}\Big)\nu_{\beta,\boldsymbol{h},G}(\sigma|\omega)\Big]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).
\end{aligned}$$

Since $\delta_{\sigma_i,\sigma_j}=\frac{1}{2}(1+\sigma_i\sigma_j)$, it follows from Corollary 1 that the rhs above is

$$\begin{aligned}
&= \sum_{\omega}\left[\sum_{\sigma}\Big(\mathbb{1}_{\{\sigma_x=\sigma_y\}}-\frac{1}{2}\Big)\frac{\Delta(\sigma,\omega)\exp\big(\beta\sum_{i\in V}h_i\sigma_i\big)}{\prod_{\alpha=1}^{k(\omega,G)}2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)\\
&= \sum_{\omega}\mathbb{1}_{\{x\leftrightarrow y\}}(\omega)\left[\sum_{\sigma}\Big(\mathbb{1}_{\{\sigma_x=\sigma_y\}}-\frac{1}{2}\Big)\frac{\Delta(\sigma,\omega)\exp\big(\beta\sum_{i\in V}h_i\sigma_i\big)}{\prod_{\alpha=1}^{k(\omega,G)}2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)\\
&+ \sum_{\omega}\mathbb{1}_{\{x\not\leftrightarrow y\}}(\omega)\left[\sum_{\sigma}\Big(\mathbb{1}_{\{\sigma_x=\sigma_y\}}-\frac{1}{2}\Big)\frac{\Delta(\sigma,\omega)\exp\big(\beta\sum_{i\in V}h_i\sigma_i\big)}{\prod_{\alpha=1}^{k(\omega,G)}2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)\\
&\equiv I_1+I_2.
\end{aligned}$$

Notice that, as long as $x\leftrightarrow y$ in ω and the pair (σ,ω) is consistent, then $\sigma_x=\sigma_y$. From this observation and Lemma 3, it follows that

$$I_1=\frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}(x\leftrightarrow y).$$

On the other hand, applying Lemma 3 again yields

$$\begin{aligned}
I_2 &= -\frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}(x\not\leftrightarrow y)\\
&\quad+\sum_{\omega}\mathbb{1}_{\{x\not\leftrightarrow y\}}(\omega)\left[\sum_{\sigma}\mathbb{1}_{\{\sigma_x=\sigma_y\}}\frac{\Delta(\sigma,\omega)\exp\big(\beta\sum_{i\in V}h_i\sigma_i\big)}{\prod_{\alpha=1}^{k(\omega,G)}2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)\\
&\equiv -\frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}(x\not\leftrightarrow y)+\tilde{I}_2. \qquad (7)
\end{aligned}$$

Now we work on $\tilde{I}_2$. By using Lemma 5, we have

$$\tilde{I}_2 = \sum_{\omega} \mathbb{1}_{\{x \not\leftrightarrow y\}} \left[\sum_{\sigma} \mathbb{1}_{\{\sigma_x = \sigma_y\}} \frac{\Delta(\sigma,\omega) \exp\left(\beta \sum_{i\in V} h_i \sigma_i\right)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)} \right] \phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)$$

$$= \sum_{\omega} \mathbb{1}_{\{x \not\leftrightarrow y\}}(\omega) \left[\frac{2\cosh\left(\boldsymbol{h}(K_t) + \boldsymbol{h}(K_u)\right) \prod_{\substack{\alpha=1 \\ \alpha\neq t,u}}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)} \right] \phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)$$

$$= \frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}\Big(\mathbb{1}_{\{x \not\leftrightarrow y\}} \cdot \frac{\cosh\left(\boldsymbol{h}(K_t) + \boldsymbol{h}(K_u)\right)}{\cosh\left(\boldsymbol{h}(K_t)\right) \cdot \cosh\left(\boldsymbol{h}(K_u)\right)}\Big)$$

$$= \frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\mathbb{1}_{\{x \not\leftrightarrow y\}} \cdot \big\{1 + \tanh\left(\boldsymbol{h}(K_t)\right) \cdot \tanh\left(\boldsymbol{h}(K_u)\right)\big\}\big).$$

Replacing the last expression in (7) we get that

$$I_2 = \frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\mathbb{1}_{\{x \not\leftrightarrow y\}} \cdot \tanh\left(\boldsymbol{h}(K_t)\right) \cdot \tanh\left(\boldsymbol{h}(K_u)\right)\big). \tag{8}$$

Since $\tau_{2\beta,\boldsymbol{h},2,V}(x,y) = I_1 + I_2$, the theorem follows. □

Remark 2. *Notice that in the absence of the magnetic field, i.e.* $\boldsymbol{h} \equiv 0$*, the conclusion of the Theorem 2 reduces to*

$$\tau_{2\beta,0,2,V}(x,y) = \frac{1}{2}\phi_{\boldsymbol{p},0,G}(x \leftrightarrow y), \quad \forall x,y \in V$$

which is a well known identity for the Ising/Potts model with $q = 2$*, see [27] Theorem 1.16, p. 11.*

Corollary 2. *The spin-spin correlation of the Ising model on the finite volume* V *satisfies the following identity for any magnetic field* $\boldsymbol{h} \in \mathbb{R}^V$

$$\lambda_{\beta,\boldsymbol{h},V}(\sigma_x\sigma_y) = \phi_{\boldsymbol{p},\boldsymbol{h},G}(x \leftrightarrow y) + \phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\mathbb{1}_{\{x \not\leftrightarrow y\}} \cdot \tanh\left(\boldsymbol{h}(K_t)\right) \cdot \tanh\left(\boldsymbol{h}(K_u)\right)\big).$$

Proof. This follows easily from the definition of the expected value and Theorem 2 since $\lambda_{\beta,\boldsymbol{h},V}(\sigma_x\sigma_y) = \lambda_{\beta,\boldsymbol{h},V}(\sigma_i = \sigma_j) - \lambda_{\beta,\boldsymbol{h},V}(\sigma_i \neq \sigma_j) = 2\lambda_{\beta,\boldsymbol{h},V}(\sigma_i = \sigma_j) - 1 = 2\left[\pi_{2\beta,\boldsymbol{h},2,V}(\hat{\sigma}_x = \hat{\sigma}_y) - \frac{1}{2}\right] = 2\tau_{2\beta,\boldsymbol{h},2,V}(x,y)$. □

Remark 3. *If we consider the Ising model on* G *without magnetic field, from Corollary 2 we get* $\lambda_{\beta,0,V}(\sigma_x\sigma_y) = \phi_{\boldsymbol{p},0,G}(x \leftrightarrow y)$*,* $\forall x,y \in V$*.*

8 Applications

Spin-spin correlations. Corollary 2 can be used to obtain some correlation inequalities. Keeping the notation of Theorem 2 and supposing that $h_i \geqslant 0$ for all $i \in V$, it follows from the monotonicity of the hyperbolic tangent that $\tanh(\beta h_x) \leqslant \tanh(\boldsymbol{h}(K_t))$ and $\tanh(\beta h_y) \leqslant \tanh(\boldsymbol{h}(K_u))$. These estimates together with Corollary 2 give us the following lower bound $\phi_{\boldsymbol{p},\boldsymbol{h},G}\ (x \not\leftrightarrow y) \tanh(\beta h_x) \tanh(\beta h_y) \leqslant \lambda_{\beta,\boldsymbol{h},V}(\sigma_x\sigma_y)$. A simple computation shows that $P_{\boldsymbol{p}}(x \not\leftrightarrow y) \leqslant \phi_{\boldsymbol{p},\boldsymbol{h},G}(x \not\leftrightarrow y)$, where $P_{\boldsymbol{p}}$ is the probability measure of the independent bond percolation model with parameter $\boldsymbol{p}$. Supposing that $\boldsymbol{p} \equiv p$ (the homogeneous model) and $p < p_c(\mathbb{V})$, for any given $\varepsilon > 0$, if the distance between x and y is large enough then $(1-\varepsilon) \tanh(\beta h_x) \tanh(\beta h_y) \leqslant \lambda_{\beta,\boldsymbol{h},V}(\sigma_x\sigma_y)$, which, of course, can also be (better) obtained by the GKS inequality.

Under the above assumptions, Corollary 2 also gives us an upper bound in terms of the iid Bernoulli bond percolation model, which is $\lambda_{\beta,\boldsymbol{h},V}(\sigma_x\sigma_y) \leqslant e^{-C(\beta)d_G(x,y)} + P_p\big(\tanh(\boldsymbol{h}(K_t)) \cdot \tanh(\boldsymbol{h}(K_u))\big)$, where at this point we are assuming $J_{ij} \equiv J$ and $p = 1 - e^{-\beta J}$. To obtain the asymptotic behavior of the second term in the rhs above, one needs to impose extra conditions on the geometry of the graph and the decay ratio of the magnetic field.

Expected value and distribution function of a single spin.

Lemma 6. *Consider a finite graph $G = (V, E)$, $x \in V$ and $\omega \in \{0,1\}^E$ a fixed edge configuration. Then*

$$\sum_{\sigma} \mathbb{1}_{\{\sigma_x = \pm 1\}} \Delta(\sigma, \omega) \exp\left(\beta \textstyle\sum_{i\in V} h_i\sigma_i\right) = \exp\left(\pm \boldsymbol{h}(K_t)\right) \prod_{\substack{\alpha=1\\ \alpha\neq t}}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha)\big),$$

where $K_t \equiv K_t(\omega)$ is the connected component containing the vertex x.

Proof. To prove this lemma we proceed, mutatis mutandis, as in the proof of Lemma 5. □

Theorem 3 (Distribution function)**.** *Let $G = (V, E)$ be a finite graph. We have, for any fixed $x \in V$, that*

$$\lambda_{\beta,\boldsymbol{h},V}(\sigma_x = \pm 1) = \frac{1}{2} \pm \frac{1}{2}\phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\tanh(\boldsymbol{h}(K_t))\big),$$

where $K_t(\omega) \equiv K_t$ is the connected component containing x.

Proof. From Theorem 1, it follows that

$$\lambda_{\beta,\boldsymbol{h},V}(\sigma_x = \pm 1) = \sum_{(\sigma,\omega)\in\{-1,+1\}^V\times\{0,1\}^E} \mathbb{1}_{\{\sigma_x=\pm1\}}\nu_{\beta,\boldsymbol{h},G}(\sigma,\omega)$$

$$= \sum_{\omega\in\{0,1\}^E}\Big[\sum_{\sigma\in\{-1,+1\}^V} \mathbb{1}_{\{\sigma_x=\pm1\}}\nu_{\beta,\boldsymbol{h},G}(\sigma|\omega)\Big]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

Using Corollary 1, the above expression can be rewritten as

$$\sum_{\omega\in\{0,1\}^E}\left[\sum_{\sigma\in\{-1,+1\}^V} \mathbb{1}_{\{\sigma_x=\pm1\}}\frac{\Delta(\sigma,\omega)\exp\big(\beta\sum_{i\in V}h_i\sigma_i\big)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\left(\boldsymbol{h}(K_\alpha)\right)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

Using now Lema 6, we can see that the above expression is equal to

$$\sum_{\omega\in\{0,1\}^E}\left[\frac{\exp\left(\pm\boldsymbol{h}(K_t)\right)\prod_{\substack{\alpha=1\\ \alpha\neq t}}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha)\big)}{\prod_{\alpha=1}^{k(\omega,G)} 2\cosh\big(\boldsymbol{h}(K_\alpha)\big)}\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega)$$

$$= \frac{1}{2}\sum_{\omega\in\{0,1\}^E}\left[1\pm\tanh\left(\boldsymbol{h}(K_t)\right)\right]\phi_{\boldsymbol{p},\boldsymbol{h},G}(\omega).$$

□

Corollary 3. *Under the hypothesis of Theorem 3, we have that*

$$\lambda_{\beta,\boldsymbol{h},V}(\sigma_x) = \phi_{\boldsymbol{p},\boldsymbol{h},G}\big(\tanh\left(\boldsymbol{h}(K_t)\right)\big).$$

Proof. The proof follows directly from Theorem 3. □

9 General Potts models in external fields

In this last section we state two propositions establishing a graphical representation for the two-point function of the q-state Potts model with general external fields, defined in the Section 4, in terms of the connectivity of the random-cluster model introduced below. The techniques employed to prove these results are similar to the ones we used in the previous section and therefore the proofs are omitted.

Given a finite graph $G = (V, E)$, coupling constants $\boldsymbol{J} = (J_{ij} \geqslant 0 : \{i,j\} \in E)$ and $\hat{\boldsymbol{h}}$ a magnetic field as defined in the Section 4, for each $\omega \in$

$\{0,1\}^E$ we define the finite-volume Gibbs measure of the (general) random-cluster model in external field by

$$\phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(\omega) = \frac{1}{\mathscr{Z}^{\mathrm{RC}}_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}} B_{\boldsymbol{J},q}(\omega) \prod_{\alpha=1}^{k(\omega,G)} \sum_{p=1}^{q} \exp\big(\beta \sum_{i\in K_\alpha} h_{i,p}\big), \tag{9}$$

where K_α is defined exactly as in the Section 5 and $B_{\boldsymbol{J},q}(\omega)$ is similar to the Bernoulli factor of the Section 5 with exception that $p_{ij} = 1-\exp(-q\beta J_{ij})$.

The Edwards-Sokal measure is generalized to

$$\nu_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(\hat{\sigma},\omega) \equiv \frac{1}{\mathscr{Z}^{\mathrm{ES}}_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}} B_{\boldsymbol{J},q}(\omega)\Delta_q(\hat{\sigma},\omega) \times \exp\big(\beta \sum_{i\in V}\sum_{p=1}^{q} h_{i,p}\delta_{\hat{\sigma}_i,p}\big). \tag{10}$$

Proposition 2. *Consider the Potts model with Hamiltonian given by (3), densities $p_{ij} \equiv 1-\exp(-q\beta J_{ij})$ and $q\in\{2,3,\ldots\}$ fixed. For any pair of vertices $x,y\in V$ we have that*

$$\tau_{q\beta,\hat{\boldsymbol{h}},q,V}(x,y) = \Big(1-\frac{1}{q}\Big)\phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(x\leftrightarrow y) + \phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}\Big(\mathbb{1}_{\{x\not\leftrightarrow y\}}\cdot\big\{H_{\hat{\boldsymbol{h}}}(K_t,K_u)-\frac{1}{q}\big\}\Big),$$

where the random variable $H_{\hat{\boldsymbol{h}}}(K_t,K_u)$ is given by

$$H_{\hat{\boldsymbol{h}}}(K_t,K_u) \equiv \frac{\sum_{r=1}^{q}\exp\big(\beta\sum_{i\in K_t}h_{i,r}+\beta\sum_{i\in K_u}h_{i,r}\big)}{\sum_{r=1}^{q}\exp\big(\beta\sum_{i\in K_t}h_{i,r}\big)\cdot\sum_{r=1}^{q}\exp\big(\beta\sum_{i\in K_u}h_{i,r}\big)},$$

with $K_t\equiv K_t(\omega)$ and $K_u\equiv K_u(\omega)$ being the disjoint connected components containing the vertices x and y, respectively.

Proof. We omit the proof of this proposition because it is similar to the one given for Theorem 2. □

Remark 4. *Notice that in case $\hat{\boldsymbol{h}}\equiv 0$, we have for any $\omega\in\{0,1\}^E$ that*

$$H_0(K_t,K_u)(\omega) = \frac{q}{q^2} = \frac{1}{q},$$

so Proposition 2 gives us the following identity

$$\tau_{q\beta,0,q,V}(x,y) = \Big(1-\frac{1}{q}\Big)\phi_{\boldsymbol{p},0,q,G}(x\leftrightarrow y).$$

This is also a very well know identity, as can be seen in [27] Theorem 1.16, p. 11. Furthermore in case $q = 2$ and $h_{i,1} = -h_{i,2} = h_i$ for all $i \in V$, we have for any pair $x, y \in V$ that

$$H_{\boldsymbol{h}}(K_t, K_u) = \frac{1}{2}\Big\{1 + \tanh\big(\beta \sum_{i\in K_t} h_i\big).\tanh\big(\beta \sum_{i\in K_u} h_i\big)\Big\}.$$

In other words, Proposition 2 generalizes Theorem 2.

Proposition 3. *Let $G = (V, E)$ be a finite graph and $x \in V$. For each $m \in \{1, \dots, q\}$ with $q \geqslant 1$, we have*

$$\pi_{q\beta,\hat{\boldsymbol{h}},q,V}(\hat{\sigma}_x = m) = \phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}\Big(\frac{\exp(\beta\sum_{i\in K_t} h_{i,m})}{\sum_{p=1}^{q}\exp(\beta\sum_{i\in K_t} h_{i,p})}\Big),$$

where $K_t \equiv K_t(\omega)$ is the connected component of x.

Sketch of the Proof. To prove this theorem one needs to compute the marginals of the Edwards-Sokal coupling given in (10). The computation is similar to the one presented in the previous sections. The next step is to prove the identity

$$\pi_{q\beta,\hat{\boldsymbol{h}},q,V}(\hat{\sigma}_x = m) = \sum_{\omega}\Big[\sum_{\hat{\sigma}} \mathbb{1}_{\{\hat{\sigma}_x = m\}}\nu_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(\hat{\sigma}|\omega)\Big]\phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(\omega)$$

and then one proves that the rhs above is exactly

$$\sum_{\omega}\left[\sum_{\hat{\sigma}} \mathbb{1}_{\{\hat{\sigma}_x = \hat{\sigma}_y\}}\frac{\Delta_q(\hat{\sigma},\omega)\exp\big(\beta\sum_{i\in V}\sum_{p=1}^{q} h_{i,p}\delta_{\hat{\sigma}_i,p}\big)}{\prod_{\alpha=1}^{k(\omega,G)}\sum_{p=1}^{q}\exp\big(\beta\sum_{i\in K_\alpha} h_{i,p}\big)}\right]\phi_{\boldsymbol{p},\hat{\boldsymbol{h}},q,G}(\omega).$$

From this point, the result follows from the combinatorial arguments presented before. □

Part III
General boundary conditions

10 The general random-cluster model

In this section we define the so called general random-cluster model on the lattice $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ (this terminology, GRC model, comes from [7]) with inhomogeneous magnetic field of the form $\hat{\boldsymbol{h}} \equiv (h_{i,p} : i \in \mathbb{V};\ p = 1, \dots, q) \in \mathbb{R}^{\mathbb{V}} \times \cdots \times \mathbb{R}^{\mathbb{V}}$ and boundary conditions.

The Bernoulli factors introduced before will be replaced in this section by (abusing notation)

$$B_{\boldsymbol{J}}(\omega) \equiv \prod_{\{i,j\}:\omega_{ij}=1} r_{ij}, \tag{11}$$

where $\boldsymbol{J} = (J_{ij} \geqslant 0 : \{i,j\} \in E)$, $r_{ij} \equiv \exp(q\beta J_{ij}) - 1$ and $q \in \mathbb{Z}^+$ fixed. Although $r_{ij} \geqslant 0$, in general, they are not bounded by one, but mind that the random-cluster measure obtained with such "Bernoulli factors" is the same one gets when considering the old Bernoulli factors, since the weights in both cases are related by an overall normalization factor that cancels out because of the partition function.

Fix a random subgraph $G = (V, E)$ on the lattice $\mathbb{L}$, let $\partial E = \{e \in \mathbb{E} : e \cap V \neq \emptyset \text{ and } e \cap \partial V \neq \emptyset\}$. We denote by $\mathbb{B}_0(V)$ the set of all edges $\{x, y\} \in \mathbb{E}$ so that $\{x, y\} \subset V$. With this definition we have $\mathbb{B}_0(V) = E$. We use the notation $\mathbb{B}(V)$ to denote the set of all edges with at least one vertex in V. Note that $\mathbb{B}(V) = E \cup \partial E$. For any $\widetilde{E} \subset \mathbb{B}_0(\mathbb{V})$, we define $\mathbb{V}(\widetilde{E})$ as the set of sites which belong to at least one edge in $\widetilde{E}$.

GRC model with general boundary condition. Fix a finite subgraph $G = (V, E)$ of the lattice $\mathbb{L}$. For each $i \in \mathbb{V}$ we define $h_{i,\max} \equiv \max\{h_{i,p} : p = 1, \dots, q\}$. If $\omega \in \{0,1\}^{\mathbb{E}}$ and $C(\omega)$ denotes a generic connected component on $(\mathbb{V}, \eta(\omega))$, the GRC measure with general boundary condition is obtained by normalizing the followings weights

$$\mathcal{W}_E^{\mathrm{GRC}}(\omega_E|\omega_{E^c}) \equiv B_{\boldsymbol{J}}(\omega) \prod_{\substack{C(\omega):\\ \mathbb{V}(C(\omega))\cap V\neq\emptyset}} \sum_{p=1}^{q} q_p \, \exp\big(-\beta \sum_{i\in C(\omega)} (h_{i,\max} - h_{i,p})\big), \tag{12}$$

where $\{q_p : p = 1, \dots, q\}$ are positive constants, $B_{\boldsymbol{J}}(\omega)$ is given by (11) and the product runs over all the connected components $C(\omega)$ of the graph $(\mathbb{V}, \eta(\omega))$. In the above expression we are using the convention $e^{-\infty} = 0$. This measure is denoted by ϕ_E^{GRC}.

GRC model with free boundary condition. Let $G = (V, E)$ be a finite graph and $\omega \in \{0,1\}^E$ a configuration. If $C(\omega)$ denotes a generic connected component on $(V, \eta(\omega))$, we define

$$\Theta_{V,\mathrm{free}}(C(\omega)) \equiv \sum_{p=1}^{q} q_p \, \exp\big(\beta \sum_{i\in C(\omega)} h_{i,p}\big).$$

The GRC measure with free boundary condition is obtained by normalizing the weights

$$\mathcal{W}^{\rm GRC}_{V,\rm free}(\omega) \equiv B_{\boldsymbol{J}}(\omega) \prod_{C(\omega)} \Theta_{V,\rm free}(C(\omega)), \tag{13}$$

where $B_{\boldsymbol{J}}(\omega)$ is given by (11) and the product runs over all the connected components $C(\omega)$ of the graph $(V, \eta(\omega))$. This measure is denoted by $\phi^{\rm GRC}_{V,\rm free}$ and for each $\omega \in \{0,1\}^E$ it satisfies $\phi^{\rm GRC}_{V,\rm free}(\omega) \propto \mathcal{W}^{\rm GRC}_{V,\rm free}(\omega)$, where the proportionality constant is exactly the (inverse of the) partition function of the GRC model.

GRC model with wired boundary condition. Fix $\mathrm{m} \in \{1,\ldots,q\}$ and a finite subgraph $G = (V,E)$ of the lattice $\mathbb{L}$. If for each $\omega \in \{0,1\}^{E\cup\partial E}$. $C(\omega)$ denotes a connected component on $(V \cup \partial V, \eta(\omega))$, then we define

$$\Theta_{V,\rm m}(C(\omega)) \equiv \begin{cases} \Theta_{V,\rm free}(C(\omega)), & \text{if } C(\omega)\cap \partial V = \emptyset \\ \exp\big(\beta \sum\limits_{i\in C(\omega)} h_{i,\rm m}\big), & \text{otherwise.} \end{cases}$$

Similarly, the GRC measure with m-wired boundary condition is obtained by normalizing the weights

$$\mathcal{W}^{\rm GRC}_{V,\rm m}(\omega) \equiv B_{\boldsymbol{J}}(\omega) \prod_{C(\omega)} \Theta_{V,\rm m}(C(\omega)), \tag{14}$$

where the product runs over all the connected components $C(\omega)$ of the graph $(V \cup \partial V, \eta(\omega))$. This measure is denoted by $\phi^{\rm GRC}_{V,\rm m}$.

Remark 5. *One can easily see that when $E \equiv \mathbb{B}(\Lambda)$, $\Lambda \subset \mathbb{V}$ finite, in* (12)

$$\mathcal{W}^{\rm GRC}_{\mathbb{B}(\Lambda)}(\omega_{\mathbb{B}(\Lambda)}|\omega^{(1)}_{\mathbb{B}(\Lambda)^c}) = e^{-\beta\sum_{i\in\Lambda} h_{i,\max}} q_{\rm m} \cdot B_{\boldsymbol{J}}(\omega) \prod_{C(\omega)} \Theta_{\Lambda,\rm m}(C(\omega))$$

$$\overset{(14)}{=} e^{-\beta\sum_{i\in\Lambda} h_{i,\max}} q_{\rm m} \cdot \mathcal{W}^{\rm GRC}_{\Lambda,\rm m}(\omega),$$

where $\omega^{(i)}$ is the configuration with $\omega^{(i)}_e = i$ for all $e \in \mathbb{B}_0(\mathbb{V})$ $(i = 0,1)$. Therefore

$$\phi^{\rm GRC}_{\mathbb{B}(\Lambda)}(\omega_{\mathbb{B}(\Lambda)}|\omega^{(1)}_{\mathbb{B}(\Lambda)^c}) = \phi^{\rm GRC}_{\Lambda,\rm m}(\omega).$$

Similarly we obtain in (12) *with $E \equiv \mathbb{B}_0(\Lambda)$*

$$\mathcal{W}^{\rm GRC}_{\mathbb{B}_0(\Lambda)}(\omega_{\mathbb{B}_0(\Lambda)}|\omega^{(0)}_{\mathbb{B}_0(\Lambda)^c}) = e^{-\beta\sum_{i\in\Lambda} h_{i,\max}} \cdot B_{\boldsymbol{J}}(\omega) \prod_{C(\omega)} \Theta_{\Lambda,\rm free}(C(\omega))$$

$$\overset{(13)}{=} e^{-\beta\sum_{i\in\Lambda} h_{i,\max}} \cdot \mathcal{W}^{\rm GRC}_{\Lambda,\rm free}(\omega),$$

then

$$\phi^{\mathrm{GRC}}_{\mathbb{B}_0(\Lambda)}(\omega_{\mathbb{B}_0(\Lambda)}|\omega^{(0)}_{\mathbb{B}_0(\Lambda)^c}) = \phi^{\mathrm{GRC}}_{\Lambda,\mathrm{free}}(\omega).$$

10.1 The FKG inequality

Throughout this section we assume that $\{q_p : p = 1, \dots, q\}$ introduced in (12) and the magnetic field $\hat{\boldsymbol{h}}$ satisfy

$$\sum_{p \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})} q_p \geqslant 1, \tag{15}$$

where $\mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}}) \equiv \big\{p \in \{1,\dots,q\} : h_{i,p} = h_{i,\max}\big\}$. We consider as usual the partial order on $\{0,1\}^{\mathbb{E}}$ where $\omega \preceq \tilde{\omega} \iff \omega_e \leqslant \tilde{\omega}_e,\ \forall\ e \in \mathbb{E}$. We also use the standard notations $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ for $(\omega_1 \vee \omega_2)_e = \max\{\omega_1(e), \omega_2(e)\}$ and $(\omega_1 \wedge \omega_2)_e = \min\{\omega_1(e), \omega_2(e)\}$ with $e \in \mathbb{E}$, respectively.

Definition 2 (FKG property). *Let $(\Omega, \preceq)$ be a partially ordered space. A measure μ over Ω said to have the* FKG *property if*

$$\mu(fg) \geqslant \mu(f)\mu(g)$$

for any increasing (with respect to $\preceq$) measurable functions $f, g : \Omega \to \mathbb{R}$. Furthermore, if Ω is a cartesian product $\Omega = \prod_{e\in B} \Omega_e$, with $|\Omega_e| < \infty$, then μ is said to have the **strong FKG property**, *if $\mu(\cdot|A)$ has the* FKG *property for each cylinder event $A = \{\omega \in \Omega : \omega_e = \alpha_e,\ \forall e \in \tilde{B}\}$, where $\tilde{B} \subset B$ is finite and $\alpha_e \in \Omega_e$ for all $e \in \tilde{B}$.*

Remark 6. *If $\mathrm{m}, \tilde{\mathrm{m}} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$, then $\Theta_{V,\tilde{\mathrm{m}}}(C) = \Theta_{V,\mathrm{m}}(C)$ and therefore $\phi^{\mathrm{GRC}}_{V,\tilde{\mathrm{m}}} = \phi^{\mathrm{GRC}}_{V,\mathrm{m}}$. This measure is denoted by $\phi^{\mathrm{GRC}}_{V,\max}$.*

Theorem 4 (Strong FKG property). *Let $q \in \mathbb{Z}^+$, $\beta \geqslant 0$, $\boldsymbol{J} = (J_{ij} : \{i,j\} \in \mathbb{E}) \in [0,\infty)^{\mathbb{E}}$, $\hat{\boldsymbol{h}} = (h_{i,p} \in \mathbb{R} : i \in \mathbb{V}, 1 \leqslant p \leqslant q)$ and $\{q_p : p = 1,\dots,q\}$ satisfying* (15)*. Then for any finite subgraph $G = (V,E)$ of $\mathbb{L}$, the measures $\phi^{\mathrm{GRC}}_{V,\mathrm{free}}$ and $\phi^{\mathrm{GRC}}_{V,\max}$ have the strong* FKG *property.*

Proof. For simplicity we assume that the magnetic field we are dealing with satisfies the following inequalities

$$h_{i,1} \leqslant h_{i,2} \leqslant \ldots \leqslant h_{i,q}, \quad \forall\ i \in \mathbb{V}. \tag{16}$$

The FKG lattice condition for the $\phi^{\mathrm{GRC}}_{V,\mathrm{free}}$ is equivalent to

$$\mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{free}}(\omega^{(1)} \vee \omega^{(2)}) \mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{free}}(\omega^{(1)} \wedge \omega^{(2)}) \geqslant \mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{free}}(\omega^{(1)}) \mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{free}}(\omega^{(2)}),$$

where $\omega^{(1)}$ and $\omega^{(2)}$ are arbitrary configurations. Similarly for $\phi^{\text{GRC}}_{V,\max}$. It is well known that such condition implies the strong FKG property, see for example Theorem 2.19, p. 25 in [27]. By defining

$$\mathcal{R}(\xi,\omega) \equiv \frac{\mathcal{W}^{\text{GRC}}_{V,\text{free}}(\xi \vee \omega)}{\mathcal{W}^{\text{GRC}}_{V,\text{free}}(\xi)},$$

one can see that the lattice condition holds if

$$\mathcal{R}(\omega^{(1)},\omega^{(2)}) \geqslant \mathcal{R}(\omega^{(1)} \wedge \omega^{(2)},\omega^{(2)}). \tag{17}$$

For a fixed configuration ω, we chose an arbitrary order for $\eta(\omega)$ and represent these open edges as $(e_1,\ldots,e_{|\eta(\omega)|})$. So for any configuration $\xi \in \{0,1\}^E$ we have that

$$\mathcal{R}(\xi,\omega) = \prod_{k=1}^{|\eta(\omega)|} \mathcal{R}(\xi \vee \omega^{(e_1)} \vee \cdots \vee \omega^{(e_{k-1})},\omega^{(e_k)}),$$

where $(\omega^{(e)})_{e'} \equiv \delta_{e,e'}$. Therefore it is enough to prove (17) for configurations ξ, $\omega^{(1)}$ and $\omega^{(2)}$ such that ξ has at least two zero coordinates or at most one zero and $\omega^{(1)} \equiv \xi \vee \omega^{(b)}$ and $\omega^{(2)} \equiv \xi \vee \omega^{(b')}$. Let us begin assuming that ξ has at least two zero coordinates and

$$\xi \equiv (*,\ldots,*,\underbrace{0}_{b-\text{th}},*,\ldots,*,\underbrace{0}_{b'-\text{th}},*,\ldots,*),$$

where $b,b' \in E \cup \partial E$, $b \neq b'$ and the stars indicate generic elements in $\{0,1\}$ (not necessarily equal). If we define

$$\xi^b \equiv (*,\ldots,*,\underbrace{1}_{b-\text{th}},*,\ldots,*,\underbrace{0}_{b'-\text{th}},*,\ldots,*)$$

and

$$\xi^{b'} \equiv (*,\ldots,*,\underbrace{0}_{b-\text{th}},*,\ldots,*,\underbrace{1}_{b'-\text{th}},*,\ldots,*),$$

then we have that $\omega^{(1)} = \xi \vee \omega^{(b)} = \xi^b, \omega^{(2)} = \xi \vee \omega^{(b')} = \xi^{b'}$and $\omega^{(1)} \wedge \omega^{(2)} = \xi$. So in order to prove (17) it is enough to prove that

$$\mathcal{R}(\xi^b,\xi^{b'}) \geqslant \mathcal{R}(\xi,\xi^{b'}), \quad \text{with } b \neq b'. \tag{18}$$

Now we concentrate on proving (18). To do this we first observe that if $\prod_{\{i,j\}:\xi_{ij}=1} r_{ij} = k$, then

$$\prod_{\{i,j\}:(\xi^b \vee \xi^{b'})_{ij}=1} r_{ij} = r_b r_{b'} k, \qquad \prod_{\{i,j\}:\xi^b_{ij}=1} r_{ij} = r_b k \quad \text{and} \qquad \prod_{\{i,j\}:(\xi \vee \xi^{b'})_{ij}=1} r_{ij} = r_{b'} k.$$

So it follows from the definition (11) that

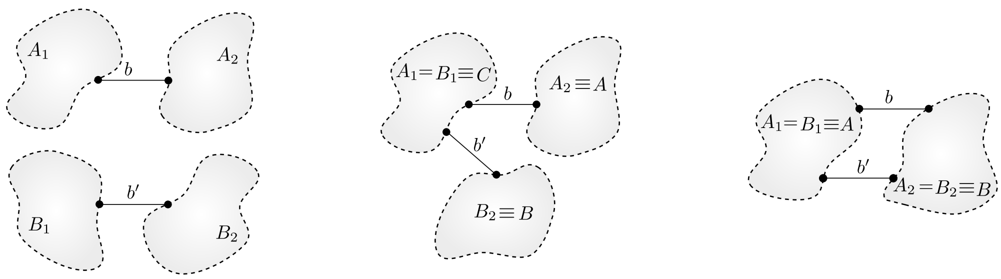
$$\frac{B_{\boldsymbol{J}}(\xi^b \vee \xi^{b'})}{B_{\boldsymbol{J}}(\xi^b)} = \frac{\prod_{\{i,j\}:(\xi^b \vee \xi^{b'})_{ij}=1} r_{ij}}{\prod_{\{i,j\}:\xi^b_{ij}=1} r_{ij}} = r_{b'} = \frac{\prod_{\{i,j\}:(\xi \vee \xi^{b'})_{ij}=1} r_{ij}}{\prod_{\{i,j\}:\xi_{ij}=1} r_{ij}} = \frac{B_{\boldsymbol{J}}(\xi \vee \xi^{b'})}{B_{\boldsymbol{J}}(\xi)}.$$

Because of the above observation and the definitions of $\mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{free}}$ and $\mathcal{W}^{\mathrm{GRC}}_{V,\mathrm{m}}$, the proof of (18) reduces to

$$\frac{\Theta_{V,\#}(C(\xi^b \vee \xi^{b'}))}{\Theta_{V,\#}(C(\xi^b))} \geqslant \frac{\Theta_{V,\#}(C(\xi \vee \xi^{b'}))}{\Theta_{V,\#}(C(\xi))}, \tag{19}$$

where # stands for "free"or "m".

Free boundary condition case.
We broke the proof of (19) in several cases. Let A_1, A_2, B_1 and B_2 be connected components of $(V, \eta(\xi))$ and consider the cases showed in the picture below



(I) (II) (III)

The case (I) represents that the end vertices of b belong to A_1 and A_2 and the end vertices of b' belong to B_1 and B_2. In this case the left and right sides of (19) are equal, since

$$\frac{\Theta_{V,\mathrm{free}}(A_1 \cup A_2)\Theta_{V,\mathrm{free}}(B_1 \cup B_2)}{\Theta_{V,\mathrm{free}}(A_1 \cup A_2)\Theta_{V,\mathrm{free}}(B_1)\Theta_{V,\mathrm{free}}(B_2)} = \frac{\Theta_{V,\mathrm{free}}(B_1 \cup B_2)\Theta_{V,\mathrm{free}}(A_1)\Theta_{V,\mathrm{free}}(A_2)}{\Theta_{V,\mathrm{free}}(A_1)\Theta_{V,\mathrm{free}}(A_2)\Theta_{V,\mathrm{free}}(B_1)\Theta_{V,\mathrm{free}}(B_2)}.$$

For the case (II), we should prove that

$$\frac{\Theta_{V,\mathrm{free}}(A \cup B \cup C)}{\Theta_{V,\mathrm{free}}(C \cup A)\Theta_{V,\mathrm{free}}(B)} \geqslant \frac{\Theta_{V,\mathrm{free}}(C \cup B)\Theta_{V,\mathrm{free}}(A)}{\Theta_{V,\mathrm{free}}(A)\Theta_{V,\mathrm{free}}(B)\Theta_{V,\mathrm{free}}(C)},$$

which is equivalent to

$$\Theta_{V,\mathrm{free}}(C)\Theta_{V,\mathrm{free}}(A \cup B \cup C) \geqslant \Theta_{V,\mathrm{free}}(C \cup A)\Theta_{V,\mathrm{free}}(C \cup B). \tag{20}$$

To help us prove inequality (20), we define for each $m \in \{1, \ldots, q\}$ the following numbers

$$a_m \equiv \exp\big(\beta \textstyle\sum_{i \in A} h_{i,m}\big),\ b_m \equiv \exp\big(\beta \textstyle\sum_{i \in B} h_{i,m}\big) \text{ and } c_m \equiv \exp\big(\beta \textstyle\sum_{i \in C} h_{i,m}\big).$$

The hypothesis (16) implies immediately that (a_m) and (b_m) are non-decreasing in m. Using this notation, (20) reads

$$\sum_{m=1}^{q} q_m c_m \sum_{m'=1}^{q} q_{m'} a_{m'} b_{m'} c_{m'} \geqslant \sum_{m=1}^{q} q_m a_m c_m \sum_{m'=1}^{q} q_{m'} b_{m'} c_{m'}. \tag{21}$$

Both sides of the above inequality can be written using a bilinear form

$$\varphi(a,b) \equiv \sum_{m,m'=1}^{q} r_{m,m'} a_{m'} b_m,$$

where $r_{m,m'} \equiv q_m c_m q_{m'} c_{m'}$, $a \equiv (a_1, \ldots, a_q)$ and $b \equiv (b_1, \ldots, b_q)$. Note that φ is a symmetric bilinear form and (21) can be written as

$$\varphi(1,c) \geqslant \varphi(a,b), \quad \text{where } c \equiv (a_1 b_1, \ldots, a_q b_q). \tag{22}$$

Therefore it remains to prove (22), which clearly holds since

$$r_{m,m'}(a_{m'} - a_m)(b_{m'} - b_m) \geqslant 0 \iff \varphi(1,c) - \varphi(a,b) - \varphi(b,a) + \varphi(c,1) \geqslant 0.$$

We proceed with (19) for the case (III). Now we have to prove that

$$1 = \frac{\Theta_{V,\mathrm{free}}(A \cup B)}{\Theta_{V,\mathrm{free}}(A \cup B)} \geqslant \frac{\Theta_{V,\mathrm{free}}(A \cup B)}{\Theta_{V,\mathrm{free}}(A)\Theta_{V,\mathrm{free}}(B)},$$

in other words $\Theta_{V,\mathrm{free}}(A)\Theta_{V,\mathrm{free}}(B) \geqslant \Theta_{V,\mathrm{free}}(A \cup B)$, or equivalently

$$\textstyle\sum_{m=1}^{q} q_m a_m \sum_{m'=1}^{q} q_{m'} b_{m'} \geqslant \sum_{m=1}^{q} q_m a_m b_m.$$

This last inequality is actually true since

$$\sum_{m=1}^{q} q_m a_m \sum_{m'=1}^{q} q_{m'} b_{m'} \geqslant \sum_{m=1}^{q} q_m a_m b_{\max} \sum_{m' \in \cap_{i \in \mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})} q_{m'} \geqslant \sum_{m=1}^{q} q_m a_m b_m.$$

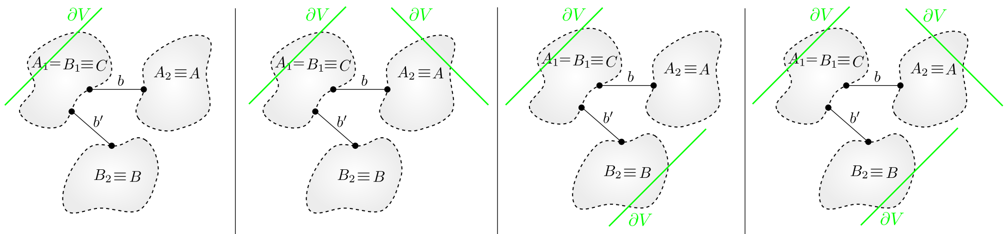
For the cases where the end vertices of b or b' are contained in the same connected component, the inequality is trivial.

Max wired boundary condition case.
Suppose that $\widetilde{\mathrm{m}} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$. To prove the inequality (19) we have to analyze again the three cases above. For case (I), analogously to the free boundary condition case, we have

$$\frac{\Theta_{V,\widetilde{\mathrm{m}}}(A_1 \cup A_2)\Theta_{V,\widetilde{\mathrm{m}}}(B_1 \cup B_2)}{\Theta_{V,\widetilde{\mathrm{m}}}(A_1 \cup A_2)\Theta_{V,\widetilde{\mathrm{m}}}(B_1)\Theta_{V,\widetilde{\mathrm{m}}}(B_2)} = \frac{\Theta_{V,\widetilde{\mathrm{m}}}(B_1 \cup B_2)\Theta_{V,\widetilde{\mathrm{m}}}(A_1)\Theta_{\widetilde{\mathrm{m}}}(A_2)}{\Theta_{V,\widetilde{\mathrm{m}}}(A_1)\Theta_{V,\widetilde{\mathrm{m}}}(A_2)\Theta_{V,\widetilde{\mathrm{m}}}(B_1)\Theta_{V,\widetilde{\mathrm{m}}}(B_2)},$$

independently on whether the components A_1, A_2, B_1 and B_2 and the possible combinations among them intersect V^c.

For the case (II) and all the configurations sketched on the figure below

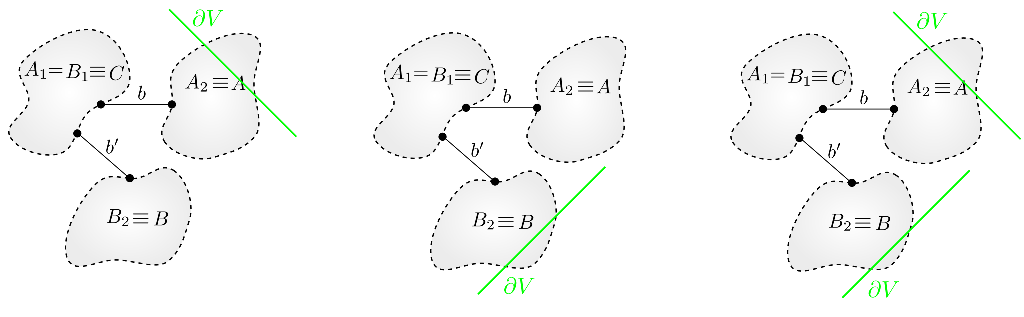


$C \cap V^c \neq \emptyset$ | $C \cap V^c \neq \emptyset$, $A \cap V^c \neq \emptyset$ | $C \cap V^c \neq \emptyset$, $B \cap V^c \neq \emptyset$ | $C \cap V^c \neq \emptyset$, $B \cap V^c \neq \emptyset$, $A \cap V^c \neq \emptyset$

we have from the definition (14) of $\Theta_{V,\widetilde{\mathrm{m}}}$ that the following equality holds

$$c_{\widetilde{\mathrm{m}}} a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} = a_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}}.$$

For the following configurations that also appears in the case (II) :



$A \cap V^c \neq \emptyset$ | $B \cap V^c \neq \emptyset$ | $A \cap V^c \neq \emptyset$, $B \cap V^c \neq \emptyset$

(a) | (b) | (c)

For (a), the inequality (19) comes from

$$\left(\sum_{m=1}^{q} q_m c_m\right) a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} \geqslant a_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} \left(\sum_{m'=1}^{q} q_{m'} b_{m'} c_{m'}\right),$$

which is always valid since we have that $b_{\widetilde{\mathrm{m}}} \geqslant b_{m'}$, $\forall m' = 1, \dots, q$. In (b) inequality (19), comes from

$$\left(\sum_{m=1}^{q} q_m c_m\right) a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} \geqslant \left(\sum_{m=1}^{q} q_m a_m c_m\right) b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}},$$

which is also true because $a_{\widetilde{\mathrm{m}}} \geqslant a_m$, $\forall m = 1, \dots, q$. Finally, in (c) inequality (19), is a consequence of

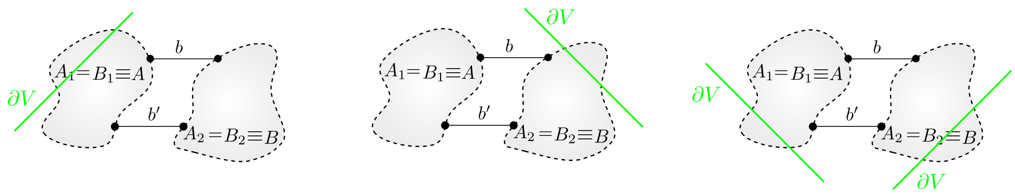
$$\left(\sum_{m=1}^{q} q_m c_m\right) a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} \geqslant a_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} c_{\widetilde{\mathrm{m}}}$$

and the validity of this inequality is ensured by

$$\sum_{m=1}^{q} q_m c_m \geqslant \sum_{m \in \cap_{i \in \mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})} q_m c_{\max} \geqslant c_{\max} = c_{\widetilde{\mathrm{m}}},$$

which follows from (15).

Now we consider the case (III), by splitting its analysis in the following sub-cases



$A \cap V^c \neq \emptyset$ — (d) $\quad$ $B \cap V^c \neq \emptyset$ — (e) $\quad$ $A \cap V^c \neq \emptyset$, $B \cap V^c \neq \emptyset$ — (f)

For (d) the inequality (19), is valid as long as

$$a_{\widetilde{\mathrm{m}}} \left(\sum_{m'=1}^{q} q_{m'} b_{m'}\right) \geqslant a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}}.$$

This is in fact true because

$$\sum_{m'=1}^{q} q_{m'} b_{m'} \geqslant \sum_{m' \in \cap_{i \in \mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})} q_{m'} b_{\max} \geqslant b_{\max} = b_{\widetilde{\mathrm{m}}}.$$

For (e), the desired inequality follows from

$$\left(\sum_{m=1}^{q} q_m a_m\right) b_{\widetilde{\mathrm{m}}} \geqslant a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}},$$

but this inequality holds because

$$\sum_{m=1}^{q} q_m a_m \geqslant \sum_{m \in \cap_{i \in \mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})} q_m a_{\max} \geqslant a_{\max} = a_{\widetilde{\mathrm{m}}}.$$

For the last sub-case (f), we have to prove that $a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}} = a_{\widetilde{\mathrm{m}}} b_{\widetilde{\mathrm{m}}}$, which is obviously true.

In the max wired boundary conditions, if the end vertices of b or b' belong to the same component, the result follows.

To finish the proof we need to address the case where ξ has at most one zero and $\omega^{(1)} \equiv \xi \vee \omega^{(b)}$ and $\omega^{(2)} \equiv \xi \vee \omega^{(b')}$. Suppose that

$$\xi \equiv (1, \dots, 1, \underbrace{0}_{b-\mathrm{th}}, 1, \dots, 1, \underbrace{1}_{b'-\mathrm{th}}, 1, \dots, 1)$$

where $b, b' \in E \cup \partial E$ with $b \neq b'$. By defining

$$\xi^b \equiv (1, \dots, 1, \underbrace{1}_{b-\mathrm{th}}, 1, \dots, 1, \underbrace{1}_{b'-\mathrm{th}}, 1, \dots, 1)$$

and

$$\xi^{b'} \equiv (1, \dots, 1, \underbrace{0}_{b-\mathrm{th}}, 1, \dots, 1, \underbrace{1}_{b'-\mathrm{th}}, 1, \dots, 1),$$

we can see that $\omega^{(1)} = \xi \vee \omega^{(b)} = \xi^b$, $\quad \omega^{(2)} = \xi \vee \omega^{(b')} = \xi^{b'}$ $\quad$ and $\quad \omega^{(1)} \wedge \omega^{(2)} = \xi$. In this case, to prove (17) for both "free" and "max" wired boundary conditions, it is enough to prove that $\mathcal{R}(\xi^b, \xi^{b'}) = \mathcal{R}(\xi, \xi^{b'})$ with $b \neq b'$, but this is trivial since

$$\begin{aligned} \mathcal{R}(\xi^b, \xi^{b'}) = \frac{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi^b \vee \xi^{b'})}{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi^b)} = \frac{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi^b)}{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi^b)} = \frac{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi)}{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi)} \\ = \frac{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi \vee \xi^{b'})}{\mathcal{W}^{\mathrm{GRC}}_{V,\#}(\xi)} = \mathcal{R}(\xi, \xi^{b'}). \end{aligned}$$

□

11 Edwards-Sokal model

Edwards-Sokal model with general boundary condition. Fix $q \in \mathbb{Z}^+$, for any finite set $V \subset \mathbb{V}$ and any fixed configurations $\sigma_{V^c}, \omega_{\mathbb{B}(V)^c}$ prescribed outside of V, we define de Edwards-Sokal measure $\phi^{\mathrm{ES}}_{V,\mathbb{B}(V)}$ as the normalization of the following weights

$$\mathcal{W}(\sigma_V, \omega_{\mathbb{B}(V)} | \sigma_{V^c}, \omega_{\mathbb{B}(V)^c}) = \prod_{\substack{\{i,j\} \in \mathbb{B}(V) \\ \omega_{ij} = 1}} r_{ij} \delta_{\sigma_i, \sigma_j} \times \exp\big(\beta \sum_{i \in V} \sum_{p=1}^{q} h_{i,p} \delta_{\sigma_i, p}\big),$$

where r_{ij} has been defined in (11).

Edwards-Sokal model with wired and free boundary conditions. From the previous definition we can observe that, for any finite volume $V \subset \mathbb{V}$, the state $\phi^{\mathrm{ES}}_{V,\mathbb{B}(V)}(\cdot|\sigma_{V^c}, \omega_{\mathbb{B}(V)^c})$ is independent of $\omega_{\mathbb{B}(V)^c}$, and we define

$$\phi^{\mathrm{ES}}_{V,\mathrm{m}}(\cdot) \equiv \phi^{\mathrm{ES}}_{V,\mathbb{B}(V)}(\cdot|\sigma^{\mathrm{m}}_{V^c}, \omega_{\mathbb{B}(V)^c}),$$

where σ^{m} is the constant configuration, $\sigma^{\mathrm{m}}_i = \mathrm{m}$ for all $i \in \mathbb{V}$, with $\mathrm{m} \in \{1, \ldots, q\}$ fixed. This state is known as the m-wired boundary condition state.

By similar reasons we have that $\phi^{\mathrm{ES}}_{V,\mathbb{B}_0(V)}(\cdot|\sigma_{V^c}, \omega_{\mathbb{B}_0(V)^c})$ does not depend on σ_{V^c}, provided that the ω-boundary condition is chosen as $\omega_{\mathbb{B}_0(V)^c} = \omega^0_{\mathbb{B}_0(V)^c}$, where ω^0 denotes the configuration with $\omega^0_{ij} = 0$ for all $\{i,j\} \in \mathbb{B}(\mathbb{V})$. In this case we introduce the notation

$$\phi^{\mathrm{ES}}_{V,\mathrm{free}}(\cdot) \equiv \phi^{\mathrm{ES}}_{V,\mathbb{B}_0(V)}(\cdot|\sigma_{V^c}, \omega^0_{\mathbb{B}_0(V)^c}).$$

12 Gibbs states and limit states

Gibbs states. Let $\mathscr{P}(\Omega)$ denote the set of probability measures defined on some probability space Ω. Since the families $\{\phi^{\mathrm{GRC}}_{\mathbb{B}}\}$ and $\{\phi^{\mathrm{ES}}_{V,\mathbb{B}(V)}\}$ are specifications (see [23]), we can define as usual the set of the Gibbs measures compatible with these specifications as follows

$$\mathscr{G}^{\mathrm{GRC}} \equiv \left\{ \phi \in \mathscr{P}(\Omega) : \begin{array}{c} \phi(f) \overset{\mathrm{DLR}}{=} \displaystyle\int \phi^{\mathrm{GRC}}_{\mathbb{B}}(f|\omega_{\mathbb{B}^c})\phi(\mathrm{d}\omega), \\ \mathrm{supp}(f) \subset \mathbb{B} \end{array} \right\} \tag{23}$$

and

$$\mathscr{G}^{\mathrm{ES}} \equiv \left\{ \nu \in \mathscr{P}(\Omega \times \Sigma) : \begin{array}{c} \nu(f) \overset{\mathrm{DLR}}{=} \displaystyle\int \phi^{\mathrm{ES}}_{V,\mathbb{B}(V)}(f|\sigma_{V^c}, \omega_{\mathbb{B}(V)^c})\, d\nu(\sigma,\omega), \\ \mathrm{supp}(f) \subset V \times \mathbb{B}(V) \end{array} \right\}.$$

That is, $\mathscr{G}^{\mathrm{GRC}}$ and $\mathscr{G}^{\mathrm{ES}}$ are the class of probability measures (Gibbs measures) that are preserved for their respective probability kernels.

Limit states. On the other hand, we define the set of the thermodynamic limits of the specification $\{\phi^{\mathrm{GRC}}_{\mathbb{B}_n}\}$, where $\{\mathbb{B}_n\}$ is a cofinal collection in $\mathbb{E}$:

$$\mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}} \equiv \left\{ \phi \in \mathscr{P}(\{0,1\}^{\mathbb{E}}) : \phi \overset{\mathrm{weak}}{=} \lim_{n\to\infty} \phi^{\mathrm{GRC}}_{\mathbb{B}_n}(\cdot|\omega_n) \right\}. \tag{24}$$

In general, it is not easy to relate the sets $\mathscr{G}^{\mathrm{GRC}}$ and $\mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}}$ due to the lack of quasilocality of the specifications $\phi^{\mathrm{GRC}}_{\mathbb{B}}$. One case where these sets can be related is the case in which we assume the existence of at most one connected component with probability one. As a consequence of Lemma 7, one can prove the following relation: $\mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}} \subset \mathscr{G}^{\mathrm{GRC}}$, see Lemma 8 below. For more details see [7].

By using the FKG property for the GRC model and the previous definitions, one can prove the following theorem which ensures the existence of thermodynamic limit.

Theorem 5 (Monotonicity and existence of limit states)**.** *Let* $\beta \geqslant 0$, $\boldsymbol{J} = (J_{ij} : \{i,j\} \in \mathbb{E}) \in [0,\infty)^{\mathbb{E}}$ *and* $\hat{\boldsymbol{h}} \equiv (h_{i,p} \in \mathbb{R} : i \in \mathbb{V};\ p = 1,\dots,q)$*. For each increasing quasilocal function* f *(see [23]),*

(i) The following limits exist

$$\phi^{\mathrm{GRC}}_{\mathrm{max}}(f) \equiv \lim_{V\uparrow\mathbb{V}} \phi^{\mathrm{GRC}}_{V,\mathrm{max}}(f) \quad \text{and} \quad \phi^{\mathrm{GRC}}_{\mathrm{free}}(f) \equiv \lim_{V\uparrow\mathbb{V}} \phi^{\mathrm{GRC}}_{V,\mathrm{free}}(f).$$

(ii) If in addition, $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\mathrm{max}}(\hat{\boldsymbol{h}})$*, then the following limits exist*

$$\phi^{\mathrm{ES}}_{\mathrm{max}}(f) \equiv \lim_{V\uparrow\mathbb{V}} \phi^{\mathrm{ES}}_{V,\mathrm{max}}(f) \quad \text{and} \quad \phi^{\mathrm{ES}}_{\mathrm{free}}(f) \equiv \lim_{V\uparrow\mathbb{V}} \phi^{\mathrm{ES}}_{V,\mathrm{free}}(f).$$

(iii) If $\phi \in \mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}}$ *or* $\phi \in \mathscr{G}^{\mathrm{GRC}}$*, then for each increasing quasilocal function* f *we have*

$$\phi^{\mathrm{GRC}}_{\mathrm{free}}(f) \leqslant \phi(f) \leqslant \phi^{\mathrm{GRC}}_{\mathrm{max}}(f).$$

Proof. The proof is similar to the proof of Theorem III.1 in [7]. □

Since we are also interested in monotonicity properties with respect to the magnetic field, it is needed to introduce a partial order between two fields [7]. Given two arbitrary magnetic fields $\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{h}}'$, we say that

$$\hat{\boldsymbol{h}} \prec \hat{\boldsymbol{h}}' \iff \forall\, i \in \mathbb{V}:\ h_{i,k} - h_{i,l} \leqslant h'_{i,k} - h'_{i,l}, \quad k,l = 1,\dots,q \tag{25}$$

whenever $h_{i,k} - h_{i,l} > 0$.

Theorem 6 (Monotonicity with respect to the magnetic field)**.** *Let* $\hat{\boldsymbol{h}}$ *and* $\hat{\boldsymbol{h}}'$ *be two arbitrary magnetic fields such that* $\hat{\boldsymbol{h}} \prec \hat{\boldsymbol{h}}'$*. Denote by* $\phi^{\mathrm{GRC},\hat{\boldsymbol{h}}}_{\#}$ *and* $\phi^{\mathrm{GRC},\hat{\boldsymbol{h}}'}_{\#}$ *their respective measures defined in Theorem 5, where* $\#$ *stands for* "free" *or* "max"*. Then, for any quasilocal increasing function* f *we have*

$$\phi^{\mathrm{GRC},\hat{\boldsymbol{h}}}_{\mathrm{free}}(f) \leqslant \phi^{\mathrm{GRC},\hat{\boldsymbol{h}}'}_{\mathrm{free}}(f) \qquad \text{and} \qquad \phi^{\mathrm{GRC},\hat{\boldsymbol{h}}}_{\mathrm{max}}(f) \leqslant \phi^{\mathrm{GRC},\hat{\boldsymbol{h}}'}_{\mathrm{max}}(f).$$

Proof. By the Holley Theorem, the stochastic domination claimed in the statement of the theorem is proved as long as the following lattice condition is satisfied

$$\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(\omega^{(1)} \vee \omega^{(2)})\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}'}(\omega^{(1)} \wedge \omega^{(2)}) \geqslant \phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}'}(\omega^{(1)})\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(\omega^{(2)}) \tag{26}$$

for all $\omega^{(1)}, \omega^{(2)} \in \{0,1\}^E$, where $\#$ denotes the "free" and "max" wired boundary conditions. For details, see Theorem 2.3, item (c), p. 20 in [27]. It is also well known that (26) is a consequence of

$$\frac{\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}'}(\zeta^e)}{\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}'}(\zeta_{(e)})} \geqslant \frac{\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(\xi^e)}{\phi_{V,\#}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(\xi_{(e)})}, \tag{27}$$

for any $\xi \preceq \zeta$ and $e \in E$, where $\xi_{(e)}$ (ξ^e) is the configuration that agrees with ξ in all edges, except in e where its value is zero (one). We shall remark that the notations ξ_e and $\xi_{(e)}$ have different meaning.

Without loss of generality, we can assume that ξ and ζ are of the form

$$\xi \equiv (*, \ldots, *, \underbrace{0}_{\mathrm{e-th}}, *, \ldots, *) \quad \text{and} \quad \zeta \equiv (*', \ldots, *', \underbrace{0}_{\mathrm{e-th}}, *', \ldots, *'),$$

with $\xi \preceq \zeta$. Let $k' \equiv \prod_{\{i,j\}:\zeta_{ij}=1} r_{ij}$ and $k \equiv \prod_{\{i,j\}:\xi_{ij}=1} r_{ij}$. From the definitions we get that

$$\zeta_{(e)} = \zeta, \quad \xi_{(e)} = \xi, \quad \prod_{\{i,j\}:\zeta^e_{ij}=1} r_{ij} = r_e k' \quad \text{and} \quad \prod_{\{i,j\}:\xi^e_{ij}=1} r_{ij} = r_e k.$$

Therefore

$$\frac{B_{\boldsymbol{j}}(\zeta^e)}{B_{\boldsymbol{j}}(\zeta_{(e)})} = \frac{\prod_{\{i,j\}:\zeta^e_{ij}=1} r_{ij}}{\prod_{\{i,j\}:\zeta_{ij}=1} r_{ij}} = r_e = \frac{\prod_{\{i,j\}:\xi^e_{ij}=1} r_{ij}}{\prod_{\{i,j\}:\xi_{ij}=1} r_{ij}} = \frac{B_{\boldsymbol{j}}(\xi^e)}{B_{\boldsymbol{j}}(\xi_{(e)})}.$$

So it follows from the equation above that (27) is a consequence of

$$\frac{\Theta_{V,\#}^{\hat{\boldsymbol{h}}'}(\zeta^e)}{\Theta_{V,\#}^{\hat{\boldsymbol{h}}'}(\zeta_{(e)})} \geqslant \frac{\Theta_{V,\#}^{\hat{\boldsymbol{h}}}(\xi^e)}{\Theta_{V,\#}^{\hat{\boldsymbol{h}}}(\xi_{(e)})}, \tag{28}$$

for both "free" and "max" wired boundary conditions.

If $e = \{x,y\}$ and $x \leftrightarrow y$ in ξ, then (28) is an equality. On the other hand, if $x \not\leftrightarrow y$ in ξ, then there are two connected components $A \equiv C(x,\xi)$ and $B \equiv C(y,\xi)$ containing the vertices x and y, respectively. If e is an open edge in ξ, then the components A and B are connected and will be denoted by $C \equiv A \cup B$. So $|C| = |A| + |B|$, from where we deduce that

$$\frac{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(\zeta^e)}{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(\zeta_{(e)})} \cdot \frac{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(\xi_{(e)})}{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(\xi^e)} = \frac{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(C)}{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(C)} \cdot \frac{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(A)\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(B)}{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(A)\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(B)},$$

for either free or max wired boundary conditions. In order to prove (28), it is enough to prove that

$$\frac{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(C)}{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(C)} . \frac{\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(A)\Theta^{\hat{\boldsymbol{h}}}_{V,\#}(B)}{\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(A)\Theta^{\hat{\boldsymbol{h}}'}_{V,\#}(B)} \geqslant 1. \tag{29}$$

To establish the above inequality, we analyze separately the "free" and "max" wired boundary condition cases.

Free boundary condition case.
Keeping the notation used in the proof of the FKG inequality, for each $m \in \{1, \ldots, q\}$ we define

$$a_m \equiv \exp\big(\beta \sum_{i\in A} h_{i,m}\big), \; b_m \equiv \exp\big(\beta \sum_{i\in B} h_{i,m}\big) \;\text{ and }\; c_m \equiv \exp\big(\beta \sum_{i\in C} h_{i,m}\big).$$

Similarly we define a'_m, b'_m and c'_m by replacing $(h_{i,m})$ for $(h'_{i,m})$. With this notation, (29) reads

$$\begin{aligned}(\textstyle\sum_{j=1}^q q_j a'_j)(\sum_{k=1}^q q_k b'_k)(\sum_{l=1}^q q_l a_l b_l)& \\ \leqslant (\textstyle\sum_{j=1}^q q_j a_j)(\sum_{k=1}^q q_k b_k)(\sum_{l=1}^q q_l a'_l b'_l).& \end{aligned} \tag{30}$$

The proof of (30) is divided in two steps.

Step 1:(move the primes from a_j's) we claim that

$$\begin{aligned}(\textstyle\sum_{j=1}^q q_j a'_j)(\sum_{k=1}^q q_k b'_k)(\sum_{l=1}^q q_l a_l b_l)& \\ \leqslant (\textstyle\sum_{j=1}^q q_j a_j)(\sum_{k=1}^q q_k b'_k)(\sum_{l=1}^q q_l a'_l b_l).& \end{aligned} \tag{31}$$

In fact, we first remark that without loss of generality we can assume that $h_{i,l} - h_{i,j} > 0$, $\forall i \in \mathbb{V}$. From the hypothesis we have $\hat{\boldsymbol{h}} \prec \hat{\boldsymbol{h}}'$, so we get $\forall\ l, j = 1, \ldots, q$ and $\forall\ i \in \mathbb{V}$ that $h_{i,l} - h_{i,j} \leqslant h'_{i,l} - h'_{i,j}$. From the last inequality, it follows that

$$\frac{a_l}{a_j} \leqslant \frac{a'_l}{a'_j}, \quad \text{which implies} \quad a'_j a_l - a_j a'_l \leqslant 0. \tag{32}$$

On the other hand, since $h_{i,l} - h_{i,j} > 0$, we have $b_l - b_j > 0$. Putting together the last two inequalities yields

$$(a'_j a_l - a_j a'_l)(b_l - b_j) \leqslant 0,$$

and we conclude that $a'_j a_l b_l \leqslant \left[a'_j a_l - a_j a'_l\right] b_j + a'_l a_j b_l \leqslant a_j a'_l b_l$, where in the last inequality we have used (32). By multiplying the above inequality for $q_j q_k q_l b'_k$ and then summing over $j, k, l = 1, \ldots, q$, we prove the claim.

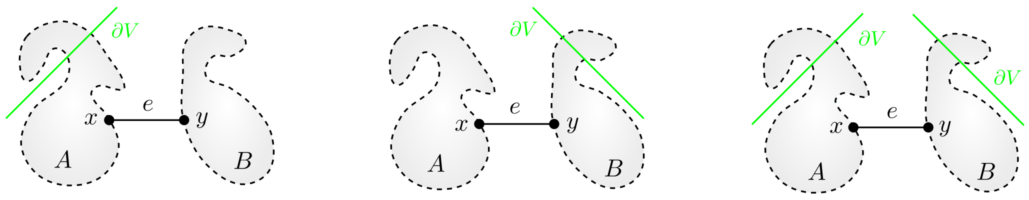
Step 2:(move the primes from b_k's) we claim that

$$\left(\sum_{j=1}^q q_j a_j\right)\left(\sum_{k=1}^q q_k b'_k\right)\left(\sum_{l=1}^q q_l a'_l b_l\right) \leqslant \left(\sum_{j=1}^q q_j a_j\right)\left(\sum_{k=1}^q q_k b_k\right)\left(\sum_{l=1}^q q_l a'_l b'_l\right). \quad (33)$$

The proof is similar to the one given for the Step 1. We assume that $h_{i,l} - h_{i,k} > 0$, $\forall\ i \in \mathbb{V}$ and prove in place of (32) that $b'_k b_l - b_k b'_l \leqslant 0$, proceeding similarly to reach the conclusion.

Finally, by piecing together the inequalities (31) and (33), we obtain (30).

Max wired boundary condition case.
We first observe that if $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,m}(\hat{\boldsymbol{h}})$ and $\widetilde{\mathrm{m}} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\widetilde{m}}(\hat{\boldsymbol{h}}')$, then $\mathrm{m} = \widetilde{\mathrm{m}}$. Given two connected components A and B, if $A \cap V^c = \emptyset$ and $B \cap V^c = \emptyset$, then the inequality follows from the free boundary condition case. The remaining cases will be analyzed by considering the following cases:



$A \cap V^c \neq \emptyset$ — (a)

$B \cap V^c \neq \emptyset$ — (b)

$A \cap V^c \neq \emptyset,\ B \cap V^c \neq \emptyset$ — (c)

In the case (a), (29) is equivalent to the inequality

$$a'_{\mathrm{m}} \left(\sum_{k=1}^q q_k b'_k\right) a_{\mathrm{m}} b_{\mathrm{m}} \leqslant a_{\mathrm{m}} \left(\sum_{k=1}^q q_k b_k\right) a'_{\mathrm{m}} b'_{\mathrm{m}}. \quad (34)$$

To see that this inequality holds, it is sufficient to observe that the ordering between the magnetic fields implies $b_{\mathrm{m}} b'_k \leqslant b'_{\mathrm{m}} b_k$. Multiplying this last inequality by $a'_m a_m$ and summing over $k = 1, \ldots, q$, we obtain (34).

In the case (b) the inequality (29) reduces to

$$\left(\sum_{j=1}^q q_j a'_j\right) b'_{\mathrm{m}} a_{\mathrm{m}} b_{\mathrm{m}} \leqslant \left(\sum_{j=1}^q q_j a_j\right) b_{\mathrm{m}} a'_{\mathrm{m}} b'_{\mathrm{m}}.$$

Now we use that the magnetic field ordering implies that $a_{\mathrm{m}}a'_j \leqslant a'_{\mathrm{m}}a_j$ and then proceed similarly to the previous case. Finally, in the case (c) the inequality (29) is equivalent to $a'_{\mathrm{m}}b'_{\mathrm{m}}a_{\mathrm{m}}b_{\mathrm{m}} = a'_{\mathrm{m}}b'_{\mathrm{m}}a'_{\mathrm{m}}b'_{\mathrm{m}}$, which is trivial. □

Our next result is the monotonicity, in the FKG sense, with respect to the coupling constants $\boldsymbol{J}$ in the special case where $J_{ij} \equiv J$, $\forall\, i,j \in \mathbb{V}$.

Theorem 7. *Suppose that $0 \leqslant J_1 < J_2$ are two coupling constants. For each finite $V \subset \mathbb{V}$ denote by $\phi^{\mathrm{GRC},J_k}_{V,\#}$, k=1,2; the measure defined by the weights* (13) *or by the weights* (14) *with* $\mathrm{m} \in \cap_{i\in\mathbb{V}} \in \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$. *Then*

$$\phi^{\mathrm{GRC},J_1}_{V,\max}(f) \leqslant \phi^{\mathrm{GRC},J_2}_{V,\mathrm{free}}(f),$$

where f is a cylindrical increasing function, and $\#$ stands for "free" *or* "max"*.*

Proof. By assuming $J_1 < J_2$, we get that $e^{q\beta J_1} - 1 < e^{q\beta J_2} - 1$. For any configuration $\omega \in \{0,1\}^{\mathbb{B}(V)}$, we define the function $g : \{0,1\}^{\mathbb{B}(V)} \to \mathbb{R}$ by

$$g(\omega) \equiv \left[\frac{e^{q\beta J_1} - 1}{e^{q\beta J_2} - 1}\right]^{o(\omega)} \times \prod_{\substack{C(\omega):\\ \mathbb{V}(C(\omega))\cap\partial V\neq\emptyset}} \exp\big(\beta \sum_{i\in C(\omega)} h_{i,\max}\big),$$

where $o(\omega)$ denotes the numbers of open edges in ω. One can easily see that the function g is decreasing since g is composed by the product of non-negative decreasing functions.

Let $f : \{0,1\}^{\mathbb{B}(V)} \to \mathbb{R}$ be a cylindrical increasing arbitrary function. Since $J_{ij} \equiv J$, we have the following expression for Bernoulli factor: $B_{J_k}(\omega) = \big(e^{q\beta J_k} - 1\big)^{o(\omega)}$, $k = 1, 2$. From the definition of the expected values we obtain

$$\begin{aligned}
\phi^{\mathrm{GRC},J_1}_{V,\max}(f) &= \frac{1}{Z^{\mathrm{GRC},J_1}_{V,\max}} \sum_{\omega\in\{0,1\}^{\mathbb{B}(V)}} f(\omega)\big(e^{q\beta J_1} - 1\big)^{o(\omega)} \prod_{C(\omega)} \Theta_{V,\max}(C(\omega))\\
&= \frac{1}{Z^{\mathrm{GRC},J_2}_{V,\max}} \sum_{\omega\in\{0,1\}^{\mathbb{B}(V)}} f(\omega)g(\omega)\,\big(e^{q\beta J_2} - 1\big)^{o(\omega)}\\
&\qquad\qquad \times \prod_{\substack{C(\omega):\\ \mathbb{V}(C(\omega))\cap\partial V=\emptyset}} \sum_{p=1}^{q} e^{\beta\sum_{i\in C(\omega)} h_{i,p}} \times \frac{Z^{\mathrm{GRC},J_2}_{V,\mathrm{free}}}{Z^{\mathrm{GRC},J_1}_{V,\max}}\\
&= \phi^{\mathrm{GRC},J_2}_{V,\mathrm{free}}(f\cdot g) \times \frac{Z^{\mathrm{GRC},J_2}_{V,\mathrm{free}}}{Z^{\mathrm{GRC},J_1}_{V,\max}},
\end{aligned} \tag{35}$$

where $Z_{V,\#}^{\mathrm{GRC},J}$ denotes the normalization constant of the measure $\phi_{V,\#}^{\mathrm{GRC},J}$ and $\#$ stands for "free" or "max". By taking $f \equiv 1$ in (35) we get the following equality

$$\phi_{V,\mathrm{free}}^{\mathrm{GRC},J_2}(g) = \frac{Z_{V,\max}^{\mathrm{GRC},J_1}}{Z_{V,\mathrm{free}}^{\mathrm{GRC},J_2}}.$$

Using the last equation, (35) and the strong FKG property (Theorem 4) we finally conclude that

$$\phi_{V,\max}^{\mathrm{GRC},J_1}(f) = \frac{\phi_{V,\mathrm{free}}^{\mathrm{GRC},J_2}(f \cdot g)}{\phi_{V,\mathrm{free}}^{\mathrm{GRC},J_2}(g)} \overset{\mathrm{FKG}}{\leqslant} \phi_{V,\mathrm{free}}^{\mathrm{GRC},J_2}(f).$$

□

Remark 7. *Note that Theorem 7 can be extended using Item* (*iii*) *of Theorem 5 for any pair of* GRC *Gibbs measures at* $J = J_1$, *resp.* $J = J_2$. *As a particular case, we obtain the following corollary.*

Corollary 4 (Monotonicity in coupling constant)**.** *Suppose that* $0 \leqslant J_1 < J_2$ *are two coupling constants. For each finite* $V \subset \mathbb{V}$ *denote by* $\phi_{V,\#}^{\mathrm{GRC},J_k}$, *k=1,2; the measure defined by the weights* (13) *or by the weights* (14) *with* $\mathrm{m} \in \cap_{i\in\mathbb{V}} \in \mathcal{Q}_{i,\max}$ $(\hat{\boldsymbol{h}})$. *Then*

$$\phi_{V,\#}^{\mathrm{GRC},J_1}(f) \leqslant \phi_{V,\#}^{\mathrm{GRC},J_2}(f),$$

where f is a cylindrical increasing function and $\#$ stands for "free" *or* "max".

13 GRC model and quasilocality

In what follows we study the quasilocality of the random-cluster model in non-homogeneous magnetic field. The next lemma tells us that the specifications $\{\phi_{\mathbb{B}}^{\mathrm{GRC}}\}$ are almost surely quasilocal (see [23, 38]). To give a precise statement of this lemma, we need to introduce some notation:

$$\mathscr{M}(\Delta, \Lambda) \equiv \Big\{\omega \in \{0,1\}^{\mathbb{E}} : \forall x, y \in \Lambda,\ x \leftrightarrow \Delta^c \text{ and } y \leftrightarrow \Delta^c \Rightarrow x \underset{\mathbb{B}_0(\Delta)}{\longleftrightarrow} y\Big\}$$

where $\Lambda \subset \Delta$ are finite subsets in $\mathbb{V}$. The following lemma is an adaptation of Lemma VI.2 in [7] for our model.

Lemma 7 (Quasilocality)**.** *Let $\mathbb{B} \subset \mathbb{B}_0(\mathbb{E})$ be a finite set and f a cylindrical function depending only on the edges in $\mathbb{B}$. Then, for each pair of finite subsets (Δ, Λ) with $\mathbb{V}(\mathbb{B}) \subset \Lambda \subset \Delta$, the function*

$$\omega \mapsto \mathbb{1}_{\mathscr{M}(\Delta,\Lambda)}(\omega)\phi_{\mathbb{B}}^{\mathrm{GRC}}(f|\omega_{\mathbb{B}^c})$$

is quasilocal. If in addition $\phi \in \mathscr{G}_{\lim}^{\mathrm{GRC}}$ or $\phi \in \mathscr{G}^{\mathrm{GRC}}$ have at most one infinite connected component and $\Lambda \subset \mathbb{V}$, then

$$\phi(\mathscr{M}(\Delta, \Lambda)) \uparrow 1, \quad \textit{whenever} \quad \Delta \uparrow \mathbb{V}.$$

Proof. Recalling the definition of $\phi_{\mathbb{B}}^{\mathrm{GRC}}(\cdot|\omega_{\mathbb{B}^c})$, we note that it is enough to prove that the function

$$\omega \mapsto \mathbb{1}_{\mathscr{M}(\Delta,\Lambda)}(\omega)\phi_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}), \quad \forall\, \overline{\omega}_{\mathbb{B}} \in \{0,1\}^{\mathbb{B}} \tag{36}$$

is quasilocal. In the sequel we shall prove the quasilocality of the mapping defined in (36). Let $\tilde{\Delta}$ be a finite subset of $\mathbb{V}$ such that $\Delta \subset \tilde{\Delta}$. Consider the following configurations:

$$\omega \equiv (*, \ldots, *, \underbrace{0}_{b-\mathrm{th}}, *, \ldots), \qquad \omega^b \equiv (*, \ldots, *, \underbrace{1}_{b-\mathrm{th}}, *, \ldots)$$

where $*$ is an arbitrary element in $\{0, 1\}$ and $b \in \mathbb{B}(\tilde{\Delta})^c$. Suppose that $\omega \in \mathscr{M}(\Delta, \Lambda)$ and that there exists a connected component C^* connecting Λ to $\mathbb{B}(\tilde{\Lambda})^c$ in ω. By definition, we have that $\omega^b \in \mathscr{M}(\Delta, \Lambda)$ and the connected component C^* is unique. Let us consider two cases:

$$1)\ \mathbb{V}(C^*) \cap \mathbb{V}(\{b\}) = \emptyset \quad \text{and} \quad 2)\ \mathbb{V}(C^*) \cap \mathbb{V}(\{b\}) \neq \emptyset.$$

In the first case we trivially have $|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}^b) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c})| = 0$. The second case is more elaborate. We consider separately two cases. We first assume that there is some $\varepsilon > 0$ such that $\varepsilon < |h_{i,\max} - h_{i,m}|$ for all $i \in \mathbb{V}$ and $m \in \{1, \ldots, q\}$. For this case let us denote by C_b^* the connected component $\mathbb{V}(C^*) \cap \mathbb{V}(\{b\})$. Then

$$\begin{aligned} &|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}^b) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c})| \leqslant \overline{B}_J(\omega)k(\omega)\times \\ &\times \sum_{m=1}^{q} q_m \Big| \exp\big(-\beta \sum_{i\in C_b^*}(h_{i,\max} - h_{i,m})\big) - \exp\big(-\beta \sum_{i\in C^*}(h_{i,\max} - h_{i,m})\big)\Big|, \end{aligned}$$

where

$$k(\omega) \equiv \prod_{\substack{C(\omega):\mathbb{V}(C)\cap\mathbb{V}(\mathbb{B})\neq\emptyset \\ |C|<\infty}} \sum_{m=1}^{q} q_m \exp\big(-\beta \sum_{i\in C}(h_{i,\max} - h_{i,m})\big) < \infty$$

and

$$\overline{B}_J(\omega) = \prod_{\{i,j\}\in\mathbb{B}:\omega_{ij}=1} r_{ij}.$$

Suppose that $m \notin \bigcap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$, then we have

$$d(b,\Lambda) \leqslant |\mathbb{V}(C_b^*)| \leqslant \sum_{i\in C_b^*} \frac{1}{\varepsilon}(h_{i,\max} - h_{i,m}).$$

The last inequalities imply that if $d(b,\Lambda) \to \infty$, then

$$\sum_{i\in C_b^*}(h_{i,\max} - h_{i,m}) \to \infty \qquad \text{and} \qquad \sum_{i\in C^*}(h_{i,\max} - h_{i,m}) \to \infty,$$

whenever $m \notin \bigcap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$. Therefore, whenever $d(b,\Lambda) \to \infty$, we have that

$$|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}^b) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c})| \to 0. \tag{37}$$

In the case $\liminf_{i\in\mathbb{V}} |h_{i,\max} - h_{i,m}| = 0$ it is enough to analyze whether

$$\sum_{i\in C^*}(h_{i,\max} - h_{i,m})$$

is finite or not. If it is infinite, then the result is trivial. Otherwise we use the continuity of the exponential function and a suitable choice of b so that $d(b,\Lambda) \to \infty$. Now we consider two different configurations:

$$\hat{\omega} \equiv (*,\ldots,*,\underbrace{0}_{b-\mathrm{th}},*,\ldots,*,\underbrace{1}_{b'-\mathrm{th}},*,\ldots)$$

and

$$\tilde{\omega} \equiv (*,\ldots,*,\underbrace{1}_{b-\mathrm{th}},*,\ldots,*,\underbrace{0}_{b'-\mathrm{th}},*,\ldots),$$

where $*$ is arbitrary in $\{0,1\}$ and $b, b' \in \mathbb{B}(\tilde{\Delta})^c$. We also denote

$$\omega \equiv (*,\ldots,*,\underbrace{0}_{b-\mathrm{th}},*,\ldots,*,\underbrace{0}_{b'-\mathrm{th}},*,\ldots,*),$$

with $*$ arbitrary in $\{0,1\}$. See that $\hat{\omega} = \omega^{b'}$ and $\tilde{\omega} = \omega^b$. Then by (37) and the triangle inequality we have

$$|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\hat{\omega}_{\mathbb{B}^c}^b) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\tilde{\omega}_{\mathbb{B}^c})|$$
$$\leqslant$$
$$|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}^{b'}) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c})| + |W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c}^b) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\omega_{\mathbb{B}^c})| \to 0,$$

when $d(b',\Lambda), d(b,\Lambda) \to \infty$. Following this reasoning, we prove that for any two distinct configurations $\hat{\omega}, \tilde{\omega}$ em $\mathbb{B}(\tilde{\Delta})^c$ we have

$$\left|W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\hat{\omega}^b_{\mathbb{B}^c}) - W_{\mathbb{B}}^{\mathrm{GRC}}(\overline{\omega}_{\mathbb{B}}|\tilde{\omega}_{\mathbb{B}^c})\right| \to 0, \quad \text{whenever} \quad \min_{b\in A} d(b,\Lambda) \to \infty$$

and $A \equiv \{e \in \mathbb{B}(\tilde{\Delta})^c : \hat{\omega}_e \neq \tilde{\omega}_e\}$, thus proving the quasilocality of the application (36).

Finally, in order to prove the second statement, it is enough to notice that $\{\mathscr{M}(\Delta,\Lambda) : \Delta \subset \mathbb{V} \text{ finite}\}$ is an increasing sequence of events. □

Lemma 8 (Subsets of Gibbs measures)**.** *Let $q \in \mathbb{Z}^+$, $\beta \geqslant 0$, $\boldsymbol{J} = (J_{ij} : \{i,j\} \in \mathbb{E}) \in [0,\infty)^{\mathbb{E}}$, $\hat{\boldsymbol{h}} = (h_{i,p} \in \mathbb{R} : i \in \mathbb{V},\ p = 1,\dots,q)$ and $\{q_p : p = 1,\dots,q\}$ satisfying* (15)*. If $\phi \in \mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}}$ and has at most one infinite connected component then $\mathscr{G}^{\mathrm{GRC}}_{\mathrm{lim}} \subset \mathscr{G}^{\mathrm{GRC}}$.*

Proof. The proof of this lemma follows from Lemma 7 and the almost sure quasilocality. See [7]. □

The next theorem gives sufficient conditions for quasilocality of the specifications under a geometric assumption of almost sure existence of an infinite connected component in the graph, thus facilitating many technical calculations.

Theorem 8 (Conditional expectations for GRC)**.** *Let $\beta \geqslant 0, J_{ij} \geqslant 0$, $h_{i,m} \in \mathbb{R}$, $\forall i,j \in \mathbb{V}$ and $q_m > 0, m = 1,\dots,q$ satisfying* (15)*. If $\phi \in \mathscr{G}^{\mathrm{GRC}}$ and has at most one infinite connected component almost surely, $\mathbb{B} \subset \mathbb{B}_0(\mathbb{V})$, and f is a cylindrical function depending on the configuration $\omega_{\mathbb{B}}$, then*

$$\phi(f|\mathscr{F}_{\mathbb{B}^c}) = \phi_{\mathbb{B}}^{\mathrm{GRC}}(f|\omega_{\mathbb{B}^c}), \quad \phi\text{−a.s.}$$

Sketch of the Proof. The idea is the same as the one employed in the proof of the Theorem III.4 in [7]. For the sake of completeness, we sketch a proof. Let $\mathbb{B}_1$, $\mathbb{B}_2$ be finite sets of bonds with $\mathbb{B}_1 \cap \mathbb{B}_2 = \emptyset$ and f and g be bounded cylinder functions depending only on the bonds in $\mathbb{B}_1$ and $\mathbb{B}_2$, respectively. Using the DLR equation (23) and the consistence of the specifications $\{\phi_{\mathbb{B}}^{\mathrm{GRC}}\}$, for $\mathbb{B} \supset \mathbb{B}_1 \cap \mathbb{B}_2$ we can easily obtain that

$$\phi(gf) = \lim_{\mathbb{B}\uparrow\mathbb{E}} \int \phi_{\mathbb{B}}^{\mathrm{GRC}}\big(g\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big|\omega_{\mathbb{B}^c}\big)\phi(\mathrm{d}\omega). \tag{38}$$

Let $\Delta \supset \mathbb{V}(\mathbb{B}_1)$, since both g and $\mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}(\cdot)\phi_{\mathbb{B}}^{\mathrm{GRC}}(f|\cdot)$ are quasilocal, the function $g \cdot \mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}\phi_{\mathbb{B}}^{\mathrm{GRC}}(f|\cdot)$ can be approximated by local functions.

Then by DLR equation (23), we have

$$\phi\big(g\cdot\mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big)$$
$$=\lim_{\mathbb{B}\uparrow\mathbb{E}}\int\phi_{\mathbb{B}}^{\mathrm{GRC}}\big(g\cdot\mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big|\omega_{\mathbb{B}^c}\big)\phi(\mathrm{d}\omega). \quad (39)$$

From Lemma 7 we get $\phi\big(\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))\big)\uparrow 1$ whenever $\Delta\uparrow\mathbb{V}$. Since f and g are bounded, using the Dominated Convergence Theorem we have

$$\lim_{\Delta\uparrow\mathbb{V}}\int\phi_{\mathbb{B}}^{\mathrm{GRC}}\big(g\cdot\mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big|\omega_{\mathbb{B}^c}\big)\phi(\mathrm{d}\omega)=\int\phi_{\mathbb{B}}^{\mathrm{GRC}}\big(g\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big|\omega_{\mathbb{B}^c}\big)\phi(\mathrm{d}\omega)$$

and

$$\lim_{\Delta\uparrow\mathbb{V}}\phi\big(g\cdot\mathbb{1}_{\mathscr{M}(\Delta,\mathbb{V}(\mathbb{B}_1))}\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big)=\phi\big(g\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big).$$

Combining the above limits, together with the items (38) and (39), we have

$$\phi(gf)=\phi\big(g\phi_{\mathbb{B}_1}^{\mathrm{GRC}}(f|\cdot)\big)$$

for all bounded g depending only on the configurations $\omega_{\mathbb{B}_1^c}$. From the almost sure uniqueness of conditional expectation with respect to ϕ, the proof follows. □

Using the general theory of thermodynamic formalism, one can prove the following lemma.

Lemma 9 (Monotonicity in the volume, [7]). *Let $q\in\mathbb{Z}^+$, $\beta\geqslant 0$, $\boldsymbol{J}=(J_{ij}:\{i,j\}\in\mathbb{E})\in[0,\infty)^{\mathbb{E}}$, the magnetic field be $\hat{\boldsymbol{h}}$ and the sequence $\{q_p:p=1,\dots,q\}$ satisfy (15). If $\Lambda\subset V$ are finite subsets of $\mathbb{V}$, then for any cylindrical increasing function f we have*

$$\phi_{\Lambda,\mathrm{free}}^{\mathrm{GRC}}(f)\leqslant\phi_{V,\mathrm{free}}^{\mathrm{GRC}}(f)\quad\text{and}\quad\phi_{\Lambda,\max}^{\mathrm{GRC}}(f)\geqslant\phi_{V,\max}^{\mathrm{GRC}}(f).$$

Remark 8. *When $q_p=1$, for all $p=1,\dots,q$ in (12) , then we call the model simply the* RC *model. In this case, we define the set of Gibbs measures $\mathscr{G}^{\mathrm{RC}}$ and $\mathscr{G}_{\lim}^{\mathrm{RC}}$ similarly to (23) and (24).*

From now on, the study turns to some fundamental properties of the RC model. The following theorem is valid only for the random-cluster model.

Theorem 9. *Let $q\in\mathbb{Z}^+$, $\beta\geqslant 0$, $\boldsymbol{J}=(J_{ij}:\{i,j\}\in\mathbb{E})\in[0,\infty)^{\mathbb{E}}$ and $\hat{\boldsymbol{h}}$ be a magnetic field as previously defined. Given $\nu\in\mathscr{G}^{\mathrm{ES}}$, let ϕ_ν denote its edge-marginal. Then for any cylindrical increasing function f we have $\phi_\nu(f)\leqslant\phi_{\max}^{\mathrm{RC}}(f)$.*

Proof. For more details see the proof of Theorem III.2 reference [7]. □

14 Uniqueness of the infinite connected component

We have so far developed the theory of the random-cluster model with non-uniform magnetic field for countably infinite graphs. We are interested in the situation in which the infinite connected component is (almost surely) unique, as is commonly the case for an "amenable graph". The amenability hypothesis is important for the uniqueness of the infinite connected component in several models, [13, 28, 31]. When the graph is non-amenable, the non-uniqueness of the infinite connected component is known for several models including the Bernoulli percolation and null magnetic field random-cluster model, see [6, 28, 31] and references therein. Therefore, from now on we assume tacitly that the lattice $\mathbb{L}$ is amenable, that is, $\inf\{|\partial_{\mathbb{E}} V|/|V|\} = 0$, where the infimum ranges over all finite connected subsets V of $\mathbb{V}$, and $\partial_{\mathbb{E}} V$ is the set of edges with one end-vertex in V and one in $\mathbb{V} \setminus V$.

In what follows we denote by N_∞ the random variable that counts the number of infinite connected components in both sample spaces $\Omega \equiv \{0,1\}^{\mathbb{E}}$ and $\Sigma_q \times \Omega$.

Theorem 10 (Uniqueness of the infinite connected component)**.** *Let $\beta > 0$ be the inverse temperature and $\hat{\boldsymbol{h}}$ a magnetic field. Then*

$$\phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(N_\infty \leqslant 1) = \phi_{\mathrm{free}}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(N_\infty \leqslant 1) = 1.$$

Proof. We only present the argument for $\phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}} \in \mathscr{G}^{\mathrm{GRC}}$, since for the free boundary condition case the proof works similarly. Let $\Lambda \subset \Delta$ be finite subsets of $\mathbb{V}$ and $D_{\Lambda,\Delta}$ the set of all $\omega \in \Omega$ with the property: there exist two points $u, v \in \partial\Lambda$ such that both u and v are joined to $\partial\Delta$ by paths using ω-open edges of $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$, but u is not joined to v by a path using ω-open edges of $\mathbb{E}_\Delta$. For any fixed configuration $\eta \in \Omega$ the mapping $\omega \mapsto \mathbb{1}_{D_{\Lambda,\Delta}}(\omega_{\mathbb{E}_\Lambda}\eta_{\mathbb{E}_{\mathbb{V}\setminus\Lambda}})$ is decreasing. Because of the definition of $D_{\Lambda,\Delta}$ we can abuse notation and simply write $\mathbb{1}_{D_{\Lambda,\Delta}}(\omega_{\mathbb{E}_\Lambda}\eta_{\mathbb{E}_{\Delta\setminus\Lambda}})$. Given $\epsilon > 0$ small enough, we consider the external magnetic field $\epsilon\hat{\boldsymbol{h}} \equiv (\epsilon h_{i,p}, \forall\, i \in \mathbb{V}, p = 1, \ldots, q)$. A straightforward computation shows that $\epsilon\hat{\boldsymbol{h}} \prec \hat{\boldsymbol{h}}$, where the partial order is given by (25). If V contains Δ then it follows from the Theorem 6 that

$$\phi_{V,\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(\mathbb{1}_{D_{\Lambda,\Delta}}(\cdot\,\eta_{\mathbb{E}_{\Delta\setminus\Lambda}})) \leqslant \phi_{V,\max}^{\mathrm{GRC},\epsilon\hat{\boldsymbol{h}}}(\mathbb{1}_{D_{\Lambda,\Delta}}(\cdot\,\eta_{\mathbb{E}_{\Delta\setminus\Lambda}})).$$

By summing over all $\eta_{\mathbb{E}_{\Delta\setminus\Lambda}}$ the above inequality we get that

$$\phi_{V,\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(D_{\Lambda,\Delta}) \leqslant \phi_{V,\max}^{\mathrm{GRC},\epsilon\hat{\boldsymbol{h}}}(D_{\Lambda,\Delta}).$$

Taking $\epsilon \to 0$, $V \uparrow \mathbb{V}$ and using the continuity of $\mathbb{1}_{D_{\Lambda,\Delta}}$ and the Theorem 5 we get from the last inequality that

$$\phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(D_{\Lambda,\Delta}) \leqslant \phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{0}}}(D_{\Lambda,\Delta}).$$

Since $\cap_{\Delta\supset\Lambda} D_{\Lambda,\Delta} \uparrow \{N_\infty > 1\}$, when $\Lambda \uparrow \mathbb{V}$ it follows from the continuity of the measure and Theorem III.3 of [7] that $\phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(N_\infty > 1) = 0$. $\square$

To state our next theorem, which is the main theorem of the next section, we need to introduce the following parameters:

$$P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) \equiv \sup_{x\in\mathbb{V}} \sup_{\phi\in\mathscr{G}^{\mathrm{GRC}}} \phi(|C_x| = \infty)$$

and

$$\widetilde{P}_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) \equiv \sup_{x\in\mathbb{V}} \inf_{\phi\in\mathscr{G}^{\mathrm{GRC}}} \phi(|C_x| = \infty),$$

where C_x is the infinite connected component containing the vertex x. For the RC model, the parameters P_∞ and $\widetilde{P}_\infty$ are defined similarly. We also define the critical parameter

$$\beta_c(\boldsymbol{J}, \hat{\boldsymbol{h}}) \equiv \inf\{\beta > 0 : P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) > 0\}.$$

To lighten the notation we introduce for each $\mathrm{m} \in \{1, \ldots, q\}$ fixed, the event

$$\mathscr{A}_{\geqslant 1,\mathrm{m}}^{\infty} \equiv \left\{ (\sigma,\omega) \in \Sigma_q \times \Omega : \begin{array}{c} N_\infty(\sigma,\omega) \geqslant 1 \text{ and all vertices in any infinite} \\ \text{connected component satisfies } \sigma_x = \mathrm{m} \end{array} \right\}.$$

15 Uniqueness and phase transition

Now we are ready to state and prove one of the main theorems of this paper. We emphasize that this theorem was inspired by Theorem II.5 in [7].

Theorem 11 (Uniqueness and phase transition)**.** *Fix* $q \in \mathbb{Z}^+$, $\beta \geqslant 0$, *a magnetic field* $\hat{\boldsymbol{h}} = (h_{i,p} \in \mathbb{R} : i \in \mathbb{V}, p = 1, \ldots, q)$ *and* $\{q_p : p = 1, \ldots, q\}$ *satisfying* (15).

(i) For all $\boldsymbol{J} \geqslant 0$ $(J_{ij} \geqslant 0,\ \forall \{i,j\} \in \mathbb{E})$, *there is at most one probability measure* μ_0 *in* $\mathscr{G}_0^{\mathrm{ES}} \equiv \{\nu \in \mathscr{G}^{\mathrm{ES}} : \nu(N_\infty = 0) = 1\}$.

(ii) If $P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) = 0$, *then* $|\mathscr{G}^{\mathrm{ES}}| = |\mathscr{G}^{\mathrm{RC}}| = 1$. *In particular, if* $\beta < \beta_c(\boldsymbol{J}, \hat{\boldsymbol{h}})$, *then* $|\mathscr{G}^{\mathrm{ES}}| = |\mathscr{G}^{\mathrm{RC}}| = 1$.

(iii) *If* $\boldsymbol{J} \geqslant 0$ *is an uniform coupling constant* ($J_{ij} \equiv J \geqslant 0$, $\forall\ \{i,j\} \in \mathbb{E}$), *then* $P_\infty(\beta, J, \hat{\boldsymbol{h}}) = \sup_{x\in\mathbb{V}}\ \phi^{\mathrm{GRC}}_{\max}(|C_x| = \infty)$ *and* $\widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) = \sup_{x\in\mathbb{V}}\ \phi^{\mathrm{GRC}}_{\mathrm{free}}(|C_x| = \infty)$.

(iv) *Let* $\boldsymbol{J} \geqslant 0$ ($J_{ij} \equiv J \geqslant 0$, $\forall\ \{i,j\} \in \mathbb{E}$). *If* $P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) > 0$ *then the states* $\phi^{\mathrm{ES}}_{\mathrm{m}}$, $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$ *are extremal* ES *Gibbs states with* $\phi^{\mathrm{ES}}_{\mathrm{m}}(\mathscr{A}^{\infty}_{\geqslant 1,\mathrm{m}}) = 1$. *Moreover under the strong assumption* $\widetilde{P}_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) > 0$ *we have that* $|\mathscr{G}^{\mathrm{ES}}| > 1$.

(v) *Let* $\boldsymbol{J} \geqslant 0$ ($J_{ij} \equiv J \geqslant 0$, $\forall\ \{i,j\} \in \mathbb{E}$). *If* $\beta < \beta_c$, *then* $P_\infty(\beta, J, \hat{\boldsymbol{h}}) = \widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) = 0$, *while both* $P_\infty(\beta, J, \hat{\boldsymbol{h}}) > 0$ *and* $\widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) > 0$ *whenever* $\beta > \beta_c$.

Proof. The whole proof follows closely reference [7]. (i) We prove that $\mathscr{G}^{\mathrm{ES}}_0 = \{\phi^{\mathrm{ES}}_{\mathrm{free}}\}$, In fact, let $\nu \in \mathscr{G}^{\mathrm{ES}}_0$ and $\{\Delta_n : n \in \mathbb{N}\}$ be a cofinal sequence of subsets of $\mathbb{V}$. Then the sequence of random sets $\{\Lambda_n : n \in \mathbb{N}\}$ defined by $\Lambda_n(\omega) \equiv \{x \in \Delta_n : x \not\leftrightarrow \Delta_n^c \text{ in } \omega\}$ is also increasing. Note that the set Λ_n is well defined due to the absence of infinite connected components. By Theorem 5, given $\epsilon > 0$, we can take Δ big enough so that for each function f with support in $(\Delta, \mathbb{B}_0(\Delta))$ we have

$$|\phi^{\mathrm{ES}}_{V,\mathrm{free}}(f) - \phi^{\mathrm{ES}}_{\mathrm{free}}(f)| \leqslant \epsilon, \quad \forall\ V \supset \Delta. \tag{40}$$

On the other hand, we have that

$$\begin{aligned}\nu(f) &= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\supset\Delta\}}) \\ &= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}).\end{aligned}$$

By using the DLR equations and their equivalent version of conditional expectations for the specification $\{\phi^{\mathrm{ES}}_{\Lambda,\mathbb{B}_0(\Lambda)}\}$, we can rewrite the above expression as

$$\begin{aligned}&= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\phi^{\mathrm{ES}}_{\overline{\Lambda}_n,\mathbb{B}_0(\overline{\Lambda}_n)}(f\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}|\cdot)\big) \\ &= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\nu(f\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}|\mathscr{F}_{\overline{\Lambda}_n^c,\mathbb{B}_0(\overline{\Lambda}_n)^c})\big).\end{aligned}$$

For each fixed n, the random variable $\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}$ depends only on the states of the sites and edges in $(\overline{\Lambda}_n, \mathbb{B}_0(\overline{\Lambda}_n))$, so we have that this random variable is independent of the σ-algebra $\mathscr{F}_{\overline{\Lambda}_n^c,\mathbb{B}_0(\overline{\Lambda}_n)^c}$. Hence the latter expression can be rewritten as

$$= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}\nu(f|\mathscr{F}_{\overline{\Lambda}_n^c,\mathbb{B}_0(\overline{\Lambda}_n)^c})\big)$$
$$= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}\phi^{\mathrm{ES}}_{\overline{\Lambda}_n,\mathbb{B}_0(\overline{\Lambda}_n)}(f|\cdot)\big)$$
$$= \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}\phi^{\mathrm{ES}}_{\overline{\Lambda}_n,\mathrm{free}}(f)\big),$$

where in the second equality we have used again the equivalent version of conditional expectation for specification $\{\phi^{\mathrm{ES}}_{\Lambda,\mathbb{B}_0(\Lambda)}\}$, and in the last one we use the definition of the measure $\phi^{\mathrm{ES}}_{\overline{\Lambda}_n,\mathrm{free}}$. So we have the identity

$$\nu(f) = \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + \sum_{\overline{\Lambda}_n(\cdot)\supset\Delta} \nu\big(\mathbb{1}_{\{\Lambda_n(\cdot)=\overline{\Lambda}_n\}}\phi^{\mathrm{ES}}_{\overline{\Lambda}_n,\mathrm{free}}(f)\big). \tag{41}$$

Combining the identities (40) and (41) we have

$$\nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + [\phi^{\mathrm{ES}}_{\mathrm{free}}(f) - \epsilon]\nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\supset\Delta\}}) \leqslant \nu(f)$$
$$\leqslant \nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\not\supset\Delta\}}) + [\phi^{\mathrm{ES}}_{\mathrm{free}}(f) + \epsilon]\nu(f\mathbb{1}_{\{\Lambda_n(\cdot)\supset\Delta\}}). \tag{42}$$

Since the sequence $\{\Lambda_n(\omega) : n \in \mathbb{N}\}$ is increasing, the sequence $\{A_n : n \in \mathbb{N}\}$, with $A_n \equiv \{\Lambda_n(\cdot) \supset \Delta\}$, is also increasing. Therefore $\mathbb{1}_{A_n} \uparrow 1$. Since f is bounded, taking $n \uparrow \infty$ in (42) and using the Dominated Convergence Theorem yields

$$|\nu(f) - \phi^{\mathrm{ES}}_{\mathrm{free}}(f)| \leqslant \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude the proof of this item.

(ii) If $P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) = 0$, then

$$\phi(|C_x| = \infty) = 0, \quad \forall\ \phi \in \mathscr{G}^{\mathrm{RC}} \text{ and } x \in \mathbb{V}. \tag{43}$$

By the uniqueness of the infinite connected component (Theorem 10) and Lemma 8, it follows that property (43) holds for $\phi^{\mathrm{RC}}_{\max}$. If ϕ_ν denotes the edge-marginal of $\nu \in \mathscr{G}^{\mathrm{ES}}$, by Theorem 9 we have

$$0 = \phi^{\mathrm{RC}}_{\max}(N_\infty > 0) \geqslant \phi_\nu(N_\infty > 0) = \nu(N_\infty > 0),$$

which implies that $\nu \in \mathscr{G}^{\mathrm{ES}}_0$. Therefore it follows from the item *(i)* that $\nu = \phi^{\mathrm{ES}}_{\mathrm{free}}$. That is, $\mathscr{G}^{\mathrm{ES}} = \mathscr{G}^{\mathrm{ES}}_0 = \{\phi^{\mathrm{ES}}_{\mathrm{free}}\}$.

On the other hand, if we denote by $\mathscr{G}_0^{\mathrm{RC}} \equiv \{\phi \in \mathscr{G}^{\mathrm{RC}} : \phi(N_\infty = 0) = 1\}$, we have from (43) and Theorem 5,

$$0 = \phi_{\max}^{\mathrm{RC}}(N_\infty > 0) \geqslant \phi(N_\infty > 0), \quad \forall\, \phi \in \mathscr{G}^{\mathrm{RC}}.$$

Thus $\phi \in \mathscr{G}_0^{\mathrm{RC}}$. By repeating the proof of item (i), using Theorem 8 and the DLR equations (23), we have that $\mathscr{G}^{\mathrm{RC}} = \{\phi_{\mathrm{free}}^{\mathrm{RC}}\}$.

(iii) Using Item (iii) of Theorem 5 gives

$$P_\infty(\beta, J, \hat{\boldsymbol{h}}) \leqslant \sup_{x\in\mathbb{V}} \phi_{\max}^{\mathrm{GRC}}(|C_x| = \infty) \ \text{ and } \ \widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) \geqslant \sup_{x\in\mathbb{V}} \phi_{\mathrm{free}}^{\mathrm{GRC}}(|C_x| = \infty).$$

To prove that the equality is attained, it is enough to show that $\phi_{\max}^{\mathrm{GRC}} \in \mathscr{G}^{\mathrm{GRC}}$ and $\phi_{\mathrm{free}}^{\mathrm{GRC}} \in \mathscr{G}^{\mathrm{GRC}}$, respectively. By using Theorem 10, we have that $\phi_{\max}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(N_\infty \leqslant 1) = \phi_{\mathrm{free}}^{\mathrm{GRC},\hat{\boldsymbol{h}}}(N_\infty \leqslant 1) = 1$, thus we conclude from Lemma 8 that $\phi_{\max}^{\mathrm{GRC}} \in \mathscr{G}^{\mathrm{GRC}}$ and $\phi_{\mathrm{free}}^{\mathrm{GRC}} \in \mathscr{G}^{\mathrm{GRC}}$.

(iv) Using the same technique employed by [7], one can prove that

$$\phi_{\max}^{\mathrm{RC}}(x \leftrightarrow \infty) = \lim_{V\uparrow\mathbb{V}} \phi_{V,\max}^{\mathrm{RC}}(x \leftrightarrow V^c).$$

As a consequence, we have that, for all $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$,

$$\phi_{\mathrm{m}}^{\mathrm{ES}}(x \leftrightarrow \infty) = \lim_{V\uparrow\mathbb{V}} \phi_{V,\mathrm{m}}^{\mathrm{ES}}(x \leftrightarrow V^c).$$

Combining the two last identities with the trivial fact

$$\phi_{V,\mathrm{m}}^{\mathrm{ES}}(x \leftrightarrow V^c, \sigma_x = \tilde{\mathrm{m}}) = \phi_{V,\mathrm{m}}^{\mathrm{ES}}(x \leftrightarrow V^c)\delta_{\mathrm{m},\tilde{\mathrm{m}}},$$

and taking the thermodynamic limit, we have

$$\phi_{\mathrm{m}}^{\mathrm{ES}}(\sigma_x = \tilde{\mathrm{m}} | x \leftrightarrow \infty) = \delta_{\mathrm{m},\tilde{\mathrm{m}}}, \quad \forall \mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}}). \tag{44}$$

Now we prove that the state $\phi_{\mathrm{m}}^{\mathrm{ES}}$ is extremal whenever $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$. To this end, let us assume that $\phi_{\mathrm{m}}^{\mathrm{ES}}(\mathscr{A}_{\geqslant 1,\mathrm{m}}^{\infty}) = 1$, this will be proved below. Suppose that $\phi_{\mathrm{m}}^{\mathrm{ES}}$ is not extremal - then there are two Gibbs measures in $\mathscr{G}^{\mathrm{ES}}$ so that

$$\phi_{\mathrm{m}}^{\mathrm{ES}} = t\phi_1^{\mathrm{ES}} + (1-t)\phi_2^{\mathrm{ES}}, \quad \text{with } \phi_i^{\mathrm{ES}}(\mathscr{A}_{\geqslant 1,\mathrm{m}}^{\infty}) = 1 \text{ and } t \in (0,1). \tag{45}$$

If ϕ_i^{RC} denotes the RC marginal of ϕ_i^{ES}, it follows from Lemma VIII.1 in [7] that $\phi_i^{\mathrm{RC}} \in \mathscr{G}^{\mathrm{RC}}$, $i = 1, 2$. This implies that

$$\phi_{\max}^{\mathrm{RC}} = t\phi_1^{\mathrm{RC}} + (1-t)\phi_2^{\mathrm{RC}}, \quad t \in (0,1).$$

By stochastic domination one can prove that $\phi^{\mathrm{RC}}_{\max}$ is an extremal probability measure, so $\phi^{\mathrm{RC}}_{1} = \phi^{\mathrm{RC}}_{2} = \phi^{\mathrm{RC}}_{\max}$. Using Lemma VIII.3 in [7], this fact implies $\phi^{\mathrm{ES}}_{1} = \phi^{\mathrm{ES}}_{2}$, hence the extremality of $\phi^{\mathrm{ES}}_{\mathrm{m}}$ is proved.

Finally, we prove that $\phi^{\mathrm{ES}}_{\mathrm{m}}(\mathscr{A}^{\infty}_{\geqslant 1,\mathrm{m}}) = 1$. Since $P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) > 0$, we get from Item (iii) of Theorem 5 that

$$0 < P_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) \leqslant \sup_{x\in\mathbb{V}} \phi^{\mathrm{RC}}_{\max}(x \leftrightarrow \infty) \leqslant \phi^{\mathrm{RC}}_{\max}(N_\infty \geqslant 1). \tag{46}$$

Since $\phi^{\mathrm{RC}}_{\max}$ is an extremal Gibbs state and $\{N_\infty \geqslant 1\}$ is a tail event it follows from the inequality (46) and the uniqueness of the infinite connected component (Theorem 10) that

$$1 = \phi^{\mathrm{RC}}_{\max}(N_\infty = 1) = \phi^{\mathrm{ES}}_{\mathrm{m}}(N_\infty = 1), \quad \forall \mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}}).$$

The previous equation together with the identity (44) implies, for each $\mathrm{m} \in \cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$, that $\phi^{\mathrm{ES}}_{\mathrm{m}}(\mathscr{A}^{\infty}_{\geqslant 1,\mathrm{m}}) = 1$.

We now prove the second statement of Item (iv). As long as the set $\cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}})$ has more than one element, the result follows from the first statement of the Item (iv). Otherwise, without loss of generality, we can assume that $\cap_{i\in\mathbb{V}} \mathcal{Q}_{i,\max}(\hat{\boldsymbol{h}}) = \{1\}$. Let $\phi^{\mathrm{RC}} \in \mathscr{G}^{\mathrm{RC}}$ be a spin-marginal of ϕ^{ES}_{2}, then

$$\begin{aligned} 0 < \widetilde{P}_\infty(\beta, \boldsymbol{J}, \hat{\boldsymbol{h}}) \leq \sup_{x\in\mathbb{V}} \phi^{\mathrm{RC}}(x \leftrightarrow \infty) &= \sup_{x\in\mathbb{V}} \phi^{\mathrm{ES}}_{2}(x \leftrightarrow \infty) \\ &= \sup_{x\in\mathbb{V}} \phi^{\mathrm{ES}}_{2}(x \leftrightarrow \infty, \sigma_x = 2) \leq \phi^{\mathrm{ES}}_{2}(\mathscr{A}^{\infty}_{\geqslant 1,2}). \end{aligned}$$

Since $\phi^{\mathrm{ES}}_{1}(\mathscr{A}^{\infty}_{\geqslant 1,1}) = 1$ and $\mathscr{A}^{\infty}_{\geqslant 1,1} \cap \mathscr{A}^{\infty}_{\geqslant 2,1} = \emptyset$, it follows from the above inequality that $\phi^{\mathrm{ES}}_{1} \neq \phi^{\mathrm{ES}}_{2}$.

(v) By Item (iii) and Corollary 4, we get that the maps $J \mapsto P_\infty(\beta, J, \hat{\boldsymbol{h}})$ and $J \mapsto \widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}})$ are increasing, and so are the maps $\hat{\boldsymbol{h}} \mapsto P_\infty(\beta, J, \hat{\boldsymbol{h}})$ and $\hat{\boldsymbol{h}} \mapsto \widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}})$, with respect to the partial order (25). From the definition, one has

$$\widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) \leqslant P_\infty(\beta, J, \hat{\boldsymbol{h}}), \quad \forall \beta, J \text{ and } \hat{\boldsymbol{h}}. \tag{47}$$

From Item (iii) and Theorem 7, we get $P_\infty(\beta, J_1, \hat{\boldsymbol{h}}) \leqslant \widetilde{P}_\infty(\beta, J_2, \hat{\boldsymbol{h}})$ for all $J_1 < J_2$. By Item (iii), we have that P_∞ and $\widetilde{P}_\infty$ are thermodynamical limits. Using the form of the Hamiltonian of this model and the monotonicity properties proved above, we get, for all $\beta_1 < \beta_2$, that

$$\begin{aligned} P_\infty(\beta_1, J, \hat{\boldsymbol{h}}) = P_\infty(1, \beta_1 J, \beta_1 \hat{\boldsymbol{h}}) &\leqslant \widetilde{P}_\infty(1, \beta_2 J, \beta_1 \hat{\boldsymbol{h}}) \\ &\leqslant \widetilde{P}_\infty(1, \beta_2 J, \beta_2 \hat{\boldsymbol{h}}) \\ &= \widetilde{P}_\infty(\beta_2, J, \hat{\boldsymbol{h}}). \end{aligned} \tag{48}$$

Combining inequalities (47) and (48) yields (v). □

We now consider the q-state Potts model where each value of the spin is coupled to a distinct and site dependent external field. The formal Hamiltonian of the model is

$$H(\hat{\sigma}) = -J \sum_{\{i,j\}} \delta_{\hat{\sigma}_i} \delta_{\hat{\sigma}_j} - \sum_{p=1}^{q} \sum_{i} \frac{h_{i,p}}{q} \delta_{\hat{\sigma}_{i,p}}. \tag{49}$$

Let $\mathscr{G}^{\text{Spin}}$ denote the set of all spin Gibbs states, defined by means of the DLR condition and the above Hamiltonian (appropriately modified to incorporate boundary conditions).

Theorem 12. *Let* $\Pi_S : \mathscr{G}^{\text{ES}} \to \mathscr{G}^{\text{Spin}}$ *denote the mapping that assigns the spin-marginal to infinite volume* ES *measure. Then* Π_S *is a linear isomorphism.*

Proof. A direct proof of this theorem can be found in [7]. □

16 Application - Ising model with power law decay external field

In this section we apply the results above obtained to prove the uniqueness of the Gibbs measures, at any positive temperature, for the Ising model in $\mathbb{L} \equiv (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d$ is the set of the nearest neighbors in the d-dimensional hypercubic lattice, with the Halmiltonian given by

$$\mathscr{H}^{\mu,\text{Ising}}_{\boldsymbol{h},V}(\sigma) \equiv -\sum_{\substack{i,j \in V \\ \{i,j\} \in \mathbb{E}}} J\, \sigma_i \sigma_j - \sum_{i \in V} h_i\, \sigma_i - \sum_{\substack{i \in V,\ j \in \partial V \\ \{i,j\} \in \mathbb{E}}} J\, \sigma_i \mu_j, \tag{50}$$

where $\alpha \geqslant 0$ and $h^* > 0$ and

$$h_i = \begin{cases} \frac{h^*}{\|i\|^\alpha}, & \text{if } i \neq 0; \\ h^*, & \text{if } i = 0. \end{cases}$$

From now on, we write $\mathscr{G}_\beta^{\text{Spin}}$ instead of $\mathscr{G}^{\text{Spin}}$ to make clear its dependence on the inverse temperature.

By Proposition 1, it follows that the set $\mathscr{G}_\beta^{\text{Spin}}$ (defined in the last section) is precisely $\mathscr{G}_{2\beta}^{\text{Potts}}(\boldsymbol{J},\boldsymbol{h}) = \mathscr{G}_\beta^{\text{Ising}}(\boldsymbol{J},\boldsymbol{h})$, the set of the Gibbs measures of the above Ising model, if we take in the Hamiltonian (49) $q = 2$ and the magnetic field given by

$$\hat{\boldsymbol{h}} = \big((h_{i,1}, h_{i,2}) \in \mathbb{R}^2 : \ h^*/\|i\|^\alpha = h_{i,1} = -h_{i,2}, \ \forall i \in \mathbb{V}\ \big).$$

In order to apply the previous results to study the uniqueness of this Ising model with magnetic field decaying to zero with polynomial rate $0 \leqslant \alpha < 1$, we will consider in this section the GRC model defined in (12) with $q = 2$, the constants $q_p \equiv 1$ and the magnetic field $\hat{\boldsymbol{h}}$ as above.

In [8] the authors proved that for any $\alpha \in [0,1)$ there is a positive inverse temperature $\beta_\alpha < +\infty$ so that, for any $\beta > 0$ such that $\beta_\alpha < \beta$, the set of the Gibbs measures for the Ising model defined by (50) is a singleton. By the Dobrushin Uniqueness Theorem, we know that for any $\beta < 1/(2dJ)$ the set of Gibbs measures for this Ising model at these inverse temperatures is also singleton. In the reference [8] it was conjectured that the set of the Gibbs measures for this model with $\alpha \in [0,1)$ is a singleton for any $\beta > 0$. In this work we settle this conjecture.

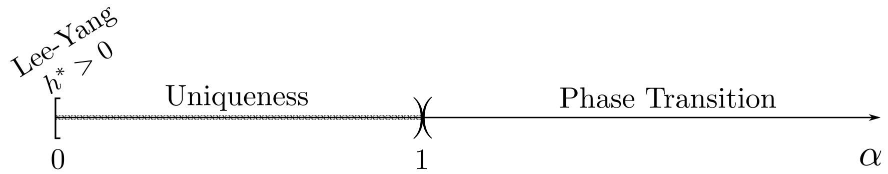


Figure 3: *Uniqueness and non-uniqueness interval for the ferromagnetic Ising model with magnetic field $h_i = h^*/\|i\|^\alpha$.*

Suppose that

$$\beta_c(\boldsymbol{J},\hat{\boldsymbol{h}}) \equiv \inf\{\beta > 0 : P_\infty(\beta,\boldsymbol{J},\hat{\boldsymbol{h}}) > 0\} = +\infty.$$

In this case, it follows from Item *(ii)* of Theorem 11 that for any $\beta > 0$ we have $|\mathscr{G}_\beta^{\text{ES}}| = 1$. By Theorem 12 we get that $|\mathscr{G}_\beta^{\text{Spin}}| = 1$.

Suppose that $\beta_c(\boldsymbol{J},\hat{\boldsymbol{h}}) < +\infty$. By using once more Item *(ii)* of Theorem 11, we obtain the uniqueness for $\beta < \beta_c(\boldsymbol{J},\hat{\boldsymbol{h}})$, that is, $|\mathscr{G}_\beta^{\text{Spin}}| = 1$ for such values of β. If $\beta > \max\{\beta_\alpha, \beta_c(\boldsymbol{J},\hat{\boldsymbol{h}})\}$ it was proved in [8] that $|\mathscr{G}_\beta^{\text{Spin}}| = 1$. We claim that $\widetilde{P}_\infty(\beta,J,\hat{\boldsymbol{h}}) \equiv 0$ for any $\beta > 0$. Indeed, take $\beta > \beta_\alpha$ if $\widetilde{P}_\infty(\beta,J,\hat{\boldsymbol{h}}) > 0$, so by Item *(iv)* of Theorem 11 we have at least two ES Gibbs measures and by Theorem 12 two Gibbs measures for the Ising model

(50) which contradicts [8]. Therefore $\widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) = 0$ whenever $\beta > \beta_\alpha$. Since the mapping $\beta \mapsto \widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}})$ is increasing, the claim follows.

From Item (v) we have for any $\beta > \beta_c(\boldsymbol{J}, \hat{\boldsymbol{h}})$ that $P_\infty(\beta, J, \hat{\boldsymbol{h}}) > 0$ and $\widetilde{P}_\infty(\beta, J, \hat{\boldsymbol{h}}) > 0$, but this contradicts the above claim. Therefore we have proved that for any $\alpha \in [0, 1)$ that $\beta_c(\boldsymbol{J}, \hat{\boldsymbol{h}}) = +\infty$, which implies by Theorem 11 that $|\mathscr{G}_\beta^{\text{Spin}}| = 1$ for any $\beta > 0$.

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