SPECTRAL TRIPLES ON THERMODYNAMIC FORMALISM
AND DIXMIER TRACE REPRESENTATIONS OF
GIBBS MEASURES

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Abstract. In this paper we construct spectral triples \((A, H, D)\) and describe
the associated Dixmier trace representations of Gibbs measures of a large class
of \(g\)-functions defined on symbolic space \(\mathcal{A}^N\). We obtain results in both con-
texts of finite and infinite countable alphabets and provide examples with
explicit computations. On finite alphabet setting one of the main contribu-
tions of this paper is to generalize this representations theorems to potentials
in the Walters class, where in general the spectral gap property is absent. As
an application we establish a noncommutative integral representation for the
DLR-Gibbs measures of the Dyson model. Similar results are proved in the
context of infinite countable alphabets, and Topological Markov Shifts, when
the spectral gap property holds. In the last section, we make some remarks
about this representation problem when \(\mathcal{A}\) is an uncountable alphabet and
discuss about its potential applications.

1. Introduction

This paper aims to contribute to the ongoing research on obtaining noncommu-
tative integral representations for Gibbs measures. The starting point of our inves-
tigations is the work of Richard Sharp [Sha16]. In his work Sharp shows, among
other things, that Gibbs Measures, appearing in the context of conformal graph di-
 rected Markov systems, can be recovered from suitable spectral triples and Dixmier
traces. These are important mathematical objects in noncommutative geometry,
where certain geometric spaces are analyzed by using operator and \(C^*\)-algebras, see
[Con85, Con94]. Although this theory has a geometric origin, there has been con-
siderable interest in finding examples where the \(C^*\)-algebra is the space continuous
functions (which is the case we consider) on suitable subspaces of infinite cartesian
products, sometimes called Cantor sets. The first examples were given by Connes
[Con85, Con89, Con94] and in the last decades this subject has attracted a lot of
attention, see for example [CI06, JKS15, JP16, KLS13, KS13, PB09, Sha12, Sha16,
Whi13] and references therein. These works concern not just Cantor sets, but also
hyperbolic dynamics, directed Markov systems, IFSs and symbolic dynamics.

Connes showed that spectral triples can be used to define a pseudo-metric on the
state space of the associated \(C^*\)-algebra. The way this pseudo-metric is defined is
analogous to the Monge-Kantorovitch metric and some general conditions ensuring
that this pseudo-metric is indeed a metric were obtained independently by Pavlović
and Rieffel, see [Pav98, Rie98]. Kesseböhmer and Samuel [KS13] study metric aspects

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of this theory, following [CI06], but in the context of Gibbs measures. They proved that Connes’ pseudo-metric is actually a metric that induces the weak-∗-topology on the space of Borel probability measures on certain subshifts of finite type. Several other geometric aspects of the spectral triples associated to Gibbs measure of Hölder potentials are studied in [KS13]. For example, they compute the dimension of their spectral triples, and show that the noncommutative volume constant is given by the inverse of Kolmogorov-Sinai entropy of the Gibbs measure. This fact was established by showing that the Gibbs measure associated to a Hölder potential admits a Dixmier trace representation, which is a formula similar to (1) without the factor two.

Here we focus on this representation problem for g-measures on the symbolic space Ω ≡ A^N. In [KS13, Sha12, Sha16] this representations are obtained when A is finite and the Gibbs measures are associated to Hölder potentials. In [JP16] similar setting is considered and this problem is investigated, for subshifts of finite type on A^Z.

We extended this Dixmier trace representation of a Gibbs measure associated to a potential in the Walter class, on the context of finite alphabets. This is a non-trivial generalization due to lack of spectral gap property of potentials in this space, which is a crucial property in analyzing the associated dynamical zeta function near its singularity at s = 1. The main idea to circumvent this problem is to analyzing suitable perturbations of the Hölder potentials and use a handy representation for the Kolmogorov-Sinai entropy of a Gibbs measure. By using this representation we overcome the lack of continuity of the entropy and take limits of appropriated sequences. The motivation to prove this generalization for potentials in Walters class is to obtain, for the first time, a noncommutative integral representation for the famous Dyson model of Classical Equilibrium Statistical Mechanics, and to develop tools with potential applications in dynamics of expanding endomorphims in metric spaces as in [Wal07].

We also obtain a similar representation result when A is an infinite countable set, and the potential has the spectral gap property (in the sense of Definition 6.1).

With the formalism above described we are able to define a momentum operator D acting on certain Hilbert space which is defined in terms of the symbolic space (see Remark 4.8 in the end of Section 4).

For the sake of simplicity, the discussion is conducted for the full shift on Ω, but it is easy to see that the arguments are similar for subshifts. Moreover, the way we adapted the arguments from [Sha16], for symbolic dynamics, facilitate the reader’s understanding on how to extend this singular trace representation of Gibbs measures to several other interesting settings. This issue has been addressed in [KS13], but we believe the ideas presented here are simpler. Nonetheless, here we work with zero dimensional spectral triples while in [KS13] the dimension is one.

Although the ideas developed in [Sha16] have shown themselves robust in handling less regular than Hölder potentials and infinite countable alphabets, there are some obstructions to use them in cases where the alphabet A is uncountable. This issue is discussed in details in Section 7 and we also comment on its potential applications in connecting noncommutative geometry ideas to the study of DLR-Gibbs measures of the general XY and Heisenberg models on the one-dimensional lattice.
We shall mention that some explicit computations related our main results are presented here for three important examples: independent Bernoulli random variables, Hofbauer model and Dyson model.

The paper is organized as follows. In Section 2, we introduce the basic notation and some concepts of Thermodynamic Formalism and spectral triples. In Section 3, we construct the spectral triples of interest in this work from H"older $g$-functions. In Section 4, we define some dynamical zeta functions and use a version of Hardy-Littlewood Tauberian theorem to obtain the Dixmier trace representation of the Gibbs measures associated to a H"older $g$-function. In Section 5, the main result of the previous section is extended to potentials in the Walters class, and as application we obtain a noncommutative integral representation for the DLR-Gibbs measure of the Dyson model in Statistical Mechanics. In Section 6, we obtain similar results in the context of infinite countable alphabets and topological Markov shifts. Finally, in Section 7, we make some remarks on the problem of obtaining such representations in the context of uncountable alphabets, and its potential applications.

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and $\mathcal{A}$ a fixed general countable metric space. We call $\mathcal{A}$ an alphabet. For the sake of simplicity in this section we assume that the alphabet $\mathcal{A}$ is a finite set. When topological concepts are concerned the symbolic space $\Omega \equiv \mathcal{A}^\mathbb{N}$ is regarded as a product space equipped with a metric $d_\Omega$ that induces the product topology. Here the dynamics $\sigma : \Omega \to \Omega$ is given by the left shift map $\sigma(x_1,x_2,\ldots) = (x_2,x_3,\ldots)$. As usual $C(\Omega, \mathbb{C})$ denotes the Banach space of all complex valued continuous functions on $\Omega$. Let $0 < \alpha < 1$. A function $f \in C(\Omega, \mathbb{C})$ is called $\alpha$-H"older continuous if

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\Omega(x,y)^\alpha} < +\infty.$$ 

A $g$-function on $\Omega$ is a real strictly positive continuous function satisfying for any $x \in \Omega$ the following condition

$$\sum_{q \in \mathcal{A}} g(qx) = 1,$$ 

where $qx \equiv (q, x_1, x_2, \ldots)$.

The topological space of all $\sigma$-invariant Borel probability measures on $\Omega$, endowed with the weak-$*$-topology, is denoted by $\mathcal{M}_\sigma(\Omega)$. On this context of finite alphabets the Kolmogorov-Sinai entropy of $\mu \in \mathcal{M}_\sigma(\Omega)$ is always well-defined and will be denoted by $h_\mu(\sigma)$. And the topological pressure of a continuous potential $f : \Omega \to \mathbb{R}$ is given by

$$P(f) \equiv \sup_{\mu \in \mathcal{M}_\sigma(\Omega)} h_\mu(\sigma) + \int_{\Omega} f \, d\mu.$$ 

A $\sigma$-invariant probability measure attaining this supremum is called an equilibrium state for $f$, see [Wal82], for details.

The Ruelle transfer operator associated to a potential $f$ is the positive linear operator $\mathcal{L}_f : C(\Omega, \mathbb{C}) \to C(\Omega, \mathbb{C})$ given by

$$\mathcal{L}_f(\varphi)(x) = \sum_{q \in \mathcal{A}} e^{f(qx)} \varphi(qx).$$
If \( J : \Omega \to \mathbb{R} \) is a \( g \)-function, then any Borel probability measure satisfying the equation \( L^*_{\log J}(\mu) = \mu \) is called a \( g \)-measure. Here \( L^*_{\log J} \) is the dual of the Ruelle operator. If \( \log J \) is a Hölder (or Walters, see Section 5) potential, then the set of its \( g \)-measures is a singleton and containing in \( \mathcal{M}_\sigma(\Omega) \). Moreover this unique \( g \)-measure is also the unique equilibrium state for the potential \( \log J \).

Let us introduce the concept of spectral triple. Our presentation follows closely the reference [Sha16]. As already observed in this reference, this way to introduce this concept is not the most general one, but it is completely satisfactory for our purposes.

**Definition 2.1.** A spectral triple is an ordered triple \((A, H, D)\), where

1. \( H \) is a Hilbert space;
2. \( A \) is a \( C^* \)-algebra where for each \( a \), we can associate a bounded linear operator \( L_a \) acting on \( H \);
3. \( D \) is an essentially self-adjoint unbounded linear operator on \( H \) with compact resolvent and such that \( \{ a \in A : \|[D, L_a]\| < +\infty \} \) is dense in \( A \), where \([D, L_a]\) is the commutator operator. The operator \( D \) is called Dirac operator.

Another important mathematical object playing a major role in this paper is the Dixmier trace. Roughly speaking the Dixmier trace of a compact operator \( B \) is given by:

\[
\text{Tr}_\omega(B) = \lim_{n \to \infty} \frac{1}{\log(n)} \sum_{k=1}^{n} \beta_k
\]

where \( \beta_k \) are the eigenvalues of the compact operator \( |B| = \sqrt{B^*B} \). The precise definition of the Dixmier trace \( \text{Tr}_\omega \) can be found, for example, in [Dix66, LSZ13, Suk08], but the above description allows for (and suitable for whom do not know about singular traces) full understanding of what is going on.

Now, we can finally state one of our first results. If \( J : \Omega \to (0, \infty) \) is a \( \alpha \)-Hölder function such that \( \log J \) is a \( g \)-function and \( \mu \) is the unique equilibrium state for \( \log J \), then there is a spectral triple \((A, H, D)\) such that

\[
\text{Tr}_\omega(L_a D^{-1}) = \frac{2}{h_\mu(\sigma)} \int_\Omega a \, d\mu.
\]

### 3. Spectral Triples and Hölder \( g \)-Functions

In this section we present the spectral triples that will be used throughout this paper to construct our singular trace representation of the Gibbs measures. This construction, as mentioned before, is inspired in [Sha16] and the proof that it satisfies all the requirements to be a spectral triple is similar to the one given there and except for the proof that \( \{ a \in A : \|[D, L_a]\| < +\infty \} \) is dense in \( A \), which will be explained at the end of this section.

The construction starts by taking the \( C^* \)-algebra \( A \) as \( C(\Omega, \mathbb{C}) \), the space of all continuous complex valued functions defined on the symbolic space \( \Omega \). The Hilbert space \( H \) is \( \ell^2(W^*) \oplus \ell^2(W^*) \), where \( W^* \) is the set of all finite strings \( w = (w_1, w_2, \ldots, w_n) \), where \( n \in \mathbb{N} \) and \( w_j \in \mathcal{A} \). The space \( \ell^2(W^*) \) is the complex...
vector space of all functions $\epsilon : W^* \to \mathbb{C}$, satisfying $\sum_{w\in W^*} |\epsilon(w)|^2 < \infty$. A generic element of $H$ will be represented as follows

$$\bigoplus_{w\in W^*} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix}.$$ 

Fix two arbitrary elements $x, y \in \Omega$. For each $a \in A$ the operator $L_a : H \to H$ is defined by

$$L_a \big( \bigoplus_{w\in W^*} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} \big) = \bigoplus_{w\in W^*} \begin{pmatrix} a(wx) \epsilon_1(w) \\ a(wy) \epsilon_2(w) \end{pmatrix}.$$ 

To construct the Dirac operator, we fix a Hölder $g$-function $J : \Omega \to \mathbb{R}$, and consider the self-adjoint operator

$$D = \big( \bigoplus_{w\in W^*} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} \big) = \bigoplus_{w\in W^*} \frac{1}{J(wx)} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix},$$

where $J(wx) = [J(w_1 w_2 x) \ldots J(w_ell w_1 x)]$, and $l_1$ is the length of a string $w \in W^*$. 

From the definitions follow that $[D, L_a]$ is given by

$$(2) \quad [D, L_a] \big( \bigoplus_{w\in W^*} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} \big) = \bigoplus_{w\in W^*} \frac{[a(wx) - a(wy)]}{J(wx)} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix}.$$ 

To prove that $\{a \in A : \|[D, L_a]\| < +\infty\}$ is dense in $A$ it is enough to prove that there is $0 < \alpha < 1$ such that the set of all Lipchitz functions, with respect to the metric $d(x, y) = \alpha^{N(x, y)}$, where $N(x, y) = \inf \{k : x_k \neq y_k\}$ (with the convention that $\inf \emptyset = +\infty$) satisfies

$$\sum_{w\in W^*} \frac{|a(wx) - a(wy)|}{J(wx)} < +\infty.$$ 

Indeed, if we $\alpha < (1/2) \inf_{x,y} J(x)$, then all Lipchitz functions satisfy the above condition and the denseness follows from the Stone-Weiertrass theorem.

Note that for any $0 < s < 1$ we have

$$L_a D^{-s} \big( \bigoplus_{w\in W^*} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} \big) = \bigoplus_{w\in W^*} \begin{pmatrix} a(wx) \epsilon_1(w) \\ a(wy) \epsilon_2(w) \end{pmatrix},$$

and therefore for any fixed $a \in C(\Omega, \mathbb{C})$ we have that the eigenvalues of $L_a D^{-s}$ are given by $\{a(wx) J^s(wx) : w \in W^*\} \cup \{a(wy) J^s(wx) : w \in W^*\}$. Since this eigenvalue set is countable, we can identify its elements with a sequence $(a_k \lambda_k^s)_{k \in \mathbb{N}}$, where $a_k$ is either $a(wx)$ or $a(wy)$, and $\lambda_k^s$ is of the form $J^s(wx)$.

For example, if $\mathcal{A} = \{0, 1\}$, then we can rearrange the elements of the form $wx$ and $wy$ as $0x, 0y, 1x, 1y, 00x, 00y, 01x, 01y, 10x, 10y, 11x, 11y, \ldots$ and from this arrangement, it is clear how to define the sequence $(a_k \lambda_k^s)_{k \in \mathbb{N}}$, above described.

If we take $\mathcal{A} = \{0, 1\}$, and $J \equiv 1/2$ then we have $J(wx) = 2^{-n}$ and so the quantity $(a(wx) - a(wy))/2^{-n}$ in rhs of $(2)$ can be thought as a kind of discrete derivative. If the norm of the commutator $\|[D, L_a]\| \leq 1$, for a certain $a$, this in some sense is saying that the Lipschitz constant of $a$ is at most 1 (see section 1.2.1 in [Haw13]). This kind of reasoning is behind the definition 1-Wasserstein distance and also the Connes distance for $C^*$-states (see [Con94, KS13]).
4. Dixmier Trace Representation of Gibbs Measures of Hölder Potentials on Finite Alphabets

Consider the following zeta function
\[ \zeta(s) = \sum_{k=1}^{\infty} a_k \lambda_k^s, \]
where \( s \in \mathbb{R} \) and \( (a_k \lambda_k^s)_{k \in \mathbb{N}} \) is as in previous section. Note that this series contains all eigenvalues of \( L_a D^{-s} \). Suppose for \( s > 1 \) the sum \( \zeta(s) \) is finite and for \( s = 1 \) the sum is infinite. In this case \( s = 1 \) is called the abscissa of convergence.

In order to prove our main result we will use the following version of Hardy-Littlewood Tauberian theorem, which claims that
\[
\lim_{s \to 1^+} (s - 1)\zeta(s) = C, \quad \text{if and only if} \quad \lim_{n \to \infty} \frac{1}{\log(n)} \sum_{k=1}^{n} a_k \lambda_k = C.
\]

To be able to work with more explicit formulas is convenient to rewrite the above zeta function in terms of following formal series
\[ \zeta^+(s) = \sum_{k=1}^{\infty} \sum_{w \in W} a(w)x^s \quad \text{and} \quad \zeta^-(s) = \sum_{k=1}^{\infty} \sum_{w \in W} a(w)x^s. \]

The main idea (inspired in Lemma 6.2 in [Sha16]) is expressing the first the zeta function \( \zeta^+(s) \) in terms of the powers of the Ruelle operator
\[
\zeta^+(s) = \sum_{k=1}^{\infty} \mathcal{L}^k_{s \log J}(a)(x),
\]
and prove its convergence, for any \( a \in C^\alpha(\Omega, \mathbb{C}) \), by using that \( \mathcal{L}^k_{s \log J}(a)(x) \), converges to zero exponentially fast, when \( k \to \infty \), for any fixed \( s < 1 \). Later, we show that \( \zeta^+(s) \) still converges for \( a \in C(\Omega, \mathbb{C}) \) and that \( s = 1 \) is the abscissa of convergence, as long as \( a \neq 0 \). Of course, similar reasoning can be used to analyze \( \zeta^- \). These information combined prove that \( s = 1 \) is the abscissa of convergence for \( \zeta(s) \).

Proposition 4.1. Let \( a \in C^\alpha(\Omega, \mathbb{C}) \). For any \( k \in \mathbb{N} \) and \( 0 < s \leq 1 \), we have
\[
\mathcal{L}^k_{s \log J}(a)(x) = \exp(kP(s \log J))(\int_{\Omega} a \ d\mu) h_{s \log J}(x) + O(\gamma(s)^k),
\]
where \( P(s \log J) \) is the topological pressure of the potential \( s \log J \), the function \( h_{s \log J} \) is a normalized maximal eigenfunction of \( \mathcal{L}_{s \log J} \), and \( 0 < \gamma(s) < 1 \). Moreover \( \sup \{ \gamma(s) : s \geq 1 \} < 1 \).

Proof. This is a classical result on Thermodynamic Formalism. For its proof see [Bal00, PP99, Sha16]. \( \square \)

Lemma 4.2. Let \( a \in C^\alpha(\Omega, \mathbb{C}) \). Then
\[
\lim_{s \to 1^+} (s - 1)\zeta^+(s) = \frac{1}{h_{\mu}(\sigma)} \int_{\Omega} a \ d\mu.
\]
Proof. From expression (3) and Proposition 4.1 we have

\[ \zeta_+(s) = \sum_{k=1}^{\infty} \mathcal{L}_{s \log J}^k(a)(x) = \frac{h_{s \log J}(x)}{1 - \exp(P(s \log J))} \int_{\Omega} a \, d\mu_s + \sum_{k=1}^{\infty} O(\gamma(s)^k) \]

Recall that

\[ \frac{d}{ds} e^{P(s \log J)} \bigg|_{s=1} = \int_{\Omega} \log J \, d\mu = h_{\mu}(\sigma) \quad \text{and} \quad P(\log J) = 0. \]

Therefore,

\[ \zeta_+(s) = \frac{h_{s \log J}(x)}{h_{\mu}(\sigma)(1-s) + r(s)} \int_{\Omega} a \, d\mu_s + \sum_{k=1}^{\infty} O(\gamma(s)^k), \]

where the reminder \( r(s) = o(s) \), when \( s \to 1 \).

Since \( \log J \) is a Hölder potential we have that \( s \mapsto h_{s \log J} \) and \( s \mapsto \mu_s \) are real analytic functions, and the series in rhs above is uniformly bounded for \( s \geq 1 \).

From these observations we get that

\[ \lim_{s \to 1^+} (s-1)\zeta_+(s) = \frac{1}{h_{\mu}(\sigma)} \int_{\Omega} a \, d\mu. \]

\[ \square \]

**Corollary 4.3.** For all \( a \in C^0(\Omega, \mathbb{C}) \) we have

\[ \lim_{s \to 1^+} (s-1)\zeta_+(s) = \lim_{s \to 1^+} (s-1)\zeta_-(s). \]

**Proof.**

\[ |\zeta_+(s) - \zeta_-(s)| \leq \sum_{k=1}^{\infty} \sum_{w \in \mathcal{W}^* \atop l(w) = k} |a(wx) - a(wy)| J(wx)^s \]

Since \( a \in C^0(\Omega, \mathbb{C}) \) for all \( w \in \mathcal{W}^* \) such that \( l(w) = k \), we have \( |a(wx) - a(wy)| \leq \text{Hol}_{a}(a)^{2^{-k\alpha}} d^*(x,y) \). We recall that \( \log J \) is a normalized potential. So for all \( s > 1 \), we have

\[ \sum_{w \in \mathcal{W}^* \atop l(w) = k} J(wx)^s \leq \sum_{w \in \mathcal{W}^* \atop l(w) = k} J(wx) = \mathcal{L}_{\log J}^k(1)(x) = 1. \]

These two inequalities combined shown that \( |\zeta_+(s) - \zeta_-(s)| \leq (1 - 2^{s-1})d(x,y) \).

Therefore \( (s-1)|\zeta_+(s) - \zeta_-(s)| \to 0 \), when \( s \to 1^+ \). Now the result follows from Lemma 4.2. \[ \square \]

**Example 4.4.** Consider the independent process on \( \Omega \equiv \{1, 2\}^N \) such that \( \log J = \log p_1 \) on the cylinder \([1]\) \equiv \{x \in \Omega : x_1 = 1\} and \( \log J = \log p_2 \) on the cylinder \([2]\), where \( p_1 + p_2 = 1 \), \( p_1, p_2 > 0 \).

If we chose \( a = 1_{[1]} \), then we have for any fixed \( x \in \Omega \) and \( k \in \mathbb{N} \)

\[ \mathcal{L}_{s \log J}(1_{[1]})(x) = p_1^s (p_1^s + p_2^s)^{(k-1)}. \]

Therefore, for \( s \geq 1 \), we get that

\[ \lim_{s \to 1^+} p_1^s (s-1) \sum_{k=1}^{\infty} \mathcal{L}_{s \log J}(1_{[1]})(x) = \lim_{s \to 1^+} p_1^s (s-1) \sum_{k=1}^{\infty} (p_1^s + p_2^s)^{k-1} \]

\[ = \lim_{s \to 1^+} \frac{p_1^s (s-1)}{1 - (p_1^s + p_2^s)} \]
Note that the function \( f(s) = p_1 s + p_2 s^2 \) is such that \( f'(1) = p_1 \log p_1 + p_2 \log p_2 \). Then,

\[
\lim_{s \to 1^+} (s - 1) \zeta^+(s) = \frac{p_1}{h_{\mu}(\sigma)} = \frac{1}{h_{\mu}(\sigma)} \int_{\Omega} 1_{[1]} \, d\mu. \tag{4}
\]

We would like to show some other examples where

\[
\lim_{s \to 1^+} (s - 1) \zeta^+(s) = \frac{1}{c} \int_{\Omega} a \, d\mu,
\]

for some \( c \) in a case where the potential is not normalized (but the maximal eigenvalue of the transfer operator is \( \lambda = 1 \)).

**Example 4.5.** The Hofbauer model - We will consider a non normalized case. For each \( k \geq 1 \) we denote by \( C_k \subset \Omega = \{0, 1\}^\mathbb{N} \), the cylinder set \([11\ldots 11] \equiv \{ x \in \Omega : x_1 = 1, \ldots, x_k = 1, x_{k+1} = 0 \} \) and define \( C_0 \) as the cylinder set \([0] \). Let \( \gamma > 2 \) be a fixed parameter, and consider the potential \( g : \Omega \to \mathbb{R} \), such that, and

\[
g(x) = \begin{cases} 
- \log(\zeta(\gamma)), & \text{if } x \in C_0; \\
- \gamma \log(\frac{\gamma+1}{k}), & \text{if } x \in C_k \text{ and } k \geq 1; \\
0, & \text{if } x = (1, 1, 1, \ldots),
\end{cases}
\]

where \( \zeta \) is the Riemann zeta function. Note that \( g \) is a continuous function on \( \Omega \), since \( g(x) \to 0 \) as \( x \to (1, 1, 1, \ldots) \). However, \( g \) is not of Holler class.

We also remark that \( L_g \) has an eigenfunction associated to the eigenvalue 1, but is not normalized.

The goal is to evaluate \( \lim_{s \to 1^+} (s - 1) \zeta^+(s) \). So in what follows \( s > 1 \) and \( z \in \Omega \) is a fixed point such that its first coordinate is zero, that is, \( z \in C_0 \). This choice of \( z \) is important in computing \( L^n_{sg}(1_{[1]}) \) in terms of lower order iterates. Basically this is obtained by using successively that \( L^n_{sg}(1_{[1]}) \) is constant in \( C_0 \). To simplify the notation we write \( L^n = L^n_{sg}(1_{[1]})(z) \) for \( n \geq 1 \).

By the property mentioned above and the binary tree structure (see [FL01]) of the inverse branches of the Ruelle operator we obtain the following renewal equation for \( L^n_s \):

\[
L^n_s = \sum_{k=1}^{n-1} \frac{1}{\zeta(\gamma)^k} L^{n-k}_{s} + \frac{1}{(n+1)\gamma^s}.
\]

From this we obtain the convergence of the following series and the identity

\[
\sum_{k=1}^{\infty} L^n_k = (\zeta(\gamma^s) - 1) + \sum_{k=1}^{\infty} L^n_k (\zeta(\gamma)^{-s}),
\]

Therefore \( \zeta^+(s) = L^1_s + L^2_s + L^3_s + \ldots + L^n_s + \ldots \) is given by

\[
\zeta^+(s) = \frac{\zeta(\gamma^s) - 1}{1 - \zeta(\gamma)^{-s}}.
\]

Clearly, \( f(s) \equiv \zeta(\gamma^s) \zeta(\gamma)^{-s} \) is differentiable at \( s = 1 \) and \( f(1) = 1 \). Therefore follow from the definition of the derivative we get

\[
f'(1) = \lim_{s \to 1} \frac{\zeta(\gamma^s) \zeta(\gamma)^{-s} - 1}{s - 1}.
\]
In [FL01, Lop93] it is shown that for the eigenprobability $\nu$ of $L^*_{g}$, we have for all $k \geq 1$, $\nu(C_0) = \zeta(\gamma)^{-1}$ and $\nu(C_k) = \zeta(\gamma)^{-1}(k + 1)^{-\gamma}$. Therefore, $\nu([1]) = \zeta(\gamma)^{-1}(\zeta(\gamma) - 1)$ and so follow from the last identity that 

$$
\lim_{s \to 1^+} (s - 1) \zeta^+(s) = \frac{\zeta(\gamma) - 1}{-f'(1)} = \frac{\zeta(\gamma) - 1}{\zeta(\gamma)c}, \text{ where } c = \frac{-f'(1)}{\zeta(\gamma)} = \frac{1}{\zeta(\gamma)^2} \left[ \gamma \left( \sum_{n=2}^{\infty} n^{-\gamma} \log n \right) - \log \zeta(\gamma) \right].
$$

This constant seems to be independent of the choice of $a \in C(\Omega)$, as suggested by the next example. But on the other hand, we have not find single relation of it with Kolmogorov-Sinai entropy of the equilibrium state associated to $g$. In fact, for the equilibrium state $\mu$ of $g$ we have 

$$
\mu(C_0) = \frac{\zeta(\gamma)}{\zeta(\gamma - 1)}, \quad \mu(C_k) = \sum_{j=k+1}^{\infty} j^{-\gamma} \quad \text{and} \quad h(\mu) = -\frac{\zeta(\gamma) \log \zeta(\gamma) + \gamma w(\gamma)}{\gamma w(\gamma - 1)},
$$

where $w(\gamma) = \sum_{j=2}^{\infty} j^{-\gamma} \log(j)$.

In the next example we show that the constant $c$ we get in last example is the same for $a \equiv 1$.

**Example 4.6.** Now we assume that $\gamma > 2$ and obtain estimates for $L^*_{g}(1)(z)$, where $z \in C_0$. Following a similar reasoning as in last example we get the following renewal equation for $L^n_{g} = L^*_{g}(1)(z)$, $n \geq 1$:

$$
L^n_{s} = \frac{1}{\zeta(\gamma)s} L^{n-1}_{s} + \frac{1}{\zeta(\gamma)s} \sum_{j=2}^{n-2} \frac{1}{(j+1)^{\gamma}s} L^{n-j}_{s} + \frac{1}{\zeta(\gamma)s n^{\gamma}s} + \frac{1}{(n+1)^{\gamma}s}.
$$

From the above equation follows that 

$$
\sum_{n=1}^{\infty} L^n_{s} = \frac{\zeta(\gamma)s}{\zeta(\gamma)} + \sum_{n=2}^{\infty} \frac{1}{n^{\gamma}s} + \frac{\zeta(\gamma)s}{\zeta(\gamma)} \sum_{n=1}^{\infty} L^n_{s}
$$

and so $\zeta^+(s) \equiv L^1_{s} + L^2_{s} + \ldots$ satisfies the following equation 

$$
\zeta^+(s) = \frac{\zeta(\gamma)s}{\zeta(\gamma)} + (\zeta(s\gamma) - 1) + \zeta^+(s) \frac{\zeta(\gamma)s}{\zeta(\gamma)}
$$

which implies 

$$
\zeta^+(s) = \frac{\zeta(\gamma)^{-s} \zeta(\gamma s) + (\zeta(s\gamma) - 1)}{1 - \zeta(s\gamma) \zeta(\gamma)^{-s}} \quad \text{and} \quad \lim_{s \to 1^+} (s - 1) \zeta^+(s) = \frac{1}{c} \int_{\Omega} 1 \, d\nu,
$$

where $c$ is again given by $c = -f'(1)/\zeta(\gamma)$ and $f$ is as in previous example. 

**Theorem 4.7.** Let $J : \Omega \to (0, \infty)$ be a $\alpha$-Hölder function such that $\log J$ is a $g$-function, $\mu$ the unique equilibrium state for $\log J$, and $(A, H, D)$ the spectral triple constructed in Section. Then for all $a \in A$ we have 

$$
\text{Tr}_a(L_a D^{-1}) = \frac{2}{h(\mu)} \int_{\Omega} a \, d\mu.
$$
Proof. From Corollary 4.3 and Lemma 4.2 we have for all \( a \in C^\alpha(\Omega, \mathbb{C}) \)

\[
\lim_{s \to 1^+} (s - 1) \zeta(s) = \frac{2}{\eta \mu(\sigma)} \int_\Omega a \, d\mu.
\]

Since the self-adjoint operator \( B = L_a^\sigma - s \) is a positive operator it follows from the above equality and the mentioned version of Hardy-Littlewood Tauberian theorem that

\[
\text{Tr}_\omega((L_a^\sigma)^{-1}) = \lim_{n \to \infty} \frac{1}{\log(n)} \sum_{k=1}^{n} a_k \lambda_k = \lim_{s \to 1^+} (s - 1) \zeta(s) = \frac{2}{\eta \mu(\sigma)} \int_\Omega a \, d\mu.
\]

The proof that the above formula still valid for a general \( a \in C(\Omega, \mathbb{C}) \) is similar to the one given in the last page of reference [Sha16]. □

Remark 4.8. Consider the operator \( f \to D^2(f) + V(f) \) acting on \( f \in \ell^2(W^*) \).
The self adjoint operator \((D^2 + V)^{-1}\) is compact and has a spectral decomposition with an orthonormal basis \( \varphi_n, n \in \mathbb{N} \), associated to eigenvalues \( \lambda_n \in \mathbb{R} \), such that, \( \lambda_n \to 0 \), when \( n \to \infty \).

In a similar fashion as in Quantum Mechanics the operator \( D^2 + V \) has a pure point spectrum and there is an orthonormal basis \( \varphi_n, n \in \mathbb{N} \), associated to eigenvalues \( \beta_n = \frac{1}{\lambda_n} \in \mathbb{R} \), such that, \( \beta_n \to \infty \), when \( n \to \infty \). The values \( \beta_n \) correspond to different possible values of energy. If the Dirac operator \( D \) is defined as above from a given Gibbs measures on the symbolic space, then thinking of \( D^2 + V \) as a kind of Schrödinger operator one can develop a Quantum Mechanics Formalism based on objects defined on the symbolic space.

5. Dixmier Trace Representation for Gibbs Measures of Walters Potentials

In [Wal07] Peter Walters developed a theory of Thermodynamic Formalism for potentials having less regularity than the Hölder potentials and provided several applications in dynamics of expanding endomorphisms in metric spaces. In what follows we show how to obtain a Dixmier trace representation for the Gibbs measures associated to a potential in the Walters class (defined below).

The main argument in this section is based on a continuity result. Before proceed we recall the needed definitions, for more details see [LSZ13, Suk08].

Definition 5.1. Let \( \mathcal{R}_n \equiv \{ A \in B(H) : \text{rank}(A) \leq n \} \). The \( n \)-th characteristic number of \( T \in K(H) \) (set of all compact operators acting on \( H \)) is by definition

\[
\gamma_n(T) \equiv \text{dist}(\mathcal{R}_{n-1}, T) = \inf_{A \in \mathcal{R}_{n-1}} \| T - A \|.
\]

For a compact operator \( T \), we associate a non-negative number (possibly infinity)

\[
\| T \|_{(1, \infty)} = \sup_{n \in \mathbb{N}} \frac{1}{\log(n)} \sum_{k=1}^{n} \gamma_k(T).
\]

We can show that \( \cdot \|_{(1, \infty)} \) defines a norm on the set

\[
\mathcal{L}^{(1, \infty)}(H) \equiv \{ T \in K(H), \| T \|_{(1, \infty)} < \infty \}
\]

and \( \mathcal{L}^{(1, \infty)}(H) \) is a Banach space, see Lemma 1.3 in [Suk08]. Actually, this is a Banach ideal (for the norm \( \cdot \|_{(1, \infty)} \)) in the algebra of bounded operators acting
on $H$. In the literature this ideal is known as the dual to the Macaev’s ideal. A crucial fact needed here about this ideal is the continuity of Dixmier trace when restrict to it, with respect to the norm $\| \cdot \|_{(1,\infty)}$. See Proposition 1.9 in [Suk08].

**Definition 5.2.** The Walters space $W(\Omega, \sigma)$ is the set of all functions $f \in C(\Omega, \mathbb{C})$ satisfying the following regularity condition

$$\sup_n \sup_{a \in A_n} |S_n(f)(ax) - S_n(f)(ay)| \xrightarrow{d_{\Omega}(x,y) \to 0} 0.$$  

**Theorem 5.3.** Let $J : \Omega \to (0, \infty)$ such that $\log J$ is a $g$-function belonging to $W(\Omega, \sigma)$. Let $\mu$ be the unique equilibrium state for $\log J$, then there is a spectral triple $(A, H, D)$ so that for all $a \in A$ we have

$$\text{Tr}_\omega(L_a D^{-1}) = \frac{2}{h_\mu(\sigma)} \int_\Omega a \, d\mu.$$  

**Proof.** Let $\log J \in W(\Omega, \sigma)$ be a normalized potential, and $(\log J_n)_{n \in \mathbb{N}} \subset C^\alpha(\Omega, \mathbb{C})$ a sequence of normalized Hölder potentials such that

$$\| \log J_n - \log J \|_{(1,\infty)} = O(e^{-n}).$$  

This sequence can be constructed by fixing in $\log J$ the coordinates after some suitable index.

Define the Dirac operator $D \equiv D(J)$ similarly as in the Hölder case, and for each $n \in \mathbb{N}$ consider the Dirac operator $D_n \equiv D(J_n)$. Note that for any $a \in C(\Omega, \mathbb{C})$, we have $\|L_a D_n^{-1} - L_a D_n^{-1}\|_{(1,\infty)} \to 0$, when $n \to \infty$.

Since $\mathcal{G}^* (\log J)$ is a singleton follows from Proposition 7 of [CL16] that $\mu_n \to \mu$, where $\mu_n$ is the unique equilibrium state associated to $\log J_n$. Since $h_{\mu_n}(\sigma) = -\int_\Omega \log J_n \, d\mu_n + \log |\mathcal{G}|$ (see [LMMS15]) follows from the weak convergence mentioned above and [CL16] that $h_{\mu_n}(\sigma) \to h_\mu(\sigma) \neq 0$, when $n \to \infty$. This information together with the continuity of the Dixmier trace, with respect to $\| \cdot \|_{(1,\infty)}$ provides

$$\text{Tr}_\omega(L_a D_n^{-1}) = \lim_{n \to \infty} \text{Tr}_\omega(L_a D_n^{-1}) = \lim_{n \to \infty} \frac{2}{h_{\mu_n}(\sigma)} \int_\Omega a \, d\mu_n = \frac{2}{h_\mu(\sigma)} \int_\Omega a \, d\mu,$$

and so the Dixmier trace representation $g$-measures associated to Walters potentials.

**Dyson Model.** The extension obtained in last section allow us to find a Dixmier trace representation for the DLR-Gibbs measure of the Dyson model on the lattice $\mathbb{N}$. This model is defined as follows. Consider Ising spins $\{-1, 1\}^\mathbb{N}$ having long-range ferromagnetic interaction, with decay parameter $\alpha > 1$, formally given by

$$H(\omega) = -\sum_{i,j \in \mathbb{N}} \frac{\omega_i \omega_j}{|i-j|^{\alpha}}.$$  

This model has been studied for a considerable time in Statistical Mechanics and more recently also in Dynamical Systems, see [BEvN17, Dys69, Dys71, JOP17]. In [CL17] it was proved that if $\alpha > 2$, then the DLR-Gibbs measure of this model is the same as the conformal measure associated with the potential

$$f(x) = x_1 \sum_{j=2}^{\infty} \frac{x_n}{(n-1)^\alpha}.$$
For such decay parameter $\alpha$ the authors also proved that this potential is in the Walters space $W(\Omega, \sigma)$, where $A = \{-1, 1\}$. Let $\nu$ be the conformal measure associate with this potential and $d\mu = h d\nu$ the equilibrium measure. Then follows from the last section the existence of a spectral triple $(A, H, D)$ so that $\text{Tr}_\omega(L_a D^{-1}) = (-2)(h_\mu(\sigma))^{-1} \int_\Omega a h d\nu$. Since $h$ is continuous and positive everywhere, we can replace the operators $L_a$ by $\tilde{L}_a \equiv L_a/h$, obtaining a new spectral triple. This provides an explicit (and constructible by using the Thermodynamic Limit) Dixmier trace representation for the DLR-Gibbs measure of the Dyson model on the one-sided lattice, which is

$$\text{Tr}_\omega(\tilde{L}_a D^{-1}) = \frac{2}{h_\mu(\sigma)} \int_\Omega a d\nu.$$  

So we have a natural way to associate a zeta function to the Dyson model. It would be a very interesting to know what happens to this construction when the decay parameter $\alpha \in (1, 2]$ and the model is considered in low temperatures where there is phase transitions. Another interesting question is how the phase transition manifests in the associated zeta function.

6. Dixmier Trace Representations for Equilibrium Measures of Topological Markov Shifts

The key ingredient in proving identity (1) is the spectral gap property of the Ruelle operator (see Definition 6.1). In this section we illustrate how to obtain a similar identity in non-compact setting. The basic setup and presentation follow closely the reference [CS09]. Now $\mathcal{A} = \mathbb{N}$ and $T = (t_{ij})_{\mathcal{A} \times \mathcal{A}}$ is a matrix of zeroes and ones. Let $\sigma : \Omega \rightarrow \Omega$ denote the left shift mapping, where

$$\Omega \equiv \{(x_1, x_2, \ldots) \in \mathcal{A}^\mathbb{N} : t_{x_i x_{i+1}} = 1\}.$$  

We think of $\Omega$ as the collection of one sided infinite admissible words. We equip it with its usual distance $d(x, y) = 2^{-N(x, y)}$, where $N(x, y) \equiv \inf\{k : x_k \neq y_k\}$ (with the convention that $\inf \emptyset = +\infty$). The resulting topology is generated by the cylinder sets $[y_1, y_2, \ldots, y_n] \equiv \{x \in \mathcal{X} : x_i = y_i, \ i = 1, \ldots, n\}$, where $n \geq 1$. A word $y \in \mathcal{A}^n$ is called admissible if the cylinder it defines is non-empty. The length of an admissible word $y = (y_1, \ldots, y_n)$ will be denoted in this section by $|y| \equiv n$.

We also assume that $\sigma : \Omega \rightarrow \Omega$ is topologically mixing and locally compact. This is the case when for any two symbols $p, q \in \mathcal{A}$, there is an $N(p, q) \in \mathbb{N}$ such that for all $n \geq N(p, q)$ there is an admissible word of length $n$ which starts at $p$ and ends at $q$, and for all $p \in \mathcal{A}$ we have $\#\{q \in \mathcal{A} : t_{pq} = 1\} < \infty$, see [Sar09].

We define the $n$-th variation of a function $f : \Omega \rightarrow \mathbb{R}$ as $\text{var}_n(f) \equiv \sup\{||f(x) - f(y)|| : x^n = y^n\}$, where $z_m \equiv (z_m, \ldots, z_n)$. A function $f : \Omega \rightarrow \mathbb{R}$ is called weakly $\theta$-Hölder continuous function for $0 < \theta < 1$ if there exists a positive number $\text{Hol}_\theta(f)$ such that $\text{var}_n(f) \leq \text{Hol}_\theta(f) \theta^n$. The Birkhoff sum of a function $f$ is denoted by $S_n(f)(x) \equiv \sum_{k=0}^{n-1} f \circ \sigma^k$.

Suppose $f$ weakly $\theta$-Hölder continuous function (or has summable variations) and $X$ is topologically mixing. The Gurevich pressure of $f$ is the limit

$$P_G(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(f, q), \quad \text{where} \quad Z_n(f, q) = \sum_{\sigma^n(x) = x} \exp(S_n(f)(x))1_{[q]}(x).$$
and \(q \in \mathcal{A}\). This limit is independent of \(q\), and if \(\sup f < \infty\) then it is equal to \(\sup \{\mu_\sigma(f) + \int f \, d\mu\}\), where the supremum ranges over all invariant probability measures such that the supremum is not of the form \(\infty - \infty\), see [Sar99].

In this setting the Ruelle operator associated with \(f\) is defined in similarly way

\[
\mathcal{L}_f(\varphi)(x) = \sum_{q \in \mathcal{A}} e^{f(qx)} \varphi(qx)
\]

This is well-defined for functions \(f\) such that the sum converges for all \(x \in X\). Let \(\text{dom}(\mathcal{L}_f)\) denote the collection of such functions.

**Definition 6.1** (Spectral Gap Property - SGP). Suppose \(f\) is \(\theta\)-weakly Hölder continuous, and \(P_\theta(f) < \infty\). We say that \(f\) has the spectral gap property (SGP) if there is a Banach space of continuous functions \(B\) such that

(a) \(B \subset \text{dom}(\mathcal{L}_f)\) and \(B \supset \{1_a : a \in \mathcal{A}^n, n \in \mathbb{N}\}\);
(b) \(f \in B\) implies \(|f| \in B\), \(\|f\|_B \leq \|f\|_B\);
(c) \(B\)-convergence implies uniform convergence on cylinders;
(d) \(\mathcal{L}_f(B) \subset B\) and \(\mathcal{L}_f : B \to B\) is bounded;
(e) \(\mathcal{L}_f = \lambda P + N\), where \(\lambda = \exp(P_\theta(f))\), and \(PN = NP = 0\), \(P^2 = P\), \(\dim(\text{im} P) = 1\), and the spectral radius of \(N\) is less than \(\lambda\);
(f) if \(g\) is \(\theta\)-Hölder, then \(\mathcal{L}_{f+\delta g} : B \to B\) is bounded and \(z \mapsto \mathcal{L}_{f+\delta g}\) is analytic on some complex neighborhood of zero.

The motivation to introduce this concept is the next theorem. This result were proved in several contexts by many authors, see [CS09] and references therein.

**Theorem 6.2.** Suppose \(X\) is a topologically mixing countable Markov shift, and \(f : X \to \mathbb{R}\) is a \(\theta\)-weakly Hölder continuous potential with finite Gurevich pressure, finite supremum, and the SGP. Write \(\mathcal{L}_f = \lambda P + N\). Then

1. \(P\) takes the form \(P f = h \int f \, d\nu\), where \(h \in B\) is a positive function, and \(\nu\) is a measure which is finite and positive on all cylinder sets;
2. the measure \(d\mu = h \nu\) is a \(\sigma\)-invariant probability measure satisfying:
   (a) if \(\mu\) has finite entropy, then \(\mu\) is the unique equilibrium measure of \(f\);
   (b) there is a constant \(0 < \kappa < 1\) such that for all \(g \in L^\infty(\mu)\) and \(f\) bounded Hölder continuous, there exists a positive constant \(C(\varphi, \psi)\) such that \(|\text{cov}_\mu(\varphi, \psi) \sigma^n)| \leq C(\varphi, \psi)\kappa^n\) (cov = covariance);
   (c) suppose \(g\) is a bounded Hölder continuous function, such that \(E_n[g] = 0\). If \(g \neq f - f \circ \sigma\) with \(f\) continuous, then there is a \(\theta > 0\) such that \(S_n(g) / \sqrt{n\theta}\) converges in distribution (w.r.t. \(\mu\)) to a standard normal distribution.
   (d) suppose \(g\) is a bounded Hölder continuous function, then the function \(t \mapsto P_\theta(f + tg)\) is real analytic on a neighborhood of zero.

Now the \(C^*\)-algebra \(A\) is taken to be as \(C_0(X, \mathbb{C})\), the set \(X\) of complex valued continuous functions \(f : X \to \mathbb{C}\) that vanish at infinity. The Hilbert space \(H\) is \(\ell^2(W^*) \oplus \ell^2(W^*)\), where \(W^*\) is the set of all finite length admissible words \(w = (w_1, w_2, \ldots, w_n)\), where \(n \in \mathbb{N}\) and \(w_j \in \mathcal{A}\). The space \(\ell^2(W^*)\) is defined as before, the complex vector space of all functions \(\epsilon : W^* \to \mathbb{C}\), satisfying \(\sum_{w \in W^*} |\epsilon(w)|^2 < \infty\). Fix two arbitrary elements \(x, y \in \Omega\). For each \(a \in A\) the operator \(L_a : H \to H\)
is defined by
\[
L_a( \bigoplus_{w \in W^+(x,y)} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix}) = \bigoplus_{w \in W^+(x,y)} \begin{pmatrix} a(wx) \epsilon_1(w) \\ a(wy) \epsilon_2(w) \end{pmatrix},
\]
where $W^+(x, y)$ is the set of all admissible words $w \in W^+$ such that $t_{w, x_1} = t_{w, y_1} = 1$. Finally, the Dirac operator is given by
\[
P = \left( \bigoplus_{w \in W^+(x,y)} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} \right) = \bigoplus_{w \in W^+(x,y)} \frac{1}{\mathcal{J}(wx)} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix},
\]
where $\mathcal{J}(wx) = [J(w_1, w_2, \ldots, w_l), J(w_1, w_2, \ldots, w_l) = l(w)]$ is the length of a string $w \in W^+(x, y)$ and \( J = f + \log h - \log \phi \sigma - \log \lambda \). We also assume there are constants $0 < \kappa_1 < \kappa_2$ such that $e^{-\kappa_1 q} \leq J(qx) \leq e^{-\kappa_2 q}$ for all $x \in \Omega$.

The verification that $(A, H, D)$ is a spectral triple is more involved. The difficult is to verify item (3) of Definition 2.4. When working with spectral triples in this context (infinite alphabets) we need to restrict ourselves to a class of functions $J$ for which the operator $D^{-1}$ is a compact operator. A simple example of a such function is $J(x) = e^{-\varepsilon_1 (1 - e)^{-1}}$. To prove the existence of a dense subset of $C_0(\Omega, \mathbb{C})$ satisfying $\{ a \in A : ||[D, L_a]|| < +\infty \}$ it is enough to observe that
\[
\left\{ a \in C_0(\Omega, \mathbb{C}) : \sum_{w \in W^+} \frac{|a(wx) - a(wy)|}{\mathcal{J}(wx)} < +\infty \right\}
\]
is a self-adjoint subalgebra of $C_0(\Omega, \mathbb{C})$, separating points in $\Omega$ and for any $x \in \Omega$ there is an element $a$ is this subalgebra such that $a(x) \neq 0$, and therefore we can apply the Stone-Weierstrass theorem for locally compact spaces. Indeed, the family of functions $(S_n)_{n \in \mathbb{N}}$, given by
\[
S_n(x) \equiv \arctan \left( \frac{1}{x_n} \right) \exp \left( -\kappa_1 \sum_{j=1}^{n-1} x_j \right)
\]
is in this subalgebra is non-vanish and separating points. Since we are assuming $P_\sigma(f) < \infty$ follows immediately that $h_\mu(\sigma)$ is finite. By using the main result of [CS09] we can find a potential $J$ having SGP and satisfying the above conditions. Therefore follows from the above theorems and similar computation as presented before, the formula
\[
\text{Tr}_\sigma(L_a D^{-1}) = \frac{2}{h_\mu(\sigma)} \int_\Omega a d\mu, \quad \forall a \in C_0(X, \mathbb{C}).
\]

7. Remarks About Uncountable Alphabets

Let us begin this section with the following observation. In case where $A$ is finite alphabet, any element of the Hilbert space $H$, considered in first section, can be written as
\[
\bigoplus_{w \in W^+} \begin{pmatrix} \epsilon_1(w) \\ \epsilon_2(w) \end{pmatrix} = \bigoplus_{n=1}^{\infty} \bigoplus_{(w_1, \ldots, w_n) \in A^n}^{\text{admissible}} \begin{pmatrix} \epsilon_1(w_1, \ldots, w_n) \\ \epsilon_2(w_1, \ldots, w_n) \end{pmatrix}.
\]
Actually such representation has a precise meaning when $A$ is either finite or infinite countable. In this section we want to use this observation to discuss what happens with our previous constructions when uncountable alphabets are taken into account.
This generalization would have natural interest as a pure mathematical problem, but it has also a potential to create a bridge between noncommutative geometry and equilibrium states of continuous spins systems in Statistical Mechanics.

In previous section, we shown that the DLR-Gibbs measures of the Dyson model admits a Dixmier trace representation. Although it is a very important model in studying ferromagnetic systems, it has some limitations due to its formulation in terms of discrete spins. To be able to extend our results for more realistic models such as XY and Heisenberg models, it is necessary to generalize the previous constructions to allow for uncountable alphabets.

Let \( E = (E, d) \) be a compact metric space and \( p : \mathcal{B}(E) \to [0, 1] \) an a priori probability measure on \( E \), fully supported. Consider the following symbolic space \( \Omega = E^\mathbb{N} \) endowed with its standard topology, metric, etc. In this setting the Ruelle operator, associated to a potential \( \log J : \Omega \to \mathbb{R} \), acts in a continuous function \( \varphi : \Omega \to \mathbb{C} \) as follows

\[
L_{\log J}(\varphi)(x) = \int_E \exp(\log J(qx)) \varphi(qx) \, dp(q),
\]

where \( qx \equiv (q, x_1, x_2, \ldots) \). Similarly, we say that \( \log J \) is a normalized potential if \( L_{\log J}(1)(x) = 1 \), for all \( x \in \Omega \). To simplify matters in what follows, we assume that \( J : \Omega \to \mathbb{R} \) is a positive and continuous function, and that \( \log J \) is a normalized Hölder potential.

For example, if \( E = S^2 \), the unit sphere in \( \mathbb{R}^3 \), then the set of \( \sigma \)-invariant DLR-Gibbs measures of the Heisenberg model on the lattice \( \mathbb{N} \) (for a large class of potentials) coincides with the conformal measures for \( L^*_{\log J} \) for some suitable potential \( \log J \), which depends on the choice of the interaction.

In the sequel we show an obstruction to obtain a spectral triple in this cases following the ideas of the previous sections.

Taking into account the expression (6) the natural way to construct a spectral triple is taking the \( \mathcal{C}^* \)-algebra \( A \) as \( \mathcal{C}(\Omega, \mathbb{C}) \) and the Hilbert space \( H \) as the Hilbert direct sum

\[
H \equiv \bigoplus_{n=1}^\infty \int_{E^n} \mathbb{C} \oplus \mathbb{C} \, dp_n,
\]

where the constant fiber direct integrals

\[
\int_{E^n} \mathbb{C} \oplus \mathbb{C} \, dp_n \equiv L^2(E^n, dp_n, \mathbb{C} \oplus \mathbb{C})
\]

which is the Hilbert space of square integrable \( \mathbb{C} \oplus \mathbb{C} \)-valued functions, with respect to the product measure \( dp_n = \prod_{i=1}^n dp \). So an element in this space \( H \) can be thought as a pair of functions \( \epsilon_1, \epsilon_2 : \cup_{n=1}^\infty E_n \to \mathbb{C} \) satisfying

\[
\sum_{n=1}^\infty \left( \int_{E^n} |\epsilon_k(w)|^2 \, dp_n(w) \right)^2 < +\infty, \quad k = 1, 2.
\]

To keep as closed as possible of (6) one would represent the elements in this space as follows

\[
\bigoplus_{n=1}^\infty \int_{E^n} \left( \epsilon_1(w) \right) \, dp_n(w).
\]
Now, for each $a \in A$ the natural way to define the operators $L_a : H \to H$ is

$$L_a \left( \bigoplus_{i=1}^{\infty} \int_{E^n} \epsilon_1(w) \right) d\mu_n(w) = \bigoplus_{i=1}^{\infty} \int_{E^n} a(wx) \epsilon_1(w) d\mu_n(w)$$

and the Dirac operator would be given by

$$D \left( \bigoplus_{n=1}^{\infty} \int_{E^n} \epsilon_1(w) \right) d\mu_n(w) = \bigoplus_{i=1}^{\infty} \int_{E^n} \frac{1}{f(wx)} \epsilon_1(w) d\mu_n(w).$$

Although this construction is natural, in the general case it will not provide a spectral triple. The main obstruction is the compactness of $D^{-1}$. For example, if we take the metric space $E = [0, 1]$ endowed with its standard distance and the probability measure $p$ as being the Lebesgue measure, then the operators $D$ has no compact resolvent, which is a requirement in the definition of a spectral triple. Note that this is not only a technical issue because in such cases it is not clear how to define even singular traces. It would be interesting to know whether a spectral triple can be construct on this setting because of its potential applications in studying DLR-Gibbs measures associated to the long-range interactions $XY$ or Heisenberg models in one-dimensional lattices.

References


SPECTRAL TRIPLES ON THERMODYNAMIC FORMALISM


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