

Lecture notes Gibbs measures and phase transitions. Part 2

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Gibbsian formalism and metastates

Longtemps les objets dont s'occupent les mathématiciens étaient pour la plupart mal définis; on croyait les connaître, parce qu'on se les représentait avec le sens ou l'imagination; mais on n'en avait qu'une image grossière et non une idée précise sur laquelle le raisonnement pût avoir prise¹.

Henri Poincaré, La valeur de la science

We now turn to the main topic of this book, disordered systems. We split this into two parts, treating in turn *lattice models* and *mean-field models*. From the physical point of view, the former should be more *realistic* and hence more relevant, so it is natural that we present the general mathematical framework in this context. However, the number of concrete problems one can to this day solve rigorously is quite limited, so that the examples we will treat can mostly be considered as random perturbations of the Ising model. In the second part we will be able to look, in a simplified setting, at more complex, genuinely random models, that rather surprisingly will turn out to produce fascinating mathematics, but also lead to applications that are beyond the standard scope of physics.

1.1 Introduction

We have seen that statistical mechanics is a theory that treats the dynamical degrees of freedom of a large system as random variables, distributed according to the Gibbs measure. The basic rationale behind this idea was that, on the times-scales on which the system is observed, the dynamics relaxes to equilibrium and, in particular, forgets the details of the initial

¹ Approximately: For a long time the objects that mathematicians dealt with were mostly ill defined; one believed to know them, because one represented them with the senses and imagination; but one had but a rough picture and not a precise idea on which reasoning could take hold.

conditions. Such an assumption can of course not always be satisfied, as it requires the microscopic degrees of freedom to vary sufficiently fast. A typical example where this would fail are solid alloys. Assume that we have a material made of a mixture of two types of atoms, say gold and iron, that at low temperatures forms a crystalline solid. Then some lattice sites will have to be occupied by iron atoms, while the others are occupied by gold atoms. If we melt or just heat the system, the atoms become mobile and quickly change places, so that we might describe the system by some Gibbs distribution. However, at low temperatures, the motion of atoms between different sites is strongly suppressed (for reasons that we will not discuss here), and, over large time scales, the microscopic realization of the gold-iron mixture will not change. One says that the positions of the atoms are ‘frozen’, and the system will not be in thermal equilibrium.

However, the positions of the atoms are not the only degrees of freedom of the system. The iron atoms have magnetic moments, and we might be interested in the magnetic properties of the system. But the orientations of the magnetic moments are not ‘frozen’, and their behaviour could very well be described by a Gibbs measure. However, the description of this system must take into account the positions of the iron atoms, as the interaction between them depends on their mutual distances. Thus, assuming that we knew the positions, x_i , of the iron atoms, we could write a (formal) Hamiltonian for the spin degrees of freedom of the form

$$H(\sigma; x) = - \sum_{i,j} \sigma_i \sigma_j \phi(x_i, x_j) \quad (1.1)$$

Again, given the positions x_i , we would then write the Gibbs measure

$$\mu_\beta(\cdot; x) = \frac{e^{-\beta H(\sigma; x)}}{Z_\beta(x)} \quad (1.2)$$

where the partition function also depends on the positions x . We would call such a system *spatially inhomogeneous*, or, *disordered*. The point is that it would be fairly impractical to study all possible systems for all possible arrangements of the x_i ; thus we should hope that the microscopic details of these arrangements do not matter too much, and that only certain statistical properties are relevant. In other words, we would like to *model* the spatial inhomogeneity by a *random process*, i.e. model a disordered system as a *random model*, by introducing some probability distribution, \mathbb{P}_x , on the space of possible realizations of the iron positions. This new type of randomness is often called *quenched* randomness, a term derived from the idea that the solid alloy has been produced by rapid cooling through immersion in water, a process that in metallurgy is called *quenching*. One should be well

aware that this new type of randomness is conceptually very different from the randomness we have encountered previously. When dealing with the dynamical variables, we hope that time averages will converge to averages with respect to the Gibbs measures, whereas for the quenched randomness we cannot expect such a thing to happen.

What we may hope – and we will learn to what extent this is justified – is that certain properties of our materials depend little on the microscopic realizations, and are the same for *almost all* realizations of the disorder (essentially through a spatial effective averaging). Thus, a first reflex is to consider *averages* with respect to the disorder. Here there are two notions, conventionally used in the physics literature, that we need to clarify from the outset.

- *Quenched average.* This is the proper way to average: one computes for fixed disorder variables thermodynamic quantities, such as the Gibbs measure of the free energy, and then performs an average over the disorder; e.g. the *quenched free energy* is given as

$$F_{\beta,\Lambda}^{\text{quenched}} \equiv \beta^{-1} \mathbb{E}_x \ln Z_{\beta,\Lambda}(x) \quad (1.3)$$

- *Annealed average.* One computes the average of the partition function and unnormalized averages of the dynamical variables first and normalizes later. This yields, e.g., the *annealed free energy*

$$F_{\beta,\Lambda}^{\text{annealed}} \equiv \beta^{-1} \ln \mathbb{E}_x Z_{\beta,\Lambda}(x) \quad (1.4)$$

Upon reflection, this procedure corresponds to treating the disorder variables as dynamical variables on equal footing with the other degrees of freedom and thus disregards the fact that they do not equilibrate on the same time-scale. Thus this is inappropriate in the situations we want to describe. Of course, one can always try and see whether this will yield by accident the same results, anyway.

After these preliminary discussions we will now proceed to a rigorous set up of the Gibbsian formalism for quenched random models.

1.2 Random Gibbs measures and metastates

We will now give a definition of disordered lattice spin systems. This will not be as general as possible, as we allow disorder only in the interactions, but not in the lattice structure or the spin spaces. As in Chapter ??, we consider a lattice, \mathbb{Z}^d , a single-site spin space, $(\mathcal{S}_0, \mathcal{F}_0, \nu_0)$, and the corresponding a-priori product space, $(\mathcal{S}, \mathcal{F}, \nu)$. As a new ingredient, we add a (rich enough) probability space, $(\Omega, \mathcal{B}, \mathbb{P})$, where Ω will always be assumed to be a Polish

space. On this probability space we construct a *random interaction* as follows:

Definition 1.2.1 A random interaction, Φ , is a family, $\{\Phi_A\}_{A \subset \mathbb{Z}^d}$, of random variables on $(\Omega, \mathcal{B}, \mathbb{P})$ taking values in $B(\mathcal{S}, \mathcal{F}_A)$, i.e. measurable maps, $\Phi_A : \Omega \ni \omega \rightarrow \Phi_A[\omega] \in B(\mathcal{S}, \mathcal{F}_A)$. A random interaction is called *regular*, if, for \mathbb{P} -almost all ω , for any $x \in \mathbb{Z}^d$, there exists a finite constant, $c_x[\omega]$, such that

$$\sum_{A \ni x} \|\Phi_A[\omega]\|_\infty \leq c_x[\omega] < \infty \quad (1.5)$$

A regular random interaction is called *continuous*, if, for each $A \subset \Lambda$, Φ_A is jointly continuous in the variables η and ω .

In the present section we discuss only regular random interactions. Some interesting physical systems do correspond to irregular random interactions. In particular, many real spin-glasses have a non-absolutely summable interaction, called the RKKY-interaction. See [51, 20, 19, 55] for some rigorous results.

Remark 1.2.1 In most examples one assumes that the random interaction has the property that Φ_A and Φ_B are *independent* if $A \neq B$, or, at least, if $A \cap B = \emptyset$. In fact, in all examples of interest, Ω is a product space of the form $\Omega = \mathbf{E}^{\mathbb{Z}^d}$, where $\mathbf{E} \subseteq \mathbb{R}^k$.

Given a random interaction, it is straightforward to define random finite-volume Hamiltonians, $H_\Lambda[\omega]$, as in the deterministic case. Note that, for regular random interactions, H_Λ is a random variable that takes values in the space $\mathcal{B}_{ql}(\mathcal{S})$, i.e. the mapping $\omega \rightarrow H_\Lambda[\omega]$ is measurable. If, moreover, the Φ_A are continuous functions of ω , then the local Hamiltonians are also continuous functions of ω .

Next we need to define the random analogue of local specifications. A natural definition is the following:

Definition 1.2.2 A *random local specification* is a family of probability kernels, $\{\mu_{\beta, \Lambda}^{(\cdot)}[\omega]\}_{\Lambda \subset \mathbb{Z}^d}$, depending on a random parameter, ω , such that:

- (i) For all $\Lambda \subset \mathbb{Z}^d$ and $\mathcal{A} \in \mathcal{F}$, $\mu_{\beta, \Lambda}^{(\cdot)}(\mathcal{A})$ is a measurable function w.r.t. the product sigma-algebra $\mathcal{F}_{\Lambda^c} \times \mathcal{B}$.
- (ii) For \mathbb{P} -almost all ω , for all $\eta \in \mathcal{S}$, $\mu_{\Lambda, \beta}^{(\eta)}[\omega](d\sigma)$ is a probability measure on \mathcal{S} .
- (iii) For \mathbb{P} -almost all ω , the family $\{\mu_{\beta, \Lambda}^{(\cdot)}[\omega]\}_{\Lambda \subset \mathbb{Z}^d}$ is a Gibbs specification for the interaction $\Phi[\omega]$ and inverse temperature β .

- (iv) The random local specification is called *continuous*, if, for any finite Λ , $\mu_{\beta,\Lambda}^\eta[\omega]$ is jointly continuous in η and ω .

A regular random interaction should naturally give rise to a *random Gibbs specifications*. In fact, we have that:

Lemma 1.2.1 *Let Φ be a regular random interaction. Then, the formula*

$$\mu_{\Lambda,\beta}^{(\eta)}[\omega](d\sigma) \equiv \frac{1}{Z_{\beta,\Lambda}^\eta[\omega]} e^{-\beta H_\Lambda[\omega](\sigma_\Lambda, \eta_{\Lambda^c})} \rho_\Lambda(d\sigma_\Lambda) \delta_{\eta_{\Lambda^c}}(d\sigma_{\Lambda^c}) \quad (1.6)$$

defines a random local specification, called random Gibbs specification. Moreover, if Φ is continuous, then the Gibbs specification is continuous.

The important point is that the maps $\omega \rightarrow \mu_{\Lambda,\beta}^{(\cdot)}[\omega]$ are again measurable in all appropriate senses. In particular:

We now feel ready to define random infinite-volume Gibbs measures. The following is surely reasonable:

Definition 1.2.3 A measurable map, $\mu_\beta : \Omega \rightarrow \mathcal{M}_1(\mathcal{S}, \mathcal{F})$, is called a *random Gibbs measure* for the regular random interaction Φ at inverse temperature β , if, for \mathbb{P} -almost all ω , $\mu_\beta[\omega]$ is compatible with the random local specification $\left\{ \mu_{\beta,\Lambda}^{(\cdot)}[\omega] \right\}_{\Lambda \subset \mathbb{Z}^d}$ for this interaction.

The first question one must ask concerns the existence of such random Gibbs measures. One would expect that, for compact state space, the same argument as in the deterministic situation should provide an affirmative answer. Indeed, it is clear that, for almost all ω , any sequence, $\mu_{\beta,\Lambda_n}^\eta[\omega]$, taken along an increasing and absorbing sequence of volumes, possesses limit points, and, therefore, there exist subsequences, $\Lambda_{n[\omega]}$, such that $\mu_{\beta,\Lambda_{n[\omega]}}^\eta[\omega]$ converges to a Gibbs measure for the interaction $\Phi[\omega]$. The only open question is then whether such limits can provide a *measurable* map from Ω to the Gibbs measures? This is non-trivial, due to the fact that the subsequences $\Lambda_n[\omega]$ must in general depend on the realization of the disorder!

This question may first sound like some irrelevant mathematical sophistication, and indeed this problem was mostly disregarded in the literature. To my knowledge, it was first discussed in a paper by van Enter and Griffiths [52] and studied in more detail by Aizenman and Wehr [2], but it is the merit of Ch. Newman and D. Stein [37, 38, 35, 39, 41, 40, 36] to have brought the intrinsic *physical relevance* of this issue to light. Note that the problem is solved immediately if there are deterministic sequences, Λ_n , along which the local specifications converge. Newman and Stein pointed out that, in very strongly disordered systems such as spin-glasses, such deterministic sequences might not exist.

In more pragmatic terms, the construction of infinite-volume Gibbs measures via limits along random subsequences can be criticised by its lack of actual approximative power. An infinite-volume Gibbs measure is supposed to approximate reasonably a very large system under controlled conditions. If, however, this approximation is only valid for certain very special finite volumes that depend on the specific realization of the disorder, while for other volumes the system is described by other measures, just knowing the set of all infinite-volume measures is surely not enough.

As far as proving existence of random Gibbs measures is concerned, there is a simple way out of the random subsequence problem. This goes by extending the local specifications to probability measures, $K_{\beta,\Lambda}^\eta$, on $\Omega \times \mathcal{S}$, in such a way that the marginal distribution of $K_{\beta,\Lambda}^\eta$ on Ω is simply \mathbb{P} , while the conditional distribution, given \mathcal{B} , is $\mu_{\beta,\Lambda}^{(\eta)}[\omega]$. The measures K_β^η are sometimes called *joint measures*.

Theorem 1.2.2 *Let Φ be a continuous regular random interaction. Let $K_{\beta,\Lambda}^{(\cdot)}$ be the corresponding measure defined as above. Then*

(i) *If, for some increasing and absorbing sequence, Λ_n , and some $\eta \in \mathcal{S}$, the weak limit, $\lim_{n \uparrow \infty} K_{\beta,\Lambda_n}^\eta \equiv K_\beta^\eta$, exists, then its regular conditional distribution, $K_\beta^\eta(\cdot | \mathcal{B} \times \mathcal{S})$, given \mathcal{B} , is a random Gibbs measure for the interaction Φ .*

(ii) *If \mathcal{S} is compact, and if \mathbb{P} is tight in the sense that $\forall \epsilon > 0, \exists \Omega_\epsilon \subset \Omega$ that is compact and $\mathbb{P}[\Omega_\epsilon] \geq 1 - \epsilon$, then there exist increasing and absorbing sequences Λ_n such that the hypothesis of (i) is satisfied.*

Proof. The proof of this theorem is both simple and instructive. Note first that the existence of a regular conditional distribution is ensured if Ω and \mathcal{S} are Polish spaces. Let $f \in C(\mathcal{S}, \mathcal{F})$ be a continuous function. We must show that, a.s.

$$K_\beta^\eta(f | \mathcal{B} \times \mathcal{S})[\omega] = K_\beta^\eta(\mu_{\beta,\Lambda}^{(\cdot)}[\omega](f) | \mathcal{B} \times \mathcal{S})[\omega] \quad (1.7)$$

Set $g(\omega, \sigma) \equiv \mu_{\beta,\Lambda}^{(\sigma)}[\omega](f(\omega, \sigma))$. Let \mathcal{B}_k , $k \in \mathbb{N}$ be a filtration of the sigma-algebra \mathcal{B} where \mathcal{B}_k is generated by the interaction potentials Φ_A with $A \subset \Lambda_k$ with Λ_k some increasing and absorbing sequence of volumes. The important point is to realize that, for continuous functions, h , on $\Omega \times \Sigma$,

$$K_\beta^\eta(h | \mathcal{B} \times \mathcal{S})[\omega] = \lim_{k \uparrow \infty} \lim_{n \uparrow \infty} K_{\beta,\Lambda_n}^\eta(f | \mathcal{B}_k \times \mathcal{S})[\omega] \quad (1.8)$$

But for any fixed Λ , and n large enough,

$$\begin{aligned}
\mathbb{E} \left[\mu_{\beta, \Lambda_n}^{(\eta)}(f) | \mathcal{B}_k \times \Sigma \right] [\omega] &= \mathbb{E} \left[\mu_{\beta, \Lambda_n}^{(\eta)} \left(\mu_{\beta, \Lambda}^{(\cdot)}(f) \right) | \mathcal{B}_k \times \Sigma \right] [\omega] \\
&= \mathbb{E} \left[\mu_{\beta, \Lambda_n}^{(\eta)} \left(\mu_{\beta, \Lambda}^{(\cdot)}[\omega](f) \right) | \mathcal{B}_k \times \Sigma \right] [\omega] \\
&+ \mathbb{E} \left[\mu_{\beta, \Lambda_n}^{(\eta)} \left(\mu_{\beta, \Lambda}^{(\cdot)}(f) - \mu_{\beta, \Lambda}^{(\cdot)}[\omega](f) \right) | \mathcal{B}_k \times \Sigma \right] [\omega]
\end{aligned} \tag{1.9}$$

The first term converges to $K_{\beta}^{\eta}(\mu_{\beta, \Lambda}^{(\eta)}[\omega](f) | \mathcal{B} \times \mathcal{S})[\omega]$, while for the last we observe that, due to the continuity of the local specifications in ω , uniformly in n ,

$$\begin{aligned}
&\mathbb{E} \left[\mu_{\beta, \Lambda_n}^{(\eta)} \left(\mu_{\beta, \Lambda}^{(\eta)}(f) - \mu_{\beta, \Lambda}^{(\eta)}[\omega](f) \right) \right] \\
&\leq \sup_{\omega' \in \mathcal{B}_k[\omega]} \sup_{\eta \in \mathcal{S}} \left| \mu_{\beta, \Lambda}^{(\eta)}[\omega'](f) - \mu_{\beta, \Lambda}^{(\eta)}[\omega](f) \right| \downarrow 0
\end{aligned} \tag{1.10}$$

as $k \uparrow \infty$. Here $\mathcal{B}_k[\omega]$ denotes the set of all $\omega' \in \Omega$ that have the same projection to \mathcal{B}_k as ω , more formally

$$\mathcal{B}_k[\omega] \equiv \{ \omega' \in \Omega \mid \forall A \in \mathcal{B}_k : \omega \in A \Rightarrow \omega' \in A \} \tag{1.11}$$

This proves (i). To prove (ii), fix any $\epsilon > 0$. If f is a bounded, continuous function on $\Omega \times \mathcal{S}$, then

$$\begin{aligned}
\int K_{\beta, \Lambda}(d\omega, d\sigma) f(\omega, \sigma) &= \mathbb{E} \int \mu_{\beta, \Lambda}[\omega](d\sigma) f(\omega, \sigma) \\
&= \mathbb{E} \mathbb{I}_{\Omega_{\epsilon}} \int \mu_{\beta, \Lambda}[\omega](d\sigma) f(\omega, \sigma) + \mathbb{E} \mathbb{I}_{\Omega_{\epsilon}^c} \int \mu_{\beta, \Lambda}[\omega](d\sigma) f(\omega, \sigma)
\end{aligned} \tag{1.12}$$

The second term is by hypothesis bounded by $C\epsilon$, i.e. as small as desired, while the first is (up to a constant) a sequence of probability measures on the compact space $\Omega_{\epsilon} \times \mathcal{S}$, and hence there are subsequences, $\Lambda_{n_k}^{\epsilon}$, such that $K_{\Lambda_{n_k}^{\epsilon}}^{\epsilon}(f) \equiv \mathbb{E} \mathbb{I}_{\Omega_{\epsilon}} \int \mu_{\beta, \Lambda_{n_k}^{\epsilon}}[\omega](d\sigma) f(\omega, \sigma) \rightarrow K_{\beta}^{\epsilon}(f)$. Now take a sequence, $\epsilon_k \downarrow 0$. By successively thinning out subsequences, one can find a sequence Λ_n such that $K_{\Lambda_n}^{\epsilon_k}(f)$ converges, for any k . Then (1.12) implies that

$$\left| \limsup_{n \uparrow \infty} \int K_{\beta, \Lambda_n}(d\omega, d\sigma) f(\omega, \sigma) - \liminf_{n \uparrow \infty} \int K_{\beta, \Lambda}(d\omega, d\sigma) f(\omega, \sigma) \right| \leq \epsilon_k \tag{1.13}$$

for any k . Thus, $\int K_{\beta, \Lambda_n}(d\omega, d\sigma) f(\omega, \sigma)$ converges, and (ii) is proven. \square

Remark 1.2.2 There has recently been some interest in the question as to whether the joint measures, K_{β}^{η} , can themselves be considered as Gibbs measures on the extended space $\mathcal{S} \times \Omega$ [48, 31, 32]. The answer turns out to be that, while they are never Gibbs measures in the strict sense, in many cases they are *weakly Gibbsian*, i.e. there exists an almost surely absolutely summable interaction, for which the $K_{\beta, \Lambda}^{\eta}$ are local specifications.

Theorem 1.2.2 appears to solve our problems concerning the proper Gibbsian set-up for random systems. We understand what a random infinite-volume Gibbs measure is and we can prove their existence in reasonable generality. Moreover, there is a constructive procedure that allows us to obtain such measures through infinite-volume limits. However, upon closer inspection, the construction is not fully satisfactory. As can be seen from the proof of Theorem 1.2.2, the measures $K_\beta^\eta(\cdot|\mathcal{B} \times \mathcal{S})$ effectively still contains an averaging over the realization of the disorder ‘at infinity’. As a result they will often be mixed states. Such states then do not describe the result of the observations of one sample of the material at given conditions, but the average over many samples that have been prepared to look alike locally.

While we have come to understand that it may not be realistic to construct a state that predicts the outcome of observations on a single (infinite) sample, it would already be more satisfactory to obtain a probability distribution for these predictions (i.e. a random probability measure), rather than just a mean prediction (and average over probability measures). This suggests the extension of the preceding construction to a measure-valued setting. That is, rather than consider measures on the space $\Omega \times \mathcal{S}$, we introduce measures, $\mathcal{K}_{\beta,\Lambda}^\eta$, on the space $\Omega \times \mathcal{M}_1(\mathcal{S})$, defined in such a way that the marginal distribution of $\mathcal{K}_{\beta,\Lambda}^\eta$ on Ω is again \mathbb{P} , while the conditional distribution, given \mathcal{B} , is $\delta_{\mu_{\beta,\Lambda}^{(\eta)}[\omega]}$, the Dirac-measure concentrated on the corresponding local specification. We will introduce the symbolic notation

$$\mathcal{K}_{\beta,\Lambda}^\eta \equiv \mathbb{P} \times \delta_{\mu_{\beta,\Lambda}^{(\eta)}[\omega]} \quad (1.14)$$

One has the following analogue of Theorem 1.2.2:

Theorem 1.2.3 *Let Φ be a continuous regular random interaction. Let $K_{\beta,\Lambda}^{(\cdot)}$ be the corresponding measure defined as above. Then*

(i) *If, for some increasing and absorbing sequence, Λ_n , and some $\eta \in \mathcal{S}$, the weak limit, $\lim_{n \uparrow \infty} K_{\beta,\Lambda_n}^\eta \equiv \mathcal{K}_\beta^\eta$ exists, then its regular conditional distribution, $\mathcal{K}_\beta^\eta(\cdot|\mathcal{B} \times \mathcal{S})$, given \mathcal{B} , is a probability distribution on $\mathcal{M}_1(\mathcal{S})$, that, for almost all ω , gives full measure to the set of infinite-volume Gibbs measures corresponding to the interaction $\Phi[\omega]$ at inverse temperature β . Moreover,*

$$K_\beta^\eta(\cdot|\mathcal{B} \times \mathcal{S}) = \mathcal{K}_\beta^\eta(\mu|\mathcal{B} \times \mathcal{S}) \quad (1.15)$$

(ii) *If \mathcal{S} is compact and \mathbb{P} is tight, then there exist increasing and absorbing sequences Λ_n such that the hypothesis of (i) is satisfied for any η .*

Remark 1.2.3 The regular conditional distribution

$$\kappa_\beta^\eta \equiv \mathcal{K}_\beta^\eta(\cdot|\mathcal{B} \times \mathcal{S}) \quad (1.16)$$

is called the Aizenman–Wehr *metastate* (following the suggestion of Newman and Stein [38]).

Proof. The proof of this theorem is even simpler than that of Theorem 1.2.2. Note that the assertion (i) will follow if for any bounded continuous function $f : \mathcal{S} \rightarrow \mathbb{R}$, and any finite $\Lambda \subset \mathbb{Z}^d$, we can show that

$$\mathbb{E} \int \kappa_\beta^\eta(d\mu)[\omega] \left| \mu(f) - \mu \left(\mu_{\beta, \Lambda}^{(\cdot)}[\omega](f) \right) \right| = 0 \quad (1.17)$$

Let us set $h(\omega, \mu) \equiv \left| \mu(f) - \mu \left(\mu_{\beta, \Lambda}^{(\cdot)}[\omega](f) \right) \right|$. We need to check that h is a continuous function on $\Omega \times \mathcal{M}_1(\mathcal{S})$. By definition of the weak topology, $\mu(g)$ is a continuous function of μ if g is continuous. By Lemma ??, $\mu_{\beta, \Lambda}^\eta[\omega](f)$ is jointly continuous in η and ω . Thus, both $\mu(f)$ and $\mu \left(\mu_{\beta, \Lambda}^{(\cdot)}[\omega](f) \right)$ are continuous in μ , and hence h is a bounded continuous function of μ and ω . But then,

$$\mathcal{K}_\beta^\eta(h) = \lim_{n \uparrow \infty} \mathcal{K}_{\beta, \Lambda_n}^\eta(h) = 0 \quad (1.18)$$

by the compatibility relations of local specifications. But h being non-negative, must be zero \mathcal{K}_β^η -almost surely, so (1.17) holds, proving (i). Assertion (ii) follows exactly as in the proof of Theorem 1.2.2. \square

Apart from the Aizenman–Wehr metastate, Newman and Stein propose another version of the metastate that they call the *empirical metastate* as follows. Define the random empirical measures, $\kappa_N^{em}(\cdot)[\omega]$, on $(\mathcal{M}_1(\mathcal{S}^\infty))$, to be given by

$$\kappa_N^{em}(\cdot)[\omega] \equiv \frac{1}{N} \sum_{n=1}^N \delta_{\mu_{\Lambda_n}[\omega]} \quad (1.19)$$

In [38] it was proven that, for sufficiently sparse sequences Λ_n and subsequences N_k , it is true that almost surely

$$\lim_{k \uparrow \infty} \kappa_{N_k}^{em}(\cdot)[\omega] = \kappa(\cdot)[\omega] \quad (1.20)$$

Newman and Stein conjectured that in many situations the use of sparse subsequences would not be necessary to achieve convergence. However, Külske [29] has exhibited some simple mean-field examples where almost sure convergence only holds for very sparse (exponentially spaced) subsequences. He also showed that, for more slowly growing sequences, convergence in law can be proven in these cases.

Illustration. At this stage the reader may rightly hold his breath and ask the question whether all this abstract formalism is really necessary, or whether, in reasonable situations, we could avoid it completely? To answer this question, we need to look at specific results, and above all, at examples. Before turning to the difficult analysis of metastates in specific spin systems, it may be worthwhile to transplant the formalism developed above to the more familiar context of sums of i.i.d. random variables.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{X_i\}_{i \in \mathbb{N}}$ be a family of i.i.d. centered random variables with variance one; let \mathcal{F}_n be the sigma algebra generated by X_1, \dots, X_n and let $\mathcal{F} \equiv \lim_{n \uparrow \infty} \mathcal{F}_n$. Define the random variables $G_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$. We may define the joint law, K_n , of G_n and the X_i , as a probability measure on $\mathbb{R} \otimes \Omega$. Clearly, this measure converges to some measure K whose marginal on \mathbb{R} will be the standard normal distribution. However, we can say more, namely,

Lemma 1.2.4 *In the example described above,*

- (i) $\lim_{n \uparrow \infty} K_n = \mathbb{P} \times \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ denotes the normal distribution, and
- (ii) the conditional measure $\kappa(\cdot)[\omega] \equiv K(\cdot | \mathcal{F})[\omega] = \mathcal{N}(0, 1)$, a.s..

Proof. All we have to understand is that indeed the limiting measure K is a product measure; then (ii) is immediate. Let f be a continuous function on $\Omega \times \mathbb{R}$, where we identify Ω with $\mathbb{R}^{\mathbb{N}}$. We must show that $K_n(f) \rightarrow \mathbb{E}_X \mathbb{E}_g f(X, g)$, where g is a standard Gaussian random variable, independent of X . Since local functions are dense in the set of continuous functions, it is enough to assume that f depends only on finitely many coordinates, say X_1, \dots, X_k , and G_n . Then

$$\begin{aligned}
 K_n(f) &= \mathbb{E} f(X_1, \dots, X_k, G_n) = \mathbb{E} f \left(X_1, \dots, X_k, \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right) \\
 &= \mathbb{E} f \left(X_1, \dots, X_k, \frac{1}{\sqrt{n-k}} \sum_{i=k+1}^n X_i \right) \\
 &+ \mathbb{E} \left[f \left(X_1, \dots, X_k, \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right) \right. \\
 &\quad \left. - f \left(X_1, \dots, X_k, \frac{1}{\sqrt{n-k}} \sum_{i=k+1}^n X_i \right) \right] \tag{1.21}
 \end{aligned}$$

Clearly, by the central limit theorem,

$$\lim_{n \uparrow \infty} \mathbb{E} f \left(X_1, \dots, X_k, \frac{1}{\sqrt{n-k}} \sum_{i=k+1}^n X_i \right) = \mathbb{E} \mathbb{E}_g f(X_1, \dots, X_k, g) \quad (1.22)$$

while the remaining terms tend to zero, as $n \uparrow \infty$, by continuity of f . This proves the lemma. \square

Let us now look at the empirical metastate in our example. Here the empirical metastate corresponds to

$$\kappa_N^{em}(\cdot)[\omega] \equiv \frac{1}{N} \sum_{n=1}^N \delta_{G_n[\omega]} \quad (1.23)$$

We will prove that the following Lemma holds:

Lemma 1.2.5 *Let G_n and $\kappa_N^{em}(\cdot)[\omega]$ be defined above. Let B_t , $t \in [0, 1]$ denote a standard Brownian motion. Then the random measures κ_N^{em} converge in law to the measure $\kappa^{em} = \int_0^1 dt \delta_{t^{-1/2} B_t}$.*

Proof. We will see that, quite clearly, this result relates to Lemma 1.2.4 as the Invariance Principle does to the CLT, and indeed, its proof is essentially an immediate consequence of Donsker's Theorem. Donsker's theorem (see [25] for a formulation in more generality than needed in this chapter) asserts the following: Let $\eta_n(t)$ denote the continuous function on $[0, 1]$ that, for $t = k/n$, is given by

$$\eta_n(k/n) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^k X_i \quad (1.24)$$

and that interpolates linearly between these values. Then, $\eta_n(t)$ converges in distribution to standard Brownian motion, in the sense that, for any continuous functional, $F : C([0, 1]) \rightarrow \mathbb{R}$, it is true that $F(\eta_n)$ converges in law to $F(B)$. We have to prove that, for any bounded continuous function f ,

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N \delta_{G_n[\omega]}(f) &\equiv \frac{1}{N} \sum_{n=1}^N f \left(\eta_n(n/N) / \sqrt{n/N} \right) \\ &\rightarrow \int_0^1 dt f(B_t / \sqrt{t}) \equiv \int_0^1 dt \delta_{B_t / \sqrt{t}}(f) \end{aligned} \quad (1.25)$$

To see this, simply define the continuous functionals F and F_N by

$$F(\eta) \equiv \int_0^1 dt f(\eta(t) / \sqrt{t}) \quad (1.26)$$

and

$$F_N(\eta) \equiv \frac{1}{N} \sum_{n=1}^N f(\eta(n/N)/\sqrt{n/N}) \quad (1.27)$$

We have to show that in distribution $F(B) - F_N(\eta_N)$ converges to zero. But

$$F(B) - F_N(\eta_N) = F(B) - F(\eta_N) + F(\eta_N) - F_N(\eta_N) \quad (1.28)$$

By the invariance principle, $F(B) - F(\eta_N)$ converges to zero in distribution while $F(\eta_N) - F_N(\eta_N)$ converges to zero since F_N is the Riemann sum approximation to F . B_t is measurable with respect to the tail sigma-algebra of the X_i , so that the conditioning on \mathcal{F} has no effect. \square

Exercise. Consider the random field version of the Curie-Weiss model, i.e. the mean-field model with Hamiltonian

$$H_N(\sigma)[\omega] \equiv -\frac{1}{2N} \sum_{i,j=1}^N \sigma_i \sigma_j - \delta \sum_{i=1}^N h_i[\omega] \sigma_i \quad (1.29)$$

where the h_i are i.i.d. random variables taking the values ± 1 with probability $1/2$.

- (i) Introduce the random sets $\Lambda_+[\omega] \equiv \{i : h_i[\omega] = +1\}$ and $\Lambda_-[\omega] = \{1, \dots, N\} \setminus \Lambda_+[\omega]$. Define the (empirical) magnetizations of these two sets, $m_{\pm}(\sigma) \equiv \frac{1}{|\Lambda_{\pm}|} \sum_{i \in \Lambda_{\pm}} \sigma_i$. Express $H_N(\sigma)$ in terms of these quantities.
- (ii) Compute an expression for the distribution of the variables $m(\sigma) \equiv (m_+(\sigma), m_-(\sigma))$ under the canonical Gibbs measure.
- (iii) Show that, if $\delta < 1$, there is a critical value $\beta_c = \beta_c(\delta)$, such that for $\beta < \beta_c$, there exist two distinct points $m^*, \bar{m}^* \in [-1, 1]^2$, with $(\bar{m}_+^*, \bar{m}_-^*) = (-m_-^*, -m_+^*)$, such that, for any $\epsilon > 0$, $\lim_{N \uparrow \infty} \mu_{\beta, N}(\{|m(\sigma) - m^*| < \epsilon\} \vee \{|m(\sigma) - \bar{m}^*| < \epsilon\}) = 1$, a.s..
- (iv) Show that, for almost all ω , there exists a random subsequence $N_k[\omega]$ such that $\lim_{k \uparrow \infty} \mu_{\beta, N_k[\omega]}(\{|m(\sigma) - m^*| < \epsilon\}) = 1$. Are there also subsequences such that the limit is $1/2$?
- (v) Now consider $q_N \equiv \mu_{\beta, N}(\{|m(\sigma) - m^*| < \epsilon\})$ as a sequence of random variables. What is the limit of its distribution as $N \uparrow \infty$? Use this result to formulate a result on the convergence in distribution of the Gibbs measures $\mu_{\beta, \delta, N}$.
- (vi*) Give an expression for the Aizenman-Wehr metastate.
- (vii*) Consider the empirical distribution of the random variables q_N , i.e. $\frac{1}{N} \sum_{n=1}^N \delta_{q_n}$. What is the limit of this probability measure as $N \uparrow \infty$?

All the concepts introduced above have been worked out explicitly for two non-trivial disordered models, the random field Curie-Weiss model and the Hopfield model with finitely many patterns (see Chapter 12), by Külske [29, 30]. Explicit constructions of metastates for lattice models are largely lacking. The only example is the two-dimensional Ising model with random boundary conditions that was studied in [49, 50].

1.3 Remarks on uniqueness conditions

As in the case of deterministic interactions, having established existence of Gibbs states, the next basic question is that of uniqueness. As in the deterministic case, uniqueness conditions can be formulated in a quite general setting for ‘weak’ enough interactions. Indeed, Theorem ?? can be applied directly for any given realization of the disorder. However, a simple application of such a criterion will not capture the particularities of a disordered system, and will therefore give bad answers in most interesting examples. The reason for this lies in the fact that the criterion of Theorem ?? is formulated in terms of a supremum over $y \in \mathbb{Z}^d$; in a translation invariant situation, this is appropriate, but, in a random system, we will often find that, while for most points the condition will be satisfied, there may exist rare random points where it is violated. Extensions of Dobrushin’s criteria have been developed by Bassalygo and Dobrushin [5], van den Berg and Maes [47], and Gielis [24]. Uniqueness for weak interactions in the class of regular interactions can be proven with the help of cluster expansion techniques rather easily. The best results in this direction are due to Klein and Masooman [28], while the basic ideas go back to Berretti [6] and Fröhlich and Imbrie [18].

It should be pointed out that the most interesting problems in high-temperature disordered systems concern the case of non-regular interactions. For example, Fröhlich and Zegarlinski [20, 19, 55] have proved uniqueness (in a weak sense), for mean zero square integrable interactions of mean zero, in the Ising case.

1.4 Phase transitions

The most interesting questions in disordered systems concern again the case of non-uniqueness of Gibbs measures, i.e. phase transitions. Already in the case of deterministic models, we have seen that there is no general theory for the classification of the extremal Gibbs states in the low-temperature

regime; in the case of disordered systems the situation is even more difficult. Basically, one should distinguish between two situations:

- (1) Small random perturbations of a deterministic model (whose phase structure is known).
- (2) Strongly disordered models.

Of course, this distinction is a bit vague. Nonetheless, we say that we are in situation (1) if we can represent the Hamiltonian in the form

$$H[\omega](\sigma) = H^{(0)}(\sigma) + H^{(1)}[\omega](\sigma) \quad (1.30)$$

where $H^{(0)}$ is a non-random Hamiltonian (corresponding to a regular interaction) and $H^{(1)}$ is a random Hamiltonian corresponding to a regular random interaction, that is ‘small’ in some sense. The main question is then, whether the phase diagram of H is a continuous deformation of that of $H^{(0)}$, or not. In particular, if $H^{(0)}$ has a first-order phase transition, will the same be true for H ?

There are situations when this question can be answered easily; they occur when the different extremal states of $H^{(0)}$ are related by a symmetry group and if, for any realization of the disorder, this symmetry is respected by the random perturbation $H^{(1)}[\omega]$. The classical example of this situation is the *Dilute Ising Model*: The Hamiltonian of this model is given (formally) by

$$H[\omega](\sigma) = - \sum_{|i-j|=1} J_{ij}[\omega] \sigma_i \sigma_j \equiv H^{Ising}(\sigma) + \sum_{|i-j|=1} (1 - J_{ij})[\omega] \sigma_i \sigma_j \quad (1.31)$$

where J_{ij} are i.i.d. random variables taking the values 0 and 1 with probabilities p and $1 - p$, respectively¹. If p is small, we may consider this as a small perturbation of the standard Ising model. We will show that the Peierls argument (Theorem ??) applies with just minor modifications, as was observed in [4].

Theorem 1.4.1 *Let μ_β be a Gibbs measure for the dilute Ising model defined by (1.31) and assume that $d \geq 2$. Then there exists $p_0 > 0$, such that, for all $p \leq p_0$, there exists $\beta(p) < \infty$, such that, for $\beta \geq \beta(p)$,*

$$\mathbb{P} \left[\mu_\beta \left[\exists_{\gamma \in \Gamma(\sigma): 0 \in \text{int} \gamma} < \frac{1}{2} > \right] > 0 \right] \quad (1.32)$$

Proof. Define the random contour energy $E(\gamma)$ by

$$E(\gamma) \equiv \sum_{\langle ij \rangle^* \in \gamma} J_{ij} \quad (1.33)$$

¹ The precise distribution of the J_{ij} plays no rôle for the arguments that follow; it is enough to have $\mathbb{E}J_{ij} = J_0 > 0$, and $\text{var}(J_{ij}) \ll J_0$.

Repeating the proof of Lemma ?? mutatis mutandis, one gets immediately the estimate

$$\mu_\beta[\omega] [\gamma \in \Gamma(\sigma)] \leq e^{-2\beta E[\omega](\gamma)} \quad (1.34)$$

By the law of large numbers, for large γ , $E(\gamma)$ will tend to be proportional to $|\gamma|$; indeed we have that

$$\mathbb{P}[E(\gamma) = x|\gamma|] = \binom{|\gamma|}{x|\gamma|} (1-p)^{x|\gamma|} p^{(1-x)|\gamma|} \quad (1.35)$$

for $x|\gamma|$ integer. Now define the event

$$\mathcal{A} \equiv \{\exists_{\gamma:0 \in \text{int}_\gamma} : E(\gamma) < |\gamma|/2\} \quad (1.36)$$

Clearly, $\mathbb{P}[\mathcal{A}] \leq \sum_{\gamma:0 \in \text{int}_\gamma} \mathbb{P}[E(\gamma) < |\gamma|/2]$, and by the crudest estimate, using (1.35), $\mathbb{P}[E(\gamma) < |\gamma|/2] \leq 2^{|\gamma|} p^{|\gamma|/2}$. Recalling (??), we get that

$$\mathbb{P}[\mathcal{A}] \leq \sum_{k=2d}^{\infty} 3^k 2^k p^{-k/2} \leq \frac{(36p)^d}{1-6\sqrt{p}} \quad (1.37)$$

if $p < 1/36$. But, if $\omega \in \mathcal{A}^c$,

$$\mu_\beta[\omega] [\exists_{\gamma \in \Gamma(\sigma):0 \in \text{int}_\gamma}] \leq \sum_{\gamma:0 \in \text{int}_\gamma} \mu_\beta[\omega] [\gamma \in \Gamma(\sigma)] \leq \sum_{\gamma:0 \in \text{int}_\gamma} e^{-\beta|\gamma|} \quad (1.38)$$

which is smaller than $1/2$ if β is large enough. Thus, for such β ,

$$\begin{aligned} \mathbb{P}[\mu_\beta [\exists_{\gamma \in \Gamma(\sigma):0 \in \text{int}_\gamma}] < \tfrac{1}{2}] &\geq \mathbb{P}[\mathcal{A}^c] \mathbb{P}[\mu_\beta [\exists_{\gamma \in \Gamma(\sigma):0 \in \text{int}_\gamma}] < \tfrac{1}{2} | \mathcal{A}^c] \\ &\geq 1 - \frac{(36p)^d}{1-6\sqrt{p}} \end{aligned} \quad (1.39)$$

which can be made as close to one as desired if p is small enough. \square

From Theorem 1.4.1 we can deduce the existence of at least two distinct random Gibbs states.

Corollary 1.4.2 *In the dilute Ising model, for any $d \geq 2$, there exists $p_0 > 0$, such that, for all $p \leq p_0$, there exists $\beta(p) > 0$, such that, for all $\beta \geq \beta(p)$, with probability one, there exist at least two extremal random Gibbs states.*

Proof. Theorem 1.4.1 implies, by the arguments put forward in Section ??, that there exist at least two extremal Gibbs states with positive probability. However, the number of extremal Gibbs measures for a given random interaction (with sufficient ergodic properties which are trivially satisfied here) is an almost sure constant ([36], Proposition 4.4.). The argument goes in two steps: first, one shows that the number of *extremal Gibbs states* for given values of the temperature is a \mathcal{B} -measurable function. Next, it is clear that the number of extremal Gibbs states, for a given realization of the disorder,

is a translation invariant quantity (where the translation group \mathbb{Z}^d acts on Ω in such a way that, for $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, $T_x\omega$ is defined such that, for all $A \subset \mathbb{Z}^d$, $\Phi_A[T_x\omega](\sigma) = \Phi_{A-x}[\omega]$). But in all cases considered, the measure \mathbb{P} is stationary and ergodic under the group of translations \mathbb{Z}^d , and thus, by the ergodic theorem (see e.g. Appendix A3 of [23]), any translation-invariant function is almost surely a constant [15]. \square

Remark 1.4.1 In the dilute Ising model, since all couplings J_{ij} are non-negative, the FKG inequalities hold, and thus, according to Corollary ??, we can construct two random Gibbs measures, $\mu_\beta^\pm[\omega]$, as limits of local specifications with pure plus, resp. pure minus boundary conditions along any deterministic sequence of increasing and absorbing finite volumes Λ_n . These two states will be distinct if there exists $x \in \mathbb{Z}^d$, such that $\mu_\beta^+[\omega][\sigma_x = +1] > 1/2$. Thus, if the translation invariant, \mathcal{B} -measurable event $\{\exists x \in \mathbb{Z}^d : \mu_\beta^+[\omega][\sigma_x = +1] > 1/2\}$ occurs, then these two measures are distinct. But if this event has strictly positive probability, by the ergodic theorem, its probability is 1, and so the two extremal states, μ_β^\pm , are then distinct, almost surely. This provides a simple alternative proof of the Corollary.

Exercise: Improve the estimates on $\beta(p)$ obtained above. Show in particular that the theorem holds with any $p_0 > 1/3$.

The Peierls approach indicated here does not give optimal results (but has the advantage of clarity and robustness). It is known that $\beta(p)$ is finite, iff p is larger than the critical value for bond (in $d \geq 3$ plaquette) percolation. This has been proven first by Georgii [21] in $d = 2$ and in more generality in [22, 1]. The latter papers also obtain precise results on the dependence of $\beta(p)$ on p . These results are all based on profound facts from percolation theory, a subject that we will not develop here.

Situations where the random perturbation respects the symmetries of the unperturbed interaction, for any realization of the disorder, is exceptional. In general, the perturbation $H^{(1)}[\omega]$ will break all symmetries of the model for typical ω and thus will render the standard Peierls argument inapplicable. The simplest example of such models is the *random field Ising model*, whose Hamiltonian is

$$H[\omega](\sigma) \equiv - \sum_{|i-j|=1} \sigma_i \sigma_j - \epsilon \sum_i h_i[\omega] \sigma_i \quad (1.40)$$

with h_i a family of i.i.d. random variables (say of mean 0 and variance 1). The issue of the RFIM is one of the first occurrences where profound probabilistic thinking has entered the field, I will devote the following chapter to the analysis of this model.

1.5 The Edwards–Anderson model.

If in the Hamiltonian of the dilute Ising model we replace the random variables J_{ij} by i.i.d. random variables that are uniformly distributed on the interval $[-1, 1]$, we obtain the *Edwards–Anderson model* of a *spin-glass* [3]. This model has proven to be one of the most elusive and difficult models to analyze, both from the analytical and numerical point of view. Consequently, the amount of rigorously known facts about this model is frighteningly small. Even on a heuristic level, there are conflicting opinions on the nature of the expected phase structure in various dimensions. I will not discuss this model in any detail (refer to Newman’s book [36] for a thorough discussion), but only indicate some basic features.

The basis of the difficulties encountered with this model lies in the fact that it is highly non-trivial to say something sensible about its *ground-states*. The reason is that the couplings take both signs, thus favouring alignment or non-alignment of the spins. Worse, it is clearly impossible to satisfy the demands of all couplings: to see this, consider, in dimension two, say, the four sites surrounding one plaquette of the lattice. It is not unusual to find that out of the four couplings around this plaquette, an odd number will be negative, while the remainders are positive. It is then impossible for the spins on the corners to be arranged in such a way that all four couplings have their way: one says that the plaquette is *frustrated*. If the couplings are Bernoulli distributed (± 1 with equal probability), we would find four arrangements contributing an equal amount of energy; one can show that this implies that the ground-states in this case must be infinitely degenerate. In fact, the number of ground state configurations in a given volume, Λ , is in this case of the order $C^{|\Lambda|}$ [4]. But even in the case of continuous distributions, we encounter plaquettes where four configurations give almost the same contribution to the energy. This allows the possibility that, on a larger scale, there can be numerous spin configurations whose energy is very similar; in particular, ground-state configuration can be very sensitive to boundary conditions and vary dramatically as the volume varies.

As a result, none of the available techniques to analyze low-temperature phases (Peierls arguments, low-temperature expansions, Pirogov–Sinai theory, etc.) is applicable, and even the most basic questions concerning the low-temperature phases of this model are wide open. It is largely believed that in two dimensions there is a unique Gibbs state at all positive temperatures, while in dimension three and higher, there should be at least two extremal states. There are several schools that predict different pictures in higher dimensions: Fisher and Huse [26, 16] predict the existence of just

two extremal states in any dimensions, while the school of Parisi [34] suggests a very complex picture based on mean-field theory (see Chapter 11) that would imply the existence of infinitely many extremal Gibbs states in high dimensions. This latter suggestion is mainly based on numerical simulations, which are, however, very difficult to interpret and do not provide unambiguous predictions. Newman and Stein have analyzed a variety of scenarios and checked their compatibility with basic principles. A very recent account summarizing the current state of understanding can be found in [43, 42]. This fascinating problem still awaits new ideas.

The random-field Ising model

Quand les physiciens nous demandent la solution d'un problème, ce n'est pas une corvée qu'ils nous imposent, c'est nous au contraire qui leur doivent des remerciements.¹

Henri Poincaré, La valeur de la science.

The random-field Ising model has been one of the big success stories of mathematical physics and deserves an entire chapter. It will give occasion to learn about many of the more powerful techniques available to the analysis of random systems. The central question heatedly discussed in the 1980's in the physics community was whether the RFIM would show spontaneous magnetization at low temperatures and weak disorder in dimension 3, or not. There were conflicting theoretical arguments, and even conflicting interpretations of experiments. Disordered systems, more than others, tend to elude common intuition. The problem was solved at the end of the decade in two rigorous papers by Bricmont and Kupiainen [12] (who proved the existence of a phase transition in $d \geq 3$ for small ϵ) and Aizenman and Wehr [2] (who showed the uniqueness of the Gibbs state in $d = 2$ for all temperatures).

2.1 The Imry–Ma argument

The earliest attempt to address the question of the phase transition in the RFIM goes back to Imry and Ma [27] in 1975. They tried to extend the beautiful and simple Peierls argument to a situation with symmetry breaking randomness. Let us recall that the Peierls argument in its essence relies on the observation that in order to deform one ground-state, $+1$, in the interior of a contour, γ , to another ground-state, -1 , costs a *surface energy* $2|\gamma|$, while, by symmetry, the 'bulk energies' of the two ground-states are the

¹ Approximately: When the physicists ask us for the solution of a problem, it is not a drudgery that they impose on us, on the contrary, it is us who owe them thanks.

same. Since the number of contours of a given length L is only of order C^L , the Boltzmann factors, $e^{-2\beta L}$, suppress such deformations sufficiently to make their existence unlikely if β is large enough. What goes wrong with the argument in the RFIM is the fact that the bulk energies of the two ground-states are no longer the same. Indeed, if all σ_i in $\text{int } \gamma$ take the value $+1$, then the random-field term gives a contribution

$$E_{bulk}(\gamma) = +\epsilon \sum_{i \in \text{int } \gamma} h_i[\omega] \quad (2.1)$$

while it is equal to minus the same quantity if all σ_i equal -1 . Thus deforming the plus state to the minus state within γ produces, in addition to the surface energy term, a bulk term of order $2\epsilon \sum_{i \in \text{int } \gamma} h_i[\omega]$ that can take on any sign. Even when the random fields, h_i , are uniformly bounded, this contribution is bounded *uniformly* only by $2\epsilon |\text{int } \gamma|$ in absolute value and thus can be considerably bigger than the surface term, no matter how small ϵ is, if $|\gamma|$ is sufficiently large. Imry and Ma argued that this uniform bound on $E_{bulk}(\gamma)$ should not be the relevant quantity to consider. Namely, by the central limit theorem, the ‘typical’ value of $E_{bulk}(\gamma)$, for large γ , would be much smaller,

$$E_{bulk}(\gamma) \sim \pm \epsilon \sqrt{|\text{int } \gamma|} \quad (2.2)$$

Since by the isoperimetric inequality on \mathbb{Z}^d $|\text{int } \gamma| \leq 2d |\gamma|^{\frac{d}{d-1}}$, this means that the typical value of the bulk energy is only $E_{bulk}(\gamma) \sim \pm \epsilon |\gamma|^{\frac{d}{2(d-1)}}$, which is small compared to $|\gamma|$ if $d > 2$. Otherwise, it is comparable or even larger. This very simple consideration led Imry and Ma to the (*correct!!*) prediction that the RFIM undergoes a phase transition in $d \geq 3$ and does not in $d \leq 2$.

Remark 2.1.1 This argument is meant to work only if $\epsilon \ll 1$; if ϵ is not small, then even in small contours the bulk energy can dominate the surface energy. In particular, it is easy to see that, if $\epsilon > 2d$, and h_i take the values ± 1 , then there is a unique random ground-state given by $\sigma_i = \text{sign } h_i$.

It is likely that this argument would have been considered sufficient by the standards of theoretical physics, had there not been a more fancy argument, based on field theoretic considerations (due to Parisi and Sourlas [44], that predicted that the RFIM in dimension d should behave like the Ising model without random field in dimension $d - 2$. This *dimensional reduction* argument would then predict the absence of a phase transition in $d = 3$, contrary to the Imry-Ma argument. The two arguments divided the community¹.

¹ And rightly so. Even though in the RFIM dimensional reduction was ultimately shown to make the wrong prediction, in another application of similar reasoning, namely in the problem

Admittedly, with an alternative option in mind, the Imry–Ma argument looks rather shaky, and anyone would be excused for not trusting it. We will thus distance us a bit from Imry and Ma and try to repeat their reasoning in a more precise way. What we would obviously want to do is to reprove something like Theorem 1.4.1. When trying to re-run the proof of Lemma ??, all works as before until the last line of (?). One obtains instead the two bounds

$$\begin{aligned}\mu_{\text{int } \gamma, \beta}^{+1} [\sigma_{\gamma^{\text{in}}} = -1] &\leq e^{-2\beta|\gamma|} \frac{Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{-1}}{Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{+1}} \\ \mu_{\text{int } \gamma, \beta}^{-1} [\sigma_{\gamma^{\text{in}}} = +1] &\leq e^{-2\beta|\gamma|} \frac{Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{+1}}{Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{-1}}\end{aligned}\tag{2.3}$$

Hence, the analogue of Lemma ?? becomes

Lemma 2.1.1 *In the random-field Ising model, for any Gibbs state μ_β ,*

$$\mu_\beta [\gamma \in \Gamma(\sigma)] \leq \exp \left(-2\beta|\gamma| + \left| \ln Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{\text{in}}, \beta}^{-1} \right| \right)\tag{2.4}$$

At this point, one may lose all hope when facing the difference of the logarithms of the two partition functions, and one may not even see how to arrive at Imry and Ma’s assertion on the ‘typical value’ of this bulk term¹. However, the situation is much better than could be feared. The reason is the so-called *concentration of measure* phenomenon that will continue to play an important rôle in the analysis of disordered systems. Roughly, concentration of measure means that in many situations, a Lipschitz continuous function of i.i.d. random variables has fluctuations that are not bigger than those of a corresponding linear function. This phenomenon has been widely investigated over the last 30 years, with culminating results due to M. Talagrand. We refer to [45, 46, 33] for a detailed presentation and references. We will use the following theorem, due to M. Talagrand, whose proof can be found in [46]:

Theorem 2.1.2 *Let $f : [-1, 1]^N \rightarrow \mathbb{R}$ be a function whose level sets are convex. Assume that f is Lipschitz continuous with uniform constant C_{Lip} , i.e. for any $X, Y \in [-1, 1]^N$,*

$$|f(X) - f(Y)| \leq C_{\text{Lip}} \|X - Y\|_2\tag{2.5}$$

of critical behaviour of *branched polymers*, Brydges and Imbrie [13] recently proved rigorously that dimensional reduction is correct!

¹ This had been considered to be the truly difficult part of the problem. Chalker in 1983 [14] and Fisher, Fröhlich, and Spencer in 1984 [17] gave a solution of the problem where this difference was ad hoc replaced by the sum over the random fields within $\text{int } \gamma$. As we will see, however, the real difficulty of the problem lies elsewhere.

Then, if X_1, \dots, X_N are i.i.d. random variables taking values in $[-1, 1]$, and $Z = f(X_1, \dots, X_N)$, and if \mathbb{M}_Z is a median² of Z , then

$$\mathbb{P}[|Z - \mathbb{M}_Z| \geq z] \leq 4 \exp\left(-\frac{z^2}{16C_{Lip}^2}\right) \quad (2.6)$$

Remark 2.1.2 In most applications, and in particular when C_{Lip} is small compared to z^2 , one can replace the median in (2.6) by the expectation $\mathbb{E}Z$ without harm.

Remark 2.1.3 If X_i are i.i.d. centred Gaussian random variables with variance 1, the conclusion of Theorem 2.1.2 hold even without the assumption of convex level sets, and, with the median replaced by the mean, and both constants 4 and 16 replaced by 2 (see [33]). There are many other situations where similar results hold [45, 33].

Remark 2.1.4 Physical quantities satisfying such concentration inequalities are often called *self-averaging*.

Theorem 2.1.2 allows us to prove the following Lemma:

Lemma 2.1.3 *Assume that the random fields have a symmetric distribution¹ and are bounded² (i.e. $|h_i| \leq 1$), or their distribution is Gaussian. Then there is a constant $C < \infty$, such that, for any $z \geq 0$,*

$$\begin{aligned} & \mathbb{P}\left[\left|\ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1}\right| > z\right] \\ & \leq C \exp\left(-\frac{z^2}{\epsilon^2 \beta^2 C |\text{int } \gamma|}\right) \end{aligned} \quad (2.7)$$

Proof. By symmetry of the distribution of h , the two partition functions we consider have, as random variables, the same distribution. In particular,

$$\mathbb{E} \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} = \mathbb{E} \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \quad (2.8)$$

Therefore,

² A median of a random variable Z is any number such that $\mathbb{P}[Z \geq \mathbb{M}_Z] \geq 1/2$ and $\mathbb{P}[Z \leq \mathbb{M}_Z] \geq 1/2$.

¹ This assumption appears necessary even for the result; otherwise the phase coexistence point could be shifted to some finite value of the external magnetic field.

² We make this assumption for convenience; as a matter of fact essentially the same result holds if we only assume that the h_i have finite exponential moments.

$$\begin{aligned}
& \mathbb{P} \left[\left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \right| > z \right] \\
& \leq \mathbb{P} \left[\left\{ \left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \mathbb{E} \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} \right| \right. \right. \\
& \quad \left. \left. + \left| \mathbb{E} \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \right| \right\} > z \right] \\
& \leq 2\mathbb{P} \left[\left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \mathbb{E} \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} \right| > z/2 \right]
\end{aligned} \tag{2.9}$$

$\ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1}$ is a function of the independent random variables h_i , with $i \in \text{int } \gamma \setminus \gamma^{in}$; Moreover, one can check (by differentiation) that it is a convex function. Thus, to use Theorem 2.1.2, we only must verify that $\ln Z$ is Lipschitz continuous and compute the Lipschitz constant. But,

$$\begin{aligned}
& \left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1}[\omega] - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1}[\omega'] \right| \\
& \leq \sup_{\omega''} \left| \sum_{i \in \text{int } \gamma \setminus \gamma^{in}} (h_i[\omega] - h_i[\omega']) \frac{\partial \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1}[\omega'']}{\partial h_i} \right| \\
& \leq \epsilon \beta \sup_{i \in \text{int } \gamma \setminus \gamma^{in}} |\mu_{\text{int } \gamma \setminus \gamma^{in}, \beta}(\sigma_i)| \sum_{i \in \text{int } \gamma \setminus \gamma^{in}} |h_i[\omega] - h_i[\omega']| \\
& \leq \epsilon \beta \sqrt{|\text{int } \gamma|} \|h_{\text{int } \gamma}[\omega] - h_{\text{int } \gamma}[\omega']\|_2
\end{aligned} \tag{2.10}$$

where in the last step we used that the expectation of σ_i is bounded by one and the Cauchy–Schwarz inequality. Theorem 2.1.2 implies (2.7). \square

Lemma 2.1.3 implies indeed that, for a given contour, γ ,

$$\mu_\beta [\gamma \in \Gamma(\sigma)] \leq \exp \left(-2\beta|\gamma| + \epsilon\beta\sqrt{|\text{int } \gamma|} \right) \tag{2.11}$$

However, the immediate attempt to prove the analogue of Theorem 1.4.1 fails. Namely, we would have to show that

$$\mathbb{P} \left[\exists_{\gamma: \text{int } \gamma \ni 0} \left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \right| > \beta|\gamma| \right] \tag{2.12}$$

is small (for small ϵ). The straightforward way to try to prove this is to write

$$\begin{aligned}
& \mathbb{P} \left[\exists_{\gamma: \text{int } \gamma \ni 0} \left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \right| > \beta|\gamma| \right] \\
& \leq \sum_{\gamma: \text{int } \gamma \ni 0} \mathbb{P} \left[\left| \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1} \right| > \beta|\gamma| \right] \\
& \leq \sum_{\gamma: \text{int } \gamma \ni 0} \exp \left(-\frac{|\gamma|^2}{C\epsilon^2|\text{int } \gamma|} \right)
\end{aligned} \tag{2.13}$$

But $\frac{|\gamma|^2}{|\text{int } \gamma|}$ can be as small (and is for many γ) as $|\gamma|^{(d-2)/(d-1)}$, and since the number of γ 's of given length is of order $C^{|\gamma|}$, the last sum in (2.13) diverges.

Some reflection shows that it is the first inequality in (2.13) that spoiled the estimates. This step would be reasonable if the partition functions for different γ were more or less independent. However, if γ and γ' are very similar, it is clear that this is not the case. A more careful analysis should exploit this fact and hopefully lead to a better bound. Such situations are quite common in probability theory, and in principle there are well-known techniques that go under the name of *chaining* to systematically improve estimates like (2.13). This was done in the papers [14] and [17], however in a model where $\ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{+1} - \ln Z_{\text{int } \gamma \setminus \gamma^{in}, \beta}^{-1}$ is ad hoc replaced by $\beta \sum_{i \in \text{int } \gamma \setminus \gamma^{in}} h_i$ (the so-called ‘no contours within contours’ approximation). In fact, they prove the following:

Proposition 2.1.4 *Assume that there is a finite positive constant, C , such that, for all $\Lambda, \Lambda' \subset \mathbb{Z}^d$,*

$$\begin{aligned} & \mathbb{P} \left[\left| \ln Z_{\Lambda, \beta}^{+1} - \ln Z_{\Lambda', \beta}^{+1} - \mathbb{E} \left[\ln Z_{\Lambda, \beta}^{+1} - \ln Z_{\Lambda', \beta}^{+1} \right] \right| \geq z \right] \\ & \leq \exp \left(- \frac{z^2}{C \epsilon^2 \beta^2 |\Lambda \Delta \Lambda'|} \right) \end{aligned} \quad (2.14)$$

where $\Lambda \Delta \Lambda'$ denotes the symmetric difference of the two sets Λ and Λ' . Then, if $d \geq 3$, there exists $\epsilon_0 > 0$, $\beta_0 < \infty$, such that for all $\epsilon \leq \epsilon_0$ and $\beta \geq \beta_0$, for \mathbb{P} almost all $\omega \in \Omega$, there exist at least two extremal infinite-volume Gibbs states μ_β^+ , and μ_β^- .

Remark 2.1.5 There are good reasons to believe that (2.14) holds, but in spite of multiple efforts, I have not been able to find an easy proof. On a heuristic level, the argument is that the difference appearing in (2.14) should depend very little on the random variables that appear in the intersection of Λ and Λ' . More precisely, when computing the Lipschitz norm, we get, instead of (2.10),

$$\begin{aligned}
& \left| \ln Z_{\Lambda,\beta}^{+1}[\omega] - \ln Z_{\Lambda,\beta}^{+1}[\omega'] - \ln Z_{\Lambda',\beta}^{+1}[\omega] + \ln Z_{\Lambda',\beta}^{+1}[\omega'] \right| \\
& \leq \sup_{\omega''} \left| \sum_{i \in \Lambda \setminus (\Lambda \cap \Lambda')} (h_i[\omega] - h_i[\omega']) \frac{\partial \ln Z_{\Lambda,\beta}^{+1}[\omega'']}{\partial h_i} \right| \\
& \quad + \left| \sum_{i \in \Lambda' \setminus (\Lambda \cap \Lambda')} (h_i[\omega] - h_i[\omega']) \frac{\partial \ln Z_{\Lambda',\beta}^{+1}[\omega'']}{\partial h_i} \right| \\
& \quad + \left| \sum_{i \in \Lambda \cap \Lambda'} (h_i[\omega] - h_i[\omega']) \left(\frac{\partial \ln Z_{\Lambda,\beta}^{+1}[\omega'']}{\partial h_i} - \frac{\partial \ln Z_{\Lambda',\beta}^{+1}[\omega'']}{\partial h_i} \right) \right| \\
& \leq \epsilon \beta \left| \sum_{i \in \Lambda \Delta \Lambda'} |h_i[\omega] - h_i[\omega']| \right| \\
& \quad + \epsilon \beta \left| \sum_{i \in \Lambda \cap \Lambda'} (h_i[\omega] - h_i[\omega']) \left(\mu_{\beta,\Lambda}^+[\omega''](\sigma_i) - \mu_{\beta,\Lambda'}^+[\omega''](\sigma_i) \right) \right| \\
& \leq \epsilon \beta \sqrt{|\Lambda \Delta \Lambda'|} \|h_{\Lambda \Delta \Lambda'}[\omega] - h_{\Lambda \Delta \Lambda'}[\omega']\|_2 \\
& \quad + \epsilon \beta \sqrt{\sum_{i \in \Lambda \cap \Lambda'} \left(\mu_{\beta,\Lambda}^+[\omega](\sigma_i) - \mu_{\beta,\Lambda'}^+[\omega'](\sigma_i) \right)^2} \|h_{\Lambda \cap \Lambda'}[\omega] - h_{\Lambda \cap \Lambda'}[\omega']\|_2
\end{aligned} \tag{2.15}$$

It is natural to believe that the expectation of σ_i with respect to the two measures, $\mu_{\beta,\Lambda}^+$ and $\mu_{\beta,\Lambda'}^+$, will be essentially the same for i well inside the intersection of Λ and Λ' , so that it should be possible to bound the coefficient in the last line by $\sqrt{|\Lambda \Delta \Lambda'|}$. If that were the case, the hypothesis of Proposition 2.1.4 would follow from Theorem 2.1.2. Unfortunately, we do not know how to prove such an estimate. The reader should realize that this argument appears nonetheless more convincing and robust than the one given in [17]; they argue that (2.14) will hold *if* the $\mu_{\beta,\Lambda}^+[\omega](\sigma_i)$ depend ‘weakly’ on ω which is essentially what we are out to prove anyway. On the other hand, smallness of (2.15) can even be expected to hold if the expectation of σ_i depends strongly on the disorder.

Proof. To simplify notation, let us set

$$F_\gamma \equiv \ln Z_{\text{int } \gamma}^{+1} - \mathbb{E} \ln Z_{\text{int } \gamma}^{+1} \tag{2.16}$$

The idea behind chaining arguments is to define a sequence of sets Γ_ℓ , $\ell \in \mathbb{N}$ of ‘coarse grained’ contours and maps $\gamma_\ell : \Gamma_0 \rightarrow \Gamma_\ell$, where Γ_0 is the original set of contours. Now write, for $k \in \mathbb{N}$ to be chosen later,

$$\begin{aligned}
F_\gamma &= F_{\gamma_k(\gamma)} + (F_{\gamma_{k-1}(\gamma)} - F_{\gamma_k(\gamma)}) \\
&\quad + (F_{\gamma_{k-2}(\gamma)} - F_{\gamma_{k-1}(\gamma)}) + \cdots + (F_\gamma - F_{\gamma_1(\gamma)})
\end{aligned} \tag{2.17}$$

Then we can write

$$\begin{aligned} \mathbb{P} \left[\sup_{\substack{\gamma: \text{int } \gamma \ni 0 \\ |\gamma|=n}} F_\gamma > z \right] &\leq \sum_{\ell=1}^{k(n)} \mathbb{P} \left[\sup_{\substack{\gamma: \text{int } \gamma \ni 0 \\ |\gamma|=n}} F_{\gamma_\ell(\gamma)} - F_{\gamma_{\ell-1}(\gamma)} > z_\ell \right] \\ &\quad + \mathbb{P} \left[\sup_{\substack{\gamma: \text{int } \gamma \ni 0 \\ |\gamma|=n}} F_{\gamma_k(\gamma)} > z_{k+1} \right] \end{aligned} \quad (2.18)$$

for any choice of $k = k(n)$ and sequences z_ℓ with $\sum_{\ell=1}^{k+1} z_\ell \leq z$. To estimate the individual probabilities, we just need to count the number, $A_{\ell,n}$, of image points in Γ_ℓ that are reached when mapping all the γ occurring in the sup (i.e. those of length n and encircling the origin), and use the assumption to get the obvious estimate

$$\begin{aligned} &\mathbb{P} \left[\sup_{\substack{\gamma: \text{int } \gamma \ni 0 \\ |\gamma|=n}} F_\gamma > z \right] \\ &\leq \sum_{\ell=1}^{k(n)} A_{\ell-1,n} A_{\ell,n} \exp \left(- \frac{z_\ell^2}{C\epsilon^2 \beta^2 \sup_\gamma |\text{int } \gamma_\ell(\gamma) \Delta \text{int } \gamma_{\ell-1}(\gamma)|} \right) \\ &\quad + A_{k(n),n} \exp \left(- \frac{z_{k+1}^2}{C\epsilon^2 \beta^2 \sup_\gamma |\text{int } \gamma_k(\gamma)|} \right) \end{aligned} \quad (2.19)$$

We must now make a choice for the sets Γ_ℓ . For this we cover the lattice \mathbb{Z}^d with squares of side-length 2^ℓ centred at the coarser lattice $(2^\ell \mathbb{Z})^d$. The set Γ_ℓ will then be nothing but collections of such squares. Next we need to define the maps γ_ℓ . This is done as follows: Let $V_\ell(\gamma)$ be the set of all cubes, c , of side-length 2^ℓ from the covering introduced above such that

$$|c \cap \text{int } \gamma| \geq 2^{d\ell-1} \quad (2.20)$$

Then let $\gamma_\ell(\gamma) \equiv \partial V_\ell(\gamma)$ be the set of boundary cubes of $V_\ell(\gamma)$. A simple example is presented in Fig. 2.1. Note that the images $\gamma_\ell(\gamma)$ are in general not connected, but one verifies that the number of connected components cannot exceed $\text{const.} |\gamma| 2^{-(d-1)(\ell-1)}$, and the maximal distance between any of the connected components is less than $|\gamma|$. This leads, after some tedious counting, to the estimate that

$$A_{\ell,n} \leq \exp \left(\frac{C\ell n}{2^{(d-1)\ell}} \right) \quad (2.21)$$

On the other hand, one readily sees that

$$|\text{int } \gamma_\ell(\gamma) \Delta \text{int } \gamma_{\ell-1}(\gamma)| \leq |\gamma| 2^\ell \quad (2.22)$$

for any γ . Finally, one chooses $k(n)$ such that $2^{k(n)} \sim n^{1/3}$ (i.e. $k(n) \sim \ln n$). Inserting these estimates into (2.19), one concludes that, for small enough ϵ , the sum in (2.20) is bounded by

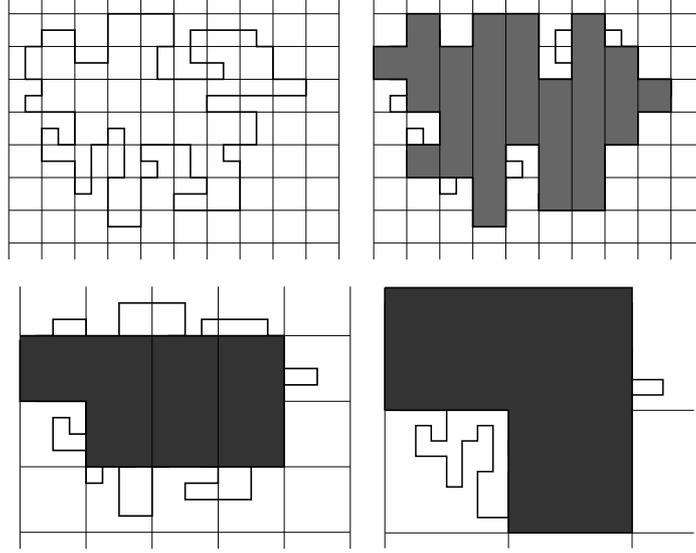


Fig. 2.1. Successive coarse graining of a contour.

$$\sum_{\ell=1}^{k(n)} \exp \left(-\frac{z_{\ell}^2}{C\beta^2\epsilon^2 2^{\ell} n} + \frac{C(\ell-1)n}{2^{(d-1)(\ell-1)}} \right) \quad (2.23)$$

and the last on the left of (2.20),

$$A_{k(n),n} \exp \left(-\frac{z_{k(n)+1}^2}{C\epsilon^2\beta^2 \sup_{\gamma} |\text{int } \gamma_k(\gamma)|} \right) \quad (2.24)$$

$$\leq \exp \left(C \ln n n^{1/3} - \frac{z_{k(n)+1}^2}{C\beta^2\epsilon^2 n^{d/(d-1)}} \right) \quad (2.25)$$

This allows us to choose $z_{\ell} = c\beta n \ell^{-2}$ to get a bound of order

$$\mathbb{P} \left[\sup_{\substack{\gamma: \text{int } \gamma \ni 0 \\ |\gamma| = n}} F_{\gamma} > c\beta n \right] \leq e^{-c'n^{2/3}\epsilon^{-2}} \quad (2.26)$$

and hence

$$\mathbb{P} \left[\sup_{\gamma: \text{int } \gamma \ni 0} F_{\gamma} > c\beta |\gamma| \right] \leq e^{-c'\epsilon^{-2}} \quad (2.27)$$

But from here follows the conclusion of Proposition 2.1.4. \square

The existence of two Gibbs states would now follow as in the Dilute Ising model, if we could verify the hypothesis of Proposition 1.4.2. The only

existing full proof of the existence of a phase transition in the RFIM is due to Bricmont and Kupiainen [12] and requires much harder work. We will return to this in Section 2.3.

2.2 Absence of phase transitions: The Aizenman–Wehr method

We have seen that in $d \geq 3$ the random energy that can be gained by flipping spins locally cannot compensate the surface energy produced in doing so if the disorder is weak and the temperature low. On the other hand, in dimension $d \leq 2$, the Imry–Ma argument predicts that the random bulk energies might outweigh surface terms, and this should imply that the particular realization of the random fields determine locally the orientation of the spins, so that the effects of boundary conditions are not felt in the interior of the system, implying a unique (random) Gibbs state. This argument was made rigorous by Aizenman and Wehr [2, 53] in 1990, using a number of clever and interesting ideas. Roughly, the proof is based on the following reasoning: Consider a volume Λ and two boundary conditions, say all plus spins and all minus spins. Then the difference between the corresponding free energies $f_{\beta, \Lambda^\pm} \equiv \ln Z_{\beta, \Lambda}^\pm$ must always be bounded by $\text{const.}|\partial\Lambda|$ (just introduce a contour right next to the boundary and proceed as in the Peierls argument). Now get a lower bound on the random fluctuations of that free energy; from the upper bound (2.7) one might guess that these can be of order $C(\beta)\sqrt{|\Lambda|}$, multiplied by a standard Gaussian random variable, say g . If this is so, there is a dilemma: by symmetry, the difference between the two free energies must be as big as the random part, and this implies that $C(\beta)\sqrt{|\Lambda|}g \leq \text{const.}|\partial\Lambda|$. In $d \leq 2$, this implies that $C(\beta) = 0$. But $C(\beta)$ will be seen to be linked to an *order parameter*, here the *magnetization*, and its vanishing will imply the uniqueness of Gibbs state. To make this rough argument precise requires, however, a delicate procedure. In what follows I will give the proof of Aizenman and Wehr only for the special case of the RFIM (actually, for any system where FKG inequalities hold).

2.2.1 Translation covariant states

A key technical idea in [2] is to carry out the argument sketched above in such a way that it gives directly information about infinite-volume states. This will allow the use of ergodicity arguments and this will force us to investigate some covariance properties of random Gibbs measures.

To do so, we equip our probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with some extra structure. First, we define the action, T , of the translation group \mathbb{Z}^d on Ω . We

will assume that \mathbb{P} is invariant under this action and that the dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, T)$ is stationary and ergodic. In the random-field model, the action of T is simply

$$(h_{x_1}[T_y\omega], \dots, h_{x_n}[T_y\omega]) \equiv (h_{x_1+y}[\omega], \dots, h_{x_n+y}[\omega]) \quad (2.28)$$

and the stationarity and ergodicity assumptions are trivially satisfied if the h_i are i.i.d.

Moreover, we will use that Ω is equipped with an affine structure, i.e. we set $(h_{x_1}[\omega + \omega'], \dots, h_{x_n}[\omega + \omega']) \equiv (h_{x_1}[\omega] + h_{x_1}[\omega'], \dots, h_{x_n}[\omega] + h_{x_n}[\omega'])$. We will introduce a subset, $\Omega_0 \subset \Omega$, of random fields that differ from zero only in some finite set, i.e.

$$\Omega_0 \equiv \{\delta\omega \in \Omega : \exists \Lambda \subset \mathbb{Z}^d, \text{finite}, \forall y \notin \Lambda, h_y[\delta\omega] = 0\} \quad (2.29)$$

We will use the convention to denote elements of Ω_0 by $\delta\omega$.

Definition 2.2.1 A random Gibbs measure μ_β is called covariant if,

- (i) for all $x \in \mathbb{Z}^d$, and any continuous function f ,

$$\mu_\beta[\omega](T_{-x}f) = \mu_\beta[T_x\omega](f), \text{ a.s.} \quad (2.30)$$

and,

- (ii) for all $\delta\omega \in \Omega_0$, for almost all ω and all bounded, continuous f ,

$$\mu_\beta[\omega + \delta\omega](f) = \frac{\mu_\beta[\omega](f e^{-\beta(H[\omega + \delta\omega] - H[\omega])})}{\mu_\beta[\omega](e^{-\beta(H[\omega + \delta\omega] - H[\omega])})} \quad (2.31)$$

Note that $H[\omega + \delta\omega] - H[\omega]$ is a finite sum: if $\delta\omega$ is supported on Λ , then $H[\omega + \delta\omega](\omega) - H\omega = -\sum_{i \in \Lambda} \sigma_i h_i[\delta\omega]$.

The properties of covariant random Gibbs measures look natural, but their verification is in general elusive (recall that even the construction of random Gibbs measures was problematic). In the context of the RFIM, we are helped, however, by the FKG inequalities that were already discussed in Section ??.

Theorem 2.2.1 Consider the random-field Ising model (2.2) with h_i a stationary and ergodic random field. Then there exist two covariant random Gibbs measures, μ_β^+ and μ_β^- , that satisfy

- (i) For almost all ω ,

$$\mu_\beta^\pm[\omega] = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_{\beta, \Lambda}^\pm[\omega] \quad (2.32)$$

- (ii) Suppose that for some β , $\mu_\beta^+ = \mu_\beta^-$. Then, for this value of β , the Gibbs measure for the RFIM model is unique for almost all ω .

Proof. In Section ??, Corollary ??, we learned that, due to the FKG inequalities (whose validity remains unaffected by the presence of the random fields), for any $\omega \in \Omega$, we can construct infinite volume Gibbs states, $\mu_\beta^\pm[\omega]$, as limits of local specifications with constant boundary conditions along arbitrary (ω -independent) increasing and absorbing sequences of finite volumes Λ_n . Thus, the functions $\omega \rightarrow \mu_\beta^\pm$ are measurable, since they are limits of measurable functions. It remains to check the covariance properties. Property (ii) follows immediately from the fact that μ_β^\pm can be represented as a limit of local specifications, and that the formula (2.31) holds trivially for local specifications with Λ large enough to contain the support of $\delta\omega$. Property (i) on the contrary requires the independence of the limit from the chosen sequence Λ_n . Indeed we have

$$\mu_{\beta,\Lambda}^+[\omega](T_{-x}f) = \mu_{\beta,\Lambda+x}^+[T_x\omega](f) \quad (2.33)$$

which implies, by Corollary ?? that $\mu_\beta^+[\omega](T_{-x}f) = \mu_\beta^+[T_x\omega](f)$ almost surely, as desired. The second assertion of the theorem follows directly from (iv) of Corollary ?. \square

Remark 2.2.1 It is remarkably hard to prove the translation covariance property in the absence of strong results like the FKG inequalities. In fact there are two difficulties. The first is that of the measurability of the limits that we have already discussed above. This can be resolved by the introduction of metastates, and it was precisely in this context that Aizenman and Wehr first applied this idea. The second problem is that without comparison results between local specifications in different volumes, the relative shift between the function and the volume implicit in (2.33) cannot be removed. A way out of this problem is to construct Gibbs states with *periodic* boundary conditions (i.e. one chooses instead of Λ a torus, i.e. $(\mathbb{Z} \bmod n)^d$). In that case, one may recover the translation covariance of the limit from translation covariance of the finite-volume measures under the automorphisms of the torus. From the point of view of the general theory as we have presented it so far, this is unsatisfactory. For this reason, we have restricted our exposition of the Aizenman–Wehr method to the RFIM and refer the reader to the original articles for more general results.

2.2.2 Order parameters and generating functions

We learned in Section ?? that, due to the monotonicity properties implied by the FKG inequalities, we will have a unique Gibbs state, for almost all ω , iff the translation covariant states μ_β^+ and μ_β^- coincide almost surely. Moreover, we know from Proposition ?? that in the translation invariant

case, uniqueness follows from the vanishing of the total magnetization. We want to extend this result to the random case. We set

$$m^\mu[\omega] \equiv \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \mu[\omega](\sigma_i) \quad (2.34)$$

provided the limit exists. We will also abuse notation and write $m_\beta^\pm = m^{\mu_\beta^\pm}$. Some simple facts follow from covariance and FKG:

Lemma 2.2.2 *Suppose that μ is a covariant Gibbs state. Then, for almost all ω , the total magnetization $m^\mu[\omega]$ exists and is independent of ω .*

Proof. By the covariance of μ ,

$$m^\mu[\omega] = \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \mu[T_{-i}\omega](\sigma_0) \quad (2.35)$$

But, $\mu_\beta[\omega](\sigma_0)$ is a bounded measurable function of ω ; since we assumed that $(\Omega, \mathcal{B}, \mathbb{P}, T)$ is stationary and ergodic, it follows from the ergodic theorem (see, e.g., Appendix A3 of [23] for a good exposition and proofs), that the limit exists, almost surely, and is given by

$$m^\mu = \mathbb{E} \mu(\sigma_0) \quad (2.36)$$

□

Lemma 2.2.3 *In the random-field Ising model,*

$$m^+ - m^- = 0 \Leftrightarrow \mu_\beta^+ = \mu_\beta^- \quad (2.37)$$

Proof. (2.36) implies that, almost surely,

$$0 = m^+ - m^- = \mathbb{E}(\mu_\beta^+(\sigma_i) - \mu_\beta^-(\sigma_i)) \quad (2.38)$$

and so, since $\mu_\beta^+(\sigma_i) - \mu_\beta^-(\sigma_i) \geq 0$, and there are only countably many sites i , almost surely, for all $i \in \mathbb{Z}^d$, $\mu_\beta^+(\sigma_i) - \mu_\beta^-(\sigma_i) = 0$. □

The order parameters introduced above can be computed as derivatives of certain *generating functions*. We set

$$G_\Lambda^\mu \equiv -\frac{1}{\beta} \ln \mu \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right) \quad (2.39)$$

Note that, if μ is a covariant Gibbs state, then

$$G_\Lambda^{\mu[\omega]} = \frac{1}{\beta} \ln \mu[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right) \quad (2.40)$$

Here, $\omega_\Lambda \in \Omega_0$ is defined such that $h_i[\omega_\Lambda] = h_i[\omega]$, if $i \in \Lambda$, and $h_i[\omega_\Lambda] = 0$ if $i \notin \Lambda$. Therefore, for $i \in \Lambda$,

$$\frac{\partial}{\partial h_i} G_\Lambda^{\mu[\omega]} = \frac{\mu[\omega - \omega_\Lambda](\sigma_i e^{\beta \sum_{i \in \Lambda} h_i \sigma_i})}{\mu[\omega - \omega_\Lambda](e^{\beta \sum_{i \in \Lambda} h_i \sigma_i})} = \mu[\omega](\sigma_i) \quad (2.41)$$

where the first equality follows from the fact that $\mu[\omega - \omega_\Lambda]$ is \mathcal{B}_Λ^c -measurable and the second one follows from (2.31). In particular, we get that

$$\mathbb{E} \frac{\partial}{\partial h_i} G_\Lambda^{\mu^\pm} = m^\pm \quad (2.42)$$

Let us now introduce the function

$$F_{\beta, \Lambda} \equiv \mathbb{E} \left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-} \mid \mathcal{B}_\Lambda \right] \quad (2.43)$$

Clearly, $\mathbb{E} \frac{\partial}{\partial h_0} F_\Lambda = m^+ - m^-$, and our goal is to prove that this quantity must be zero. The important point is the following a-priori upper bound:

Lemma 2.2.4 *For any value of β , and any volume, Λ ,*

$$|F_\Lambda| \leq 2|\partial\Lambda| \quad (2.44)$$

Proof. The first step in the proof makes use of (2.40) to express F_Λ in terms of measures that no longer depend on the disorder within Λ . Namely,

$$\begin{aligned} F_\Lambda &= \beta^{-1} \mathbb{E} \left[\ln \frac{\mu_\beta^-[\omega] \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)}{\mu_\beta^+[\omega] \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)} \mid \mathcal{B}_\Lambda \right] \\ &= \mathbb{E} \left[\ln \frac{\mu_\beta^+[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)}{\mu_\beta^-[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)} \mid \mathcal{B}_\Lambda \right] \end{aligned} \quad (2.45)$$

Next, we use the spin-flip symmetry, which implies that $\mu_\beta^+[\omega](f(\sigma)) = \mu_\beta^-[-\omega](f(-\sigma))$, and the symmetry of the distribution of the h_i to show that

$$\begin{aligned} &\mathbb{E} \left[\ln \frac{\mu_\beta^+[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)}{\mu_\beta^-[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)} \mid \mathcal{B}_\Lambda \right] \\ &= \mathbb{E} \left[\ln \frac{\mu_\beta^-[-(\omega - \omega_\Lambda)] \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)}{\mu_\beta^-[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)} \mid \mathcal{B}_\Lambda \right] \\ &= \mathbb{E} \left[\ln \frac{\mu_\beta^-[\omega - \omega_\Lambda] \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)}{\mu_\beta^-[\omega - \omega_\Lambda] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right)} \mid \mathcal{B}_\Lambda \right] \end{aligned} \quad (2.46)$$

We are left with the ratio of two expectations with respect to the same measure. Here we use the DLR equations to compare them:

$$\begin{aligned}
& \mu_{\beta}^{-}[\omega - \omega_{\Lambda}] \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right) \\
&= \mathbb{E}_{\sigma_{\Lambda^c}} \mu_{\beta}^{-}[\omega - \omega_{\Lambda}] (\sigma_{\Lambda^c}) \mu_{\beta, \Lambda}^{\sigma_{\Lambda^c}} \left(e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i} \right) \\
&= \mathbb{E}_{\sigma_{\Lambda^c}} \frac{\mu_{\beta}^{-}[\omega - \omega_{\Lambda}] (\sigma_{\Lambda^c})}{Z_{\beta, \Lambda}^{\sigma_{\Lambda^c}}} \mathbb{E}_{\sigma_{\Lambda}} e^{\beta \left(\sum_{\substack{i, j \in \Lambda \\ |i-j|=1}} \sigma_i \sigma_j + \sum_{\substack{i \in \Lambda, j \in \Lambda^c \\ |i-j|=1}} \sigma_i \sigma_j - \sum_{i \in \Lambda} h_i \sigma_i \right)} \\
&= \mathbb{E}_{\sigma_{\Lambda^c}} \frac{\mu_{\beta}^{-}[\omega - \omega_{\Lambda}] (\sigma_{\Lambda^c})}{Z_{\beta, \Lambda}^{\sigma_{\Lambda^c}}} \mathbb{E}_{\sigma_{\Lambda}} e^{\beta \left(\sum_{\substack{i, j \in \Lambda \\ |i-j|=1}} \sigma_i \sigma_j - \sum_{\substack{i \in \Lambda, j \in \Lambda^c \\ |i-j|=1}} \sigma_i \sigma_j + \sum_{i \in \Lambda} h_i \sigma_i \right)} \\
&\leq e^{2\beta |\partial \Lambda|} \mathbb{E}_{\sigma} \frac{\mu_{\beta}^{-}[\omega - \omega_{\Lambda}] (\sigma_{\Lambda^c})}{Z_{\beta, \Lambda}^{\sigma_{\Lambda^c}}} e^{\beta \left(\sum_{\substack{i, j \in \Lambda \\ |i-j|=1}} \sigma_i \sigma_j + \sum_{\substack{i \in \Lambda, j \in \Lambda^c \\ |i-j|=1}} \sigma_i \sigma_j + \sum_{i \in \Lambda} h_i \sigma_i \right)} \\
&= \mu_{\beta}^{-}[\omega - \omega_{\Lambda}] \left(e^{\beta \sum_{i \in \Lambda} h_i \sigma_i} \right) e^{2\beta |\partial \Lambda|}
\end{aligned} \tag{2.47}$$

Inserting this bound into (2.46) gives the desired estimate. \square

Next, we prove a lower bound on the fluctuations of F_{Λ} , or, more precisely, on its Laplace transform. Namely:

Lemma 2.2.5 *Assume that for some $\epsilon > 0$, the distribution of the random fields h satisfies $\mathbb{E}|h|^{2+\epsilon} < \infty$. Then, for any $t \in \mathbb{R}$, we have that*

$$\liminf_{\Lambda = [-L, L]^d; L \uparrow \infty} \mathbb{E} \exp \left(t F_{\Lambda} / \sqrt{|\Lambda|} \right) \geq \exp \left(\frac{t^2 b^2}{2} \right) \tag{2.48}$$

where

$$b^2 \geq \mathbb{E} \left[\mathbb{E} [F_{\Lambda} | \mathcal{B}_0]^2 \right] \tag{2.49}$$

Remark 2.2.2 It is easy to see that Lemmata 2.2.4 and 2.2.5 contradict each other in $d \leq 2$, unless $b = 0$. On the other hand, we will see that $b = 0$ implies $m^+ = m^-$, and thus the uniqueness of the Gibbs state.

Proof. The proof of this lemma uses a decomposition of F_{Λ} as a martingale-difference sequence. That is, we order all the points $i \in \Lambda$ and denote by $\mathcal{B}_{\Lambda, i}$ the sigma-algebra generated by the variables $\{h_j\}_{j \in \Lambda: j \leq i}$. Then we have trivially that

$$F_{\Lambda} = \sum_{i=1}^{|\Lambda|} (\mathbb{E} [F_{\Lambda} | \mathcal{B}_{\Lambda, i}] - \mathbb{E} [F_{\Lambda} | \mathcal{B}_{\Lambda, i-1}]) \equiv \sum_{i=1}^{|\Lambda|} Y_i \tag{2.50}$$

(note that $\mathbb{E} F_{\Lambda} = 0!$). Using this, we can represent the generating function as

$$\begin{aligned} \mathbb{E}e^{tF_\Lambda} &= \mathbb{E}\left[\mathbb{E}\left[\dots\mathbb{E}\left[\mathbb{E}\left[e^{tY_{|\Lambda|}}|\mathcal{B}_{\Lambda,|\Lambda|-1}\right]e^{tY_{|\Lambda|-1}}|\mathcal{B}_{\Lambda,|\Lambda|-2}\right]\dots\right.\right. \\ &\quad \left.\left.\dots e^{tY_2}|\mathcal{B}_{\Lambda,1}\right]e^{tY_1}\right] \end{aligned} \quad (2.51)$$

We want to work up the conditional expectation from the inside out. To do so, we need a lower bound for any of the terms $\mathbb{E}\left[e^{tY_i}|\mathcal{B}_{\Lambda,i-1}\right]$. To get such a bound, we use the observation (see [2], Lemma A.2.2) that there exists a continuous function, $g(a)$, with $g(a) \downarrow 0$ as $a \downarrow 0$, such that, for all real x and all $a \geq 0$, $e^x \geq 1 + x + \frac{1}{2}(1 - g(a))x^2 \mathbb{1}_{|x| \leq a}$. Since, moreover, for all $|x| \leq a$, $e^{x^2 e^{-a^2/2}/2} \leq 1 + x^2/2$, it follows that, if $\mathbb{E}X = 0$, then, for $f(a) = 1 - (1 - g(a))e^{-a^2/2}$,

$$\mathbb{E}e^X \geq e^{\frac{1}{2}(1-f(a))\mathbb{E}[X^2 \mathbb{1}_{|X| \leq a}]} \quad (2.52)$$

Using this estimate, we see that

$$\mathbb{E}\left[e^{tY_i}|\mathcal{B}_{\Lambda,i-1}\right] \exp\left(-\frac{t^2}{2}(1-f(a))\mathbb{E}\left[Y_i^2 \mathbb{1}_{|Y_i| \leq a}|\mathcal{B}_{\Lambda,i-1}\right]\right) \geq 1 \quad (2.53)$$

Since this quantity is $\mathcal{B}_{\Lambda,i}$ -measurable, we can proceed as in (2.51) to see that (we switch to the desired normalization)

$$1 \leq \mathbb{E}e^{tF_\Lambda/\sqrt{|\Lambda|} - \frac{t^2}{2|\Lambda|}(1-f(a))\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2 \mathbb{1}_{|Y_i| \leq a\sqrt{|\Lambda|}}|\mathcal{B}_{\Lambda,i-1}\right]} \quad (2.54)$$

We will show in a moment that the term

$$V_\Lambda(a) \equiv |\Lambda|^{-1} \sum_{i=1}^{|\Lambda|} \mathbb{E}\left[Y_i^2 \mathbb{1}_{|Y_i| \leq a\sqrt{|\Lambda|}}|\mathcal{B}_{\Lambda,i-1}\right] \quad (2.55)$$

appearing in the exponent in the last factor, converges in probability to a constant, C , independent of $a > 0$. Since, by Lemma 2.2.4, in $d = 2$, $F_\Lambda/\sqrt{|\Lambda|} \leq C'$, this implies easily that

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{E}e^{tF_\Lambda/\sqrt{|\Lambda|}} \geq e^{t^2 C/2} \quad (2.56)$$

We are left with controlling and identifying the limit of $V_\Lambda(a)$. This will be done by a clever use of the ergodic theorem.

To do so, we introduce new sigma-algebras \mathcal{B}_i^{\leq} , generated by the random variables h_j with $j \leq i$, where \leq refers to the lexicographic ordering. Define

$$W_i \equiv \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-}|\mathcal{B}_i^{\leq}\right] - \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-}|\mathcal{B}_{i-1}^{\leq}\right] \quad (2.57)$$

Using (2.41) one may show that, for all i in Λ , W_i is independent of Λ (the proof uses (2.41) to represent G_Λ^μ in terms of integrals over $\mu(\sigma_i)$, which is independent of Λ). On the other hand, we have the obvious relation that

$$Y_i = \mathbb{E}[W_i|\mathcal{B}_\Lambda] \quad (2.58)$$

We use this first to show that the indicator function in the conditional expectation can be removed, i.e. for all $\epsilon > 0$,

$$\lim_{\Lambda \uparrow \infty} \left[|\Lambda|^{-1} \sum_{i=1}^{|\Lambda|} \mathbb{E} \left[Y_i^2 \mathbb{1}_{|Y_i| > a\sqrt{|\Lambda|}} | \mathcal{B}_{\Lambda, i-1} \right] > \epsilon \right] = 0 \quad (2.59)$$

To see this, compute the expectation of the left-hand side in the probability, and use the Hölder inequality to get

$$\begin{aligned} \mathbb{E} |\Lambda|^{-1} \sum_{i=1}^{|\Lambda|} \mathbb{E} \left[Y_i^2 \mathbb{1}_{|Y_i| > a\sqrt{|\Lambda|}} | \mathcal{B}_{\Lambda, i-1} \right] &= |\Lambda|^{-1} \sum_{i=1}^{|\Lambda|} \mathbb{E} \left[Y_i^2 \mathbb{1}_{|Y_i| > a\sqrt{|\Lambda|}} \right] \\ &\leq |\Lambda|^{-1} \sum_{i=1}^{|\Lambda|} \left(\mathbb{E} Y_i^{2q} \right)^{1/q} \left(\mathbb{P} \left[|Y_i| > a\sqrt{|\Lambda|/t} \right] \right)^{1/p} \end{aligned} \quad (2.60)$$

with $1/p + 1/q = 1$. Now, using Jensen's inequality and (2.58), we see that for any $q > 1$ $\mathbb{E} Y_i^{2q} \leq \mathbb{E} W_0^{2q}$. However, using e.g. (2.41), it is easy to see that $|W_0| \leq C|h_0|$, so that, if the $2q$ -th moment of h is finite, then $\mathbb{E} W_0^{2q} < \infty$. Using the Chebyshev inequality and the same argument as before, we also conclude that $\mathbb{P} \left[|Y_i| > a\sqrt{|\Lambda|/t} \right] \leq \frac{t^2 \mathbb{E} W_0^2}{a^2 |\Lambda|}$ which tends to zero as $\Lambda \uparrow \infty$. We see that (2.60) tends to zero whenever $p < \infty$, for any $a > 0$. By Chebyshev's inequality, this in turn allows us to conclude (2.59)¹.

Next observe that W_i is shift covariant, i.e.

$$W_i[\omega] = W_0[T_{-i}\omega] \quad (2.61)$$

Therefore, by the ergodic theorem, we can conclude that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{i \in \Lambda} \mathbb{E} \left[W_i^2 | \mathcal{B}_{i-1}^{\leq} \right] = \mathbb{E} W_0^2, \text{ in Prob.} \quad (2.62)$$

Now we will be done if we can show that

$$E \left[Y_i^2 | \mathcal{B}_{\Lambda, i} \right] - E \left[W_i^2 | \mathcal{B}_i^{\leq} \right] \quad (2.63)$$

goes to zero as Λ goes to infinity, in probability. This follows by estimating the expectation of the square of (2.63), some simple estimates using the Cauchy-Schwarz inequality and the fact that for any square integrable function f , $\mathbb{E}[(f - \mathbb{E}[f | \mathcal{B}_\Lambda])^2]$ tends to zero as Λ approaches \mathbb{Z}^d .

To arrive at the final conclusion, note that

$$\mathbb{E} W_0^2 \geq \mathbb{E}[(\mathbb{E}[W_0 | \mathcal{B}_0])^2] \quad (2.64)$$

(where \mathcal{B}_0 is the sigma-algebra generated by the single variable h_0), and $\mathbb{E}[W_0 | \mathcal{B}_0] = \mathbb{E}[F_\Lambda | \mathcal{B}_0]$. \square

¹ In [2], only two moments are required for h . However, the proof given there is in error as it pretends that the function $x^2 \mathbb{1}_{|x| > a}$ is convex which is manifestly not the case.

Remark 2.2.3 The assertion of the preceding Lemma amounts to a central limit theorem; the basic idea of the proof, namely the use of a martingale difference sequence obtained by successive conditioning is also the basis of a useful proof of concentration of measure estimates due to Yurinskii [54].

Finally we observe that by (2.41),

$$\frac{\partial}{\partial h_0} \mathbb{E}[F_\Lambda | \mathcal{B}_0] = \mathbb{E} \left[(\mu_\beta^+(\sigma_0) - \mu_\beta^-(\sigma_0)) | \mathcal{B}_0 \right] \quad (2.65)$$

Let us denote by $f(h) = \mathbb{E}[F_\Lambda | \mathcal{B}_0]$ (where $h = h_0[\omega]$). Since $1 \geq f'(h) \geq 0$ for all h , $\mathbb{E}f^2 = 0$ implies that $f(h) = 0$ (for \mathbb{P} -almost all points) on the support of the distribution of h . But then $f'(h)$ must also vanish on the convex hull of the support of the distribution of h (except if the distribution is concentrated on a single point). Therefore, barring that case $\mathbb{E}[F_\Lambda | \mathcal{B}_0] = 0 \Rightarrow m^+ - m^- = 0$.

Collecting our results we can now prove the following

Theorem 2.2.6 [2] *In the random-field Ising model with i.i.d. random fields whose distribution is not concentrated on a single point and possesses at least $2 + \epsilon$ finite moments, for some $\epsilon > 0$, if $d \leq 2$, there exists a unique infinite-volume Gibbs state.*

Proof. Lemma 2.2.4 implies that for any Λ , $Ee^{tF_{\beta,\Lambda}} \leq e^{t|\partial\Lambda|}$. Combining this with Lemma 2.2.5, we deduce that, if $d \leq 2$, then necessarily $b = 0$. But from what was just shown, this implies $m^+ = m^-$, which in turn implies uniqueness of the Gibbs state. \square

With this result we conclude our discussion of the random-field Ising model in $d = 2$. We may stress that Theorem 2.2.6 is in some sense a soft result that gives uniqueness without saying anything more precise about the properties of the Gibbs state. Clearly, there are many interesting questions that could still be asked. What does the Gibbs state at high temperatures distinguish itself from the one at low temperatures, or how does the low temperature Gibbs state look like in dependence on the strength of the random fields? It is clear that for very low temperatures and very large ϵ , the Gibbs state will be concentrated near configurations $\sigma_i = \text{sign } h_i$. For small ϵ , on the contrary, a more complicated behaviour is expected. Results of this type are available in $d = 1$ [7], but much less is known in $d = 2$ [53].

2.3 The Bricmont-Kupiainen renormalization group

In 1988 a remarkable article by Bricmont and Kupiainen [12] settled the long-standing dispute on the lower critical dimension of the random-field

Ising model through a rigorous mathematical proof of the existence of at least two phases at low temperatures in dimension three and above. Their proof was based on a renormalization group (RG) analysis. In this section we will give an outline of their proof, following mostly the version given in [10], developed for the related problem of interfaces in random environments. The details of the proof are cumbersome, and we will focus here on the structural aspects, omitting the proofs of many of the lemmata that are mainly of combinatorial nature. All the omitted proofs can be found in [10]. A simpler problem where all cluster expansions can be avoided are the hierarchical models, see [11, 8, 9].

The central result of [12] is the following:

Theorem 2.3.1 *Let $d \geq 3$ and assume that the random variables h_x are i.i.d., symmetrically distributed and satisfy $\mathbb{P}[|h_x| > h] \leq \exp(-h^2/\Sigma^2)$ for Σ sufficiently small. Then there exists $\beta_0 < \infty$, $\Sigma_0 > 0$, such that for all $\beta \geq \beta_0$ and $\Sigma \leq \Sigma_0$ for any increasing and absorbing sequence of volumes $\Lambda_n \uparrow \mathbb{Z}^d$, the sequence of measures $\mu_{\Lambda_n, \beta}^\pm$ converges to disjoint Gibbs measures μ_β^\pm , \mathbb{P} -almost surely.*

Before entering the details of the proof of this theorem, we explain some of the main ideas and features of the RG approach. The principal idea is to use a low-temperature contour expansion as explained in Chapter ???. As opposed to many deterministic systems, the first (and in some sense *main*) difficulty in most disordered systems is that the ground-state configuration depends in general on the particular realization of the disorder, and, worse, may in principle depend strongly on the shape and size of the finite volume Λ ! In dimension $d \geq 3$, we expect, from the arguments given before, that there exist translation covariant ground states that look more or less like the plus or the minus configuration, with a few small deviations.

The crucial observation that forms the ideological basis for the renormalization group approach is that while for large volumes Λ we have no a priori control on the ground-states, for sufficiently small volumes we can give conditions on the random variables h that are fulfilled with large probability under which the ground-state in this volume is actually the same as the one without randomness. Moreover, the size of the regions for which this holds true will depend on the variance of the r.v.'s and increases to infinity as the latter decreases. This allows us to find 'conditioned' ground-states, where the conditioning is on some property of the configuration on this scale, except in some small region of space. Re-summing then over the fluctuations about these conditioned ground-states one obtains a new effective model for the conditions (the coarse grained variables) with effective random variables

that have smaller variance than the previous ones. In this case, this procedure may be iterated, as now conditioned ground states on a larger scale can be found.

To implement these ideas one has to overcome two major difficulties. The first one is to find a formulation of the model whose form remains invariant under the renormalization group transformation. The second and more serious one, is that the re-summation procedure as indicated above can only be performed outside a small random region of space, called the *bad region*. While in the first step this may not look too problematic, in the process of iteration even a very thin region will potentially ‘infect’ a larger and larger portion of space. This requires us to get some control also in the bad regions and to develop a precise notion of how regions with a certain degree of badness can be reintegrated as ‘harmless’ on the next scale. For the method to succeed the bad regions must ‘die out’ over the scales much faster than new ones are produced.

2.3.1 Renormalization group and contour models

This subsection is intended to serve two purposes. First, we want to describe the principal ideas behind the renormalization group approach for disordered systems in the low-temperature regime. Second, we want to present the particular types of contour models on which the renormalization group will act and to introduce the notation for the latter.

The renormalization group for measure spaces. Let us explain what is generally understood by a renormalization group transformation in a statistical mechanics system. We consider a probability space $(\mathcal{S}, \mathcal{F}, \mu)$, where μ is an (infinite volume) Gibbs measure. One may think for the moment of \mathcal{S} as the ‘spin’-state over the lattice \mathbb{Z}^d , but we shall need more general spaces later. What we shall, however, assume is that \mathcal{S} is associated with the lattice \mathbb{Z}^d in such a way that for any finite subset $\Lambda \subset \mathbb{Z}^d$ there exists a subset $\mathcal{S}_\Lambda \subset \mathcal{S}$ and sub-sigma algebras, \mathcal{F}_Λ , relative to \mathcal{S}_Λ that satisfy $\mathcal{F}_\Lambda \subset \mathcal{F}_{\Lambda'}$ if and only if $\Lambda \subset \Lambda'$. Note that in this case, any increasing and absorbing sequence of finite volumes, $\{\Lambda_n\}_{n \in \mathbb{Z}_+}$, induces a filtration $\{\mathcal{F}_n \equiv \mathcal{F}_{\Lambda_n}\}_{n \in \mathbb{Z}_+}$ of \mathcal{F} .

Ideally, a *renormalization group transformation* is a measurable map, \mathbb{T} , that maps $\mathbb{Z}^d \rightarrow \mathbb{Z}^d$ and $(\mathcal{S}, \mathcal{F}) \rightarrow (\mathcal{S}, \mathcal{F})$ in such a way that for any $\Lambda \subset \mathbb{Z}^d$,

- (i) $\mathbb{T}(\Lambda) \subset \Lambda$, and moreover $\exists_{n < \infty} : \mathbb{T}^n(\Lambda) = \{0\}$, where n may depend on Λ .

$$(ii) \mathbb{T}(\mathcal{S}_\Lambda) = \mathcal{S}_{\mathbb{T}(\Lambda)}$$

The action of \mathbb{T} on space will generally be blocking, e.g. $\mathbb{T}(x) = \mathcal{L}^{-1}x \equiv \text{int}(x/L)$. The action on \mathcal{S} has to be compatible with this blocking but needs to be defined carefully.

Having the action of \mathbb{T} on the measure space $(\mathcal{S}, \mathcal{F})$ we get a canonical action on measures via

$$(\mathbb{T}\mu)(\mathcal{A}') = \mu(\mathbb{T}^{-1}(\mathcal{A}')) \quad (2.66)$$

for any Borel-set $\mathcal{A} \in \mathbb{T}(\mathcal{F})$. The fundamental relation of the renormalization group allows to decompose the measure μ into a conditional expectation and the renormalized measure on the condition, i.e. for any Borel-set $\mathcal{A} \in \mathcal{F}$ we have

$$\mu(\mathcal{A}) = \int_{\mathbb{T}(\mathcal{S})} \mu(\mathcal{A}|\mathbb{T} = \omega') (\mathbb{T}\mu)(d\omega') \quad (2.67)$$

A renormalization group transformation is useful if this decomposition has the virtue that the measure $\mathbb{T}\mu$ is ‘simpler’ than the measure μ itself, and if the conditioned expectations are more easy to control at least on a subspace of large measure w.r.t. $\mathbb{T}\mu$.

So far, we have not made reference to the specific situation in random systems. In such a situation the specific choice of the renormalization group transformation has to be adapted to the particular realization of the disorder, i.e. will itself have to be a – complicated – random function. In particular, the renormalization group transformation cannot be simply iterated, since after each step the properties of the new measure have to be taken into account when specifying the new map. We will even allow the underlying spaces \mathcal{S} to be random and to change under the application of the renormalization group map.

A final aspect that should be kept in mind is that the renormalized measures (or even their local specifications) can only *in principle* be computed exactly, while in practice our knowledge is limited to certain bounds.

Contour models. The concept of ‘*contours*’ has already been introduced in the context of low-temperature expansions in Chapter 5. The idea is that the support of a contour indicates where a configuration deviates from a ground-state configuration. In our situation, the true ground states are not known, but we will proceed *as if* the constant configurations were ground-states. The trick introduced by Bricmont and Kupiainen is to correct for this sloppiness by incorporating into the support also those parts of space where the disorder is so big that this assumption is questionable, the so-called *bad*

regions. Section 2.3.2 will pinpoint this idea by dealing exclusively with the ground-state problem.

Definition 2.3.1 A contour, Γ , is a pair $(\underline{\Gamma}, \sigma)$, where $\underline{\Gamma}$ is a subset of \mathbb{Z}^d , called the support of Γ , and $\sigma \equiv \sigma(\Gamma) : \mathbb{Z}^d \rightarrow \{-1, 1\}$ is a map that is constant on connected components of $\underline{\Gamma}^c$.

In the sequel, \mathcal{S} shall denote the space of all contours. Also, \mathcal{S}_Λ will denote the space of contours in the finite volume Λ .

We will also need spaces of contours satisfying some constraints. To explain this, we must introduce some notation. Let D be a subset of \mathbb{Z}^d . We denote by $\mathcal{S}(D)$ all contours whose support contains D , i.e.

$$\mathcal{S}(D) \equiv \{\Gamma \in \mathcal{S} \mid D \subset \underline{\Gamma}\} \quad (2.68)$$

As we have indicated above, a renormalization group transformation may depend on the realization of the disorder, and in particular on a bad region D . The bad regions will be affected by the renormalization, so that we will have to construct maps, \mathbb{T}_D , that map the spaces $\mathcal{S}(D)$ into $\mathcal{S}(D')$ for suitably computed D' . The resulting structure will then be a measurable map, $\mathbb{T}_D : (\mathcal{S}(D), \mathcal{F}(D)) \rightarrow (\mathcal{S}(D'), \mathcal{F}(D'))$, that can be lifted to the measure μ s.t., for any $\mathcal{A} \in \mathcal{F}(D')$,

$$(\mathbb{T}_D \mu)(\mathcal{A}) = \mu(\mathbb{T}_D^{-1}(\mathcal{A})) \quad (2.69)$$

We want to iterate this procedure. As a first step, let us rewrite the original RFIM as a contour model.

The RFIM as a contour model. We need to introduce some more notation. We always use the metric $d(x, y) = \max_{i=1}^d |x_i - y_i|$ for points in \mathbb{Z}^d . We call a set, $A \subset \mathbb{Z}^d$, connected, iff, for all $x \in A$, $d(x, A \setminus \{x\}) \leq 1$. A maximal connected subset of a set A will be called a connected component of A . We write \overline{A} for the set of points whose distance from A is not bigger than 1, and we write $\partial A \equiv \overline{A} \setminus A$ and call ∂A the boundary of A . A further important notion is the *interior* of a set A , $\text{int}A$. It is defined as follows: For any set $A \subset \mathbb{Z}^d$, let $\hat{A} \subset \mathbb{R}^d$ denote the set in \mathbb{R}^d obtained by embedding A into \mathbb{R}^d and surrounding each of its points by the unit cube in d dimensions. Then the complement of \hat{A} may have finite connected components. Their union with \hat{A} is called $\text{int}\hat{A}$, and the intersection of this set with \mathbb{Z}^d is defined to be $\text{int}A$.

Important operations will consist of the cutting and gluing of contours. First, for any contour Γ we may decompose its support, $\underline{\Gamma}$, into connected components, $\underline{\gamma}_i$ in the sense described above. Note that a contour is uniquely described by specifying its support $\underline{\Gamma}$, the values of σ on the support and the

values σ takes on each of the connected components of the boundary of the support. This makes it possible to associate with each connected component $\underline{\gamma}_i$ of the support a contour, γ_i , by furnishing the additional information of the signs on $\underline{\gamma}_i$ and on the connected components of $\partial\underline{\gamma}_i$. We will call a contour with connected support a *connected contour*. In the same spirit we call a connected contour γ_i obtained from a connected component of the support of a contour Γ a *connected component* of Γ . A collection, $\{\gamma_1, \dots, \gamma_n\}$, of connected contours is called *compatible* if there exists a contour, Γ , such that $\gamma_1, \dots, \gamma_n$ are the connected components of Γ . This contour will also be called $(\gamma_1, \dots, \gamma_n)$.

We will also use a notion of *weak connectedness*: a set $\mathcal{A} \subset \mathbb{Z}^d$ is *weakly connected* if $\text{int } A$ is connected. All the notions of the previous paragraph then find their weak analogues.

Defining

$$E_s(\Gamma) = \frac{1}{2} \sum_{\substack{x, y \in \underline{\Gamma} \\ |x-y|=1}} (\sigma_x(\Gamma) - \sigma_y(\Gamma))^2 \quad (2.70)$$

we could write

$$H(\sigma) = E_s(\Gamma) + (h, \sigma(\Gamma)) \quad (2.71)$$

with $\underline{\Gamma}$ defined for a given function σ as the set of x that possess a nearest neighbour, y , for which $\sigma_y \neq \sigma_x$. Then the term $E_s(\Gamma)$ could be written as a sum over connected components, $E_s(\Gamma) = \sum_i E_s(\gamma_i)$. This would be reasonable if the constant configurations were indeed ground-states. But the field terms may deform the ground-states. What we want to do is to indicate where such deformations may have occurred in space. To implement this, we allow only h_x that are small enough to remain in the field term. For a fixed $\delta > 0$, to be chosen later, we set

$$S_x \equiv h_x \mathbb{I}_{|h_x| < \delta} \quad (2.72)$$

For fields that are not small, we introduce a *control field* that keeps rough track of their size,

$$N_x \equiv \delta^{-1} \mathbb{I}_{|h_x| \geq \delta} |h_x| \quad (2.73)$$

The prefactor δ^{-1} is such that non-zero control fields have minimal size one. The region D , the *bad region*, is then defined as

$$D \equiv D(N) \equiv \{x \in \mathbb{Z}^d | N_x > 0\} \quad (2.74)$$

The bad region will always be considered part of the contour of a configuration, irrespective of the signs. We define the mass of a contour Γ as

$$\mu(\Gamma) \equiv \rho(\Gamma) e^{-\beta(S, \sigma(\Gamma))} \mathbb{I}_{\underline{\Gamma} = \{x | \exists y: |x-y|=1: \sigma_y(\Gamma) \neq \sigma_x(\Gamma)\} \cup D(\Gamma)} \quad (2.75)$$

where

$$\rho(\Gamma) = e^{-\beta(E_s(\Gamma) + (h, \sigma(\Gamma))_{D(\Gamma) \cap \mathbb{L}})} \quad (2.76)$$

The important fact is that $\rho(\Gamma)$ factors over the connected components of Γ , i.e. if $\Gamma = (\gamma_1, \dots, \gamma_n)$, then

$$\rho(\Gamma) = \prod_{i=1}^n \rho(\gamma_i) \quad (2.77)$$

Note that (2.75) implies a one-to-one relation between spin-configurations and contours with non-zero weight.

We would wish that the form of the measures on the contours would remain in this form under renormalization, i.e. activities factorizing over connected components plus a small-field contribution. Unfortunately, except in the case of zero temperature, things will get a bit more complicated. In general, the renormalization will introduce *non-local interactions* between connected components of supports as well as a (small) *non-local random field* $\{S_C\}$ indexed by the connected subsets C of \mathbb{Z}^d . We will also introduce the notations

$$V_{\pm}(\Gamma) \equiv \{x \in \mathbb{Z}^d \mid \sigma_x(\Gamma) = \pm\} \quad (2.78)$$

and

$$(S, V(\Gamma)) \equiv \sum_{C \subset V_+(\Gamma)} S_C^+ + \sum_{C \subset V_-(\Gamma)} S_C^- \quad (2.79)$$

where sums over C here and henceforth are over connected sets, and the superscript \pm on S refers to whether C is contained in the plus or the minus phase. If C is a single site, $C = x$, we set $S_{\pm x} = \pm S_x$. The final structure of the contour measures will be the following:

$$\mu(\Gamma) = \frac{1}{Z_{\beta, \Lambda}} e^{-\beta(S, V(\Gamma))} \sum_{\mathbb{Z}^d \supset G \supset \mathbb{L}} \rho(\Gamma, G) \quad (2.80)$$

where the activities, $\rho(\Gamma, G)$, factor over connected components of G .

The functions S , the activities ρ and the fields N will be the parameters on which the action of the renormalization group will finally be controlled.

Renormalization of contours. We will now define the action of the renormalization group on contours. This cannot yet be done completely, since, as indicated above, the renormalization group map will depend on the bad regions, basically through the fields N_x . These details will be filled in later.

The renormalization group transformation consists of three steps:

- (i) Summation of small connected components of contours
- (ii) Blocking of the remaining large contours

(iii) Dressing of the supports by the new bad region

Note that step (iii) is to some extent cosmetic and requires already the knowledge of the renormalized bad regions. We note that this causes no problem, as the bad regions may already be computed after step (i).

Let us now give a brief description of the individual steps.

STEP 1: We would like to sum in this step over all those classes of contours for which we can get a convergent expansion in spite of the random fields. In practice, we restrict ourselves to a much smaller, but sufficiently large, class of contours. We define a connected component as ‘small’ if it is geometrically small (in the sense that $d(\underline{\gamma}_i) < L$), and if its support does not intersect the bad region, with the exception of a suitably defined ‘harmless’ subset of the bad region. This latter point is important since it will allow us to forget about this harmless part in the next stage of the iteration and this will assure that the successive bad regions become sparser and sparser. Precise definitions are given in Section 2.3.2.

A contour that contains no small connected component is called large, and we denote by $\mathcal{S}^l(D)$ the subspace of large contours. The first step of RG transformation is the canonical projection from $\mathcal{S}(D)$ to $\mathcal{S}^l(D)$, i.e. to any contour in \mathcal{S} we associated the large contour composed of only the large components of Γ .

STEP 2: In this step the large contours are mapped to a coarse-grained lattice. We choose the simplest action of \mathbb{T} on \mathbb{Z}^d , namely $(\mathbb{T}x)_i = \mathcal{L}^{-1} \equiv \text{int}(x_i/L)$. We will denote by $\mathcal{L}x$ the set of all points, y , s.t. $\mathcal{L}^{-1}y = x$. The action of \mathcal{L}^{-1} on spin configurations is defined as averaging, i.e.

$$(\mathcal{L}^{-1}\sigma)_y = \text{sign} \sum_{x \in \mathcal{L}y} \sigma_x \quad (2.81)$$

With this definition the action of \mathcal{L}^{-1} on large contours is

$$\mathcal{L}^{-1}\Gamma \equiv (\mathcal{L}^{-1}\underline{\Gamma}, \mathcal{L}^{-1}\sigma) \quad (2.82)$$

STEP 3: The action of \mathbb{T} given by (2.19) does not yet give a contour in $\mathcal{S}(D')$. Thus, the last step in the RG transformation consists of enlarging the supports of the contours by the newly created bad regions, which requires that we compute those. This will in fact be the most subtle and important part of the entire renormalization program. Given a new region D' , the effect on the contours is to replace $\underline{\mathcal{L}^{-1}\Gamma}$ by $\underline{\mathcal{L}^{-1}\Gamma} \cup D'(\mathcal{L}^{-1}\Gamma)$, so that finally the full RG transformation on the contours can be written as

$$\mathbb{T}_D(\Gamma) \equiv (D'(\mathcal{L}^{-1}\Gamma^l(\Gamma)) \cup \mathcal{L}^{-1}\underline{\Gamma}^l(\Gamma), \mathcal{L}^{-1}h(\Gamma^l(\Gamma))) \quad (2.83)$$

2.3.2 The ground-states

The crucial new feature in the analysis of the low-temperature phase of the RFIM lies in the fact that even the analysis of the properties of the ground-state becomes highly non-trivial. We therefore present the analysis of the ground-states first.

Formalism and set-up. To simplify things, we will only show that, with probability larger than $1/2$ (and in fact tending to one if $\Sigma \downarrow 0$), the spin at the origin is $+1$ in any ground-state configuration with $+$ boundary conditions. More precisely, define

$$\mathcal{G}_\Lambda^{(\Gamma)} \equiv \left\{ \Gamma^* \in \mathcal{S} \mid \Gamma_{\Lambda^c}^* = \Gamma_{\Lambda^c} \wedge H_\Lambda(\Gamma^*) = \inf_{\Gamma': \Gamma_{\Lambda^c} = \Gamma_{\Lambda^c}} H_\Lambda(\Gamma') \right\} \quad (2.84)$$

Here Γ_{Λ^c} denotes the restriction of Γ to Λ^c . We want to show that for a suitable sequence of cubes Λ_n ,

$$\mathbb{P} \left[\liminf_{n \uparrow \infty} \min_{\Gamma \in \mathcal{G}_{\Lambda_n}^{\emptyset, +1}} \sigma_0(\Gamma) = +1 \right] > 1/2 \quad (2.85)$$

Let us introduce the abbreviation $\mathcal{S}_n \equiv \mathcal{S}_{\Lambda_n}$. The analysis of ground-states via the renormalization group method then consists of the following inductive procedure. Let \mathbb{T} be a map $\mathbb{T} : \mathcal{S}_n \rightarrow \mathcal{S}_{n-1}$. Then clearly

$$\inf_{\Gamma \in \mathcal{S}_n} H_{\Lambda_n} = \inf_{\tilde{\Gamma} \in \mathcal{S}_{n-1}} \left(\inf_{\Gamma \in \mathbb{T}^{-1}\tilde{\Gamma}} H_{\Lambda_n}(\Gamma) \right) \quad (2.86)$$

which suggests to define

$$(\mathbb{T}H_{\Lambda_{n-1}})(\tilde{\Gamma}) \equiv \inf_{\Gamma \in \mathbb{T}^{-1}\tilde{\Gamma}} H_{\Lambda_n}(\Gamma) \quad (2.87)$$

Here $\mathbb{T}^{-1}\tilde{\Gamma}$ denotes the set of pre-images of $\tilde{\Gamma}$ in \mathcal{S}_n . Defining $\mathbb{T}\mathcal{G}_{\Lambda_{n-1}}^{(0)}$ to be the set of ground-states with respect to the energy function $\mathbb{T}H$, we have that

$$\mathcal{G}_{\Lambda_n}^{\emptyset, +1} = \left\{ \Gamma^* \mid H_{\Lambda_n}(\Gamma^*) = \inf_{\Gamma \in \mathbb{T}^{-1}(\mathbb{T}\mathcal{G}_{\Lambda_{n-1}}^{\emptyset, +1})} H_{\Lambda_n}(\Gamma) \right\} \quad (2.88)$$

that is, if we can determine the ground-states with respect to $\mathbb{T}H$ in the smaller volume Λ_{n-1} , then we have to search for the true ground-state only within the inverse image of this set.

We will now give a precise description of the class of admissible energy functions. The original energy function describing the RFIM was already introduced. To describe the general class of models that will appear in the RG process, we begin with the ‘control’ fields N . We let $\{N_x\}_{x \in \Lambda_n}$ be a family of non-negative real numbers. They will later be assumed to be a

random field satisfying certain specific probabilistic assumption. Given N , $D(N)$ is defined in (2.74). We denote by $\mathcal{S}_n(D) \subset \mathcal{S}_n$ the space

$$\mathcal{S}_n(D) \equiv \{\Gamma \in \mathcal{S}_n \mid D(N) \subset \underline{\Gamma}\} \quad (2.89)$$

Definition 2.3.2 An N -bounded contour energy, ϵ , of level k is a map $\epsilon : \mathcal{S}_n(D) \rightarrow \mathbb{R}$, s.t.

(i) If $\gamma_1, \dots, \gamma_m$ are the connected components of Γ , then

$$\epsilon(\Gamma) = \sum_{i=1}^m \epsilon(\gamma_i) \quad (2.90)$$

(ii) If γ is a connected contour in $\mathcal{S}_n(D)$ then

$$\epsilon(\gamma) \geq E_s(\gamma) + L^{-(d-2)k} |\underline{\gamma} \setminus \overline{D}(\gamma)| - (N, V(\gamma) \cap \underline{\gamma}) \quad (2.91)$$

where $E_s(\gamma)$ is defined in (2.70).

(iii) Let $C \subset D$ be connected and γ be the connected component of a contour $\Gamma \subset \mathcal{S}_n(D)$. Then

$$\epsilon(\gamma) \leq \sum_{x \in C} N_x \quad (2.92)$$

An N -bounded energy function of level k is a map $H_{\Lambda_n} : \mathcal{S}_n \rightarrow \mathbb{R}$ of the form

$$H_{\Lambda_n}(\Gamma) = \epsilon(\Gamma) + (S, V(\Gamma)) \quad (2.93)$$

where S_x are bounded random fields (see (2.72)) and ϵ is a N -bounded contour energy of level k .

Remark 2.3.1 The appearance of the dimension- and k -dependent constant in the lower bound (2.91) is due to the fact that in the RG process no uniform constant suppressing supports of contours outside the bad region is maintained.

We now define the notion of a proper RG transformation.

Definition 2.3.3 For a given control field N , a proper renormalization group transformation, $\mathbb{T}^{(N)}$, is a map from $\mathcal{S}_n(D(N))$ into $\mathcal{S}_{n-1}(D(N'))$, such that, if H_{Λ_n} is of the form (2.93) with ϵ an N -bounded contour energy of level k , then $H'_{\Lambda_{n-1}} \equiv \mathbb{T}^{(N)} H_{\Lambda_n}$ is of the form

$$H'_{\Lambda_{n-1}}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \quad (2.94)$$

where ϵ' is an N' -bounded contour energy of level $k+1$, and S' is a new bounded random field and N' is a new control field.

In order to make use of a RG transformation, it is crucial to study the action of the RG on the random and control fields. As both are random fields, this control will be probabilistic. We must therefore specify more precisely the corresponding assumptions.

Recall that the energy functions H are random functions on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and that H_{Λ_n} is assumed to be \mathcal{B}_{Λ_n} -measurable (this is evident, e.g., in the original model, where H_{Λ_n} is a function of the stochastic sequences h_x with $x \in \Lambda_n$ only, and \mathcal{B}_{Λ_n} is the sigma-algebra generated by these sequences). The renormalized energy functions are still random variables on this same probability space. It is useful to consider an action of the RG map on the sigma-algebras and to introduce $\mathcal{B}^{(k)} = \mathbb{T}^k \mathcal{B}$, where $\mathbb{T} \mathcal{B}_{\Lambda}^{(k)} \subset \mathcal{B}_{\mathcal{L}\Lambda}^{(k-1)}$, such that after k iterations of the RG the resulting energy function is $\mathcal{B}^{(k)}$ -measurable. Naturally, $\mathcal{B}^{(k)}$ is endowed with a filtration with respect to the renormalized lattice. In the general step we will drop the reference to the level in the specification of this sigma-algebra and write simply \mathcal{B} . We need to maintain certain locality properties that we state as follows:

- (i) The stochastic sequences $\{N_x\}$ and $\{S_x\}$ are measurable w.r.t. the sigma-algebras $\mathcal{B}_{\bar{x}}$.
- (ii) For connected contours, $\gamma \in \mathcal{S}_n(D)$, $\epsilon(\gamma)$ is measurable w.r.t. $\mathcal{B}_{\bar{\gamma}}$.
- (iii) The distribution of $\{S_x\}_{x \in \bar{\Lambda}_n}$ is jointly symmetric, and the distribution of the contour energies $\{\epsilon(\gamma)\}_{\gamma \subset \bar{\Lambda}_n}$ is symmetric under a global-spin flip.

Finally, we need assumptions on the smallness of the disorder. Here the S -fields are centred and bounded, i.e.

- (iv) $|S_x| \leq \delta$, for δ small enough (for instance $\delta = \frac{1}{8L}$ will work).
- (v)

$$\mathbb{P}[S_y \geq \epsilon] \leq \exp\left(-\frac{\epsilon^2}{2\Sigma^2}\right) \quad (2.95)$$

The control fields N_x should also satisfies bounds like (2.95), but actually the situation there is more complicated. Notice that in the original model the N -fields as defined in (2.73) satisfy bounds $\mathbb{P}(N_x > z) \leq 2 \exp\left(-\frac{z^2}{2\Sigma^2}\right)$, and, moreover, the smallest non-zero value they take is δ . The precise formulation of the conditions on N are postponed to the end of this section.

Absorption of small contours. The first part of the RG map consists of the re-summing of ‘small contours’. These can be defined as connected components of small size (on scale L) with support outside the bad regions.

The definition of the bad regions excludes the existence of such small components in a ground-state contour. Actually, there is even a large portion of the bad region that may be removed if we are willing to allow for the appearance of ‘flat’ small contours, i.e contours with non-empty supports but constant sign even on their support. It is crucial to take advantage of this fact. The following definition describes this ‘harmless’ part of the bad region.

Definition 2.3.4 Let D_i denote the $L^{1/2}$ -connected¹ components of D . Such a connected component is called small, on level k , if

- (i) $|D_i| < L^{(1-\alpha)/2}$
- (ii) $d(D_i) \leq L/4$
- (iii) $\sum_{y \in D_i} N_y < LL^{-(d-2)k} \Sigma^2$

Here $\alpha > 0$ is a constant that will be fixed later and $\Sigma^2 \equiv \Sigma_0^2$ refers to the variance of the original random fields, not to those at level k . Define

$$\mathcal{D} \equiv \bigcup_{D_i \text{ small}} D_i \quad (2.96)$$

Remark 2.3.2 The definition of \mathcal{D} is ‘local’: If we consider a point, x , and a set, $E \subset \Lambda_n$, containing x , then the event $\{E \text{ is a component of } \mathcal{D}\}$ depends only on $N_{x'}$ -fields such that $d(x, x') \leq L/3$.

Definition 2.3.5 A connected contour $\gamma \in \mathcal{S}_n(D)$ is called small, iff

- (i) $d(\gamma) < L$, and
- (ii) $(\bar{D} \setminus \mathcal{D}) \cap \text{int } \underline{\gamma} = \emptyset$

A contour Γ is called small, iff the maximal connected component of each weakly connected component is small. A contour that is not small is called *large*. We denote by $\mathcal{S}_n^s(D)$ the set of small contours and by $\mathcal{S}_n^l(D)$ the set of large contours.

Remark 2.3.3 Notice that $\mathcal{S}_n^l(D) \subset \mathcal{S}_n(D \setminus \mathcal{D})$, but in general it is *not* a subset of $\mathcal{S}_n(D)$!

Definition 2.3.6 The map $T_1 : \mathcal{S}_n(D) \rightarrow \mathcal{S}_n^l(D)$ is the canonical projection, i.e. if $\Gamma = (\gamma_1, \dots, \gamma_r, \gamma_{r+1}, \dots, \gamma_q)$ with γ_i large for $i = 1, \dots, r$ and small for $i = r+1, \dots, q$, then

$$T_1(\Gamma) \equiv \Gamma^l \equiv (\gamma_1, \dots, \gamma_r) \quad (2.97)$$

¹ It should be clear what is meant by $L^{1/2}$ -connectedness: A set A is called $L^{1/2}$ -connected if there exists a path in A with steps of length less than or equal to $L^{1/2}$ joining each point in A .

To give a precise description the conditioned ground-states under the projection T_1 , we need to define the following sets. First let $\overline{\mathcal{D}}_i$ denote the *ordinary* connected components of $\overline{\mathcal{D}}$ (in contrast to the definition of $D_i!$). Given a contour $\Gamma^l \in \mathcal{S}_n^l(D)$ we write $\mathcal{B}_i(\Gamma^l) \equiv \overline{\mathcal{D}}_i \setminus \underline{\Gamma}^l$ for all those components such that $\overline{\mathcal{D}}_i \subset V_{\pm}(\Gamma^l) \setminus \underline{\Gamma}^l$. Let $\mathcal{B}(\Gamma^l) \equiv \bigcup_i \mathcal{B}_i(\Gamma^l) = \overline{\mathcal{D}}(\Gamma^l) \setminus \underline{\Gamma}^l$. Finally we set $\mathcal{D}_i = \overline{\mathcal{D}}_i \cap \mathcal{D}$. Note that \mathcal{D}_i need not be connected.

Let $\mathcal{G}_{\Gamma^l, 1}$ be the set of contours in $\mathcal{S}_n(D)$ that minimize H_n under the condition $T_1\Gamma = \Gamma^l$. Then:

Lemma 2.3.2 *Let $\Gamma^l \in \mathcal{S}_\Lambda^l(D)$ Then, for any $\Gamma \in \mathcal{G}_{1, \Gamma^l}$:*

- (i) $\underline{\Gamma} \setminus \underline{\Gamma}^l \subset \mathcal{B}(\Gamma^l)$, and
- (ii) For all x , $\sigma_x(\Gamma) \equiv \sigma_x(\Gamma^l)$.

Remark 2.3.4 This Lemma is the crucial result of the first step of the RG transformation. It makes manifest that fluctuations on length scale L can only arise due to ‘large fields in the bad regions’. Since this statement will hold in each iteration of the RG, it shows that the spins are constant outside the bad regions.

The next Lemma gives a formula for the renormalized energy function under T_1 . We set

$$\epsilon^{\pm}(\mathcal{B}_i(\Gamma^l)) \equiv \inf_{\gamma: \mathcal{D}_i \subset \underline{\gamma} \subset \mathcal{B}_i(\Gamma^l), \gamma = (\underline{\gamma}, \sigma_x \equiv \pm)} \epsilon(\gamma) \quad (2.98)$$

Note that γ here is not necessarily connected.

Lemma 2.3.3 *Let for any $\Gamma^l \in \mathcal{S}_n^l(D)$ denote*

$$(T_1 H_n)(\Gamma^l) \equiv \inf_{\Gamma \in \mathcal{S}_n(D): T_1(\Gamma) = \Gamma^l} H_n(\Gamma) \quad (2.99)$$

Then

$$(T_1 H_n)(\Gamma^l) - H_n(\Gamma^l) = \sum_i \epsilon^{\pm}(\mathcal{B}_i(\Gamma^l)) \quad (2.100)$$

where the sign \pm is such that $\mathcal{B}_i(\Gamma^l) \subset V_{\pm}(\Gamma^l)$.

Note that in the expression $H_n(\Gamma^l)$, we view Γ^l as a contour in $\mathcal{S}_n(D \setminus \mathcal{D})$; that is, the contributions to the energy in the regions $\mathcal{D} \setminus \underline{\Gamma}^l$ are ignored.

We will skip the proof that is essentially book-keeping and using the isoperimetric inequality

$$E_s(\gamma) \geq \frac{d}{L} \sum_{x \in \text{int}(\underline{\gamma})} (\sigma_x(\gamma) - \sigma_\gamma)^2 \quad (2.101)$$

where γ is a weakly connected contour, s.t. $d(\text{int} \underline{\gamma}) \leq L$ and σ_γ denotes the sign of γ on $\partial \text{int} \underline{\gamma}$.

Remark 2.3.5 The proof of Lemma 2.3.3 requires a smallness condition on Σ^2 w.r.t. L , which is the reason for the constant $1/8L$ in (iii) of Definition 2.3.4.

From the preceding Lemmata, and Definition 2.3.4, we obtain the following uniform bounds on the ϵ^\pm :

Lemma 2.3.4 For any Γ^l , and any component $\mathcal{B}_i(\Gamma^l)$

$$|\epsilon^\pm(\mathcal{B}_i(\Gamma^l))| \leq LL^{-(d-2)k}\Sigma^2 \quad (2.102)$$

Here we see the rationale for the definition of the harmless part of the large field region, namely that the ground-state contours supported in them only introduce an extremely small correction to the energy, which can, as we will see in the next step, be absorbed locally in the small fields.

The blocking. We now want to map the configuration space \mathcal{S}_n to \mathcal{S}_{n-1} . The corresponding operator, T_2 , will be chosen as $T_2 \equiv \mathcal{L}^{-1}$, with \mathcal{L}^{-1} defined in Eq. (2.81) and (2.82). We will use the name \mathcal{L}^{-1} when referring to the purely geometric action of T_2 . Notice that \mathcal{L}^{-1} is naturally a map from $\mathcal{S}_n(D \setminus \mathcal{D})$ into $\mathcal{S}_{n-1}(\mathcal{L}^{-1}(D \setminus \mathcal{D}))$, where $\mathcal{L}^{-1}(D \setminus \mathcal{D})$ is defined as the union of the sets $\mathcal{L}^{-1}(D \setminus \mathcal{D})$. We must construct the induced action of this map on the energy functions and on the random fields S and N . Consider first the small fields. Recall that we wanted to absorb the contributions of the small contours into the renormalized small fields. This would be trivial if the $\mathcal{B}_i(\Gamma^l)$ did not depend on Γ^l . To take this effect into account, we write

$$\epsilon^\pm(\mathcal{B}_i(\Gamma^l)) = \epsilon^\pm(\overline{\mathcal{D}}_i) + (\epsilon^\pm(\mathcal{B}_i(\Gamma^l)) - \epsilon^\pm(\overline{\mathcal{D}}_i)) \quad (2.103)$$

and adding the first term to the small fields while the second is non-zero only for \mathcal{D}_i that touch the contours of Γ^l and will later be absorbed in the new contour energies. Thus we define the (preliminary) new small fields by

$$\tilde{S}'_y \equiv L^{-(d-1-\alpha)} \left(\sum_{x \in \mathcal{L}y} S_x + \sum_{i: \overline{\mathcal{D}}_i \cap \mathcal{L}y \neq \emptyset} \frac{\epsilon^\pm(\overline{\mathcal{D}}_i)}{|\mathcal{L}^{-1}\overline{\mathcal{D}}_i|} \right) \quad (2.104)$$

The pre-factor in this definition anticipates the scaling factor of the surface energy term under blocking. Note here that the \tilde{S}'_y satisfy the locality conditions (i): \tilde{S}'_y and $\tilde{S}'_{y'}$ are independent stochastic sequences if $|y - y'| > 1$, since the $\overline{\mathcal{D}}_i$ cannot extend over distances larger than L .

The (preliminary) new control field is defined as

$$\tilde{N}'_y \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}} N_x \quad (2.105)$$

Note here that the summation over x excludes the regions \mathcal{D} , as the contributions there are dealt with elsewhere. This is crucial, as otherwise the regions with positive \tilde{N}' would grow, rather than shrink, in the RG process.

The induced energy function $T_2 T_1 H_n$ on $\mathcal{S}_{n-1}(\mathcal{L}^{-1}(D \setminus \mathcal{D}))$ is

$$(T_2 T_1 H_n)(\Gamma') \equiv \inf_{\Gamma^l: \mathcal{L}^{-1}\Gamma^l = \Gamma'} (T_1 H_n)(\Gamma^l) \quad (2.106)$$

The following Lemma states that this energy function is essentially of the same form as H_n :

Lemma 2.3.5 *For any $\Gamma' \in \mathcal{S}_{n-1}(\mathcal{L}^{-1}(D \setminus \mathcal{D}))$ we have*

$$(T_2 T_1 H_n)(\Gamma') = L^{d-1-\alpha} \left(\sum_{i=1}^q \bar{\epsilon}(\gamma'_i) + (\tilde{S}', V(\Gamma')) \right) \quad (2.107)$$

where the γ'_i are the connected components of Γ' , and $\bar{\epsilon}$ satisfies the lower bound

$$\begin{aligned} \bar{\epsilon}(\gamma') &\geq c_1 L^\alpha E_s(\gamma') + c_2 L^\alpha L L^{-(d-2)(k+1)} |\underline{\gamma'} \setminus \overline{\tilde{D}'(\gamma')}| \\ &\quad - (\tilde{N}', V(\gamma') \cap \underline{\gamma'}) \end{aligned} \quad (2.108)$$

where $\tilde{D}' \equiv D(\tilde{N}')$ is the preliminary bad field region. Moreover, for flat contours of the form $\gamma' = (C, \sigma_y \equiv s)$ with $C \subset \tilde{D}'$ connected, we have the upper bound

$$\bar{\epsilon}(\gamma') \leq (\tilde{N}', V(\gamma') \cap C) \quad (2.109)$$

We will again skip the details of the proof, which is largely a matter of book-keeping, i.e. suitably distributing the various terms to the new energy functionals and the new field terms. To obtain the desired estimates on the energy terms, we need the following isoperimetric inequalities:

Lemma 2.3.6 *Let $\sigma' = \text{sign} \sum_{x \in \mathcal{L}0} \sigma_x$. Then*

$$\sum_{\langle x, y \rangle: x, y \in \mathcal{L}0} |\sigma_x - \sigma_y| \geq \frac{1}{L} \sum_{x \in \mathcal{L}0} |\sigma_x - \sigma'| \quad (2.110)$$

Lemma 2.3.7 *Let $\Gamma \in \mathcal{L}^{-1}\gamma'$. Then*

$$E_s(\Gamma) \geq \frac{L^{d-1}}{d+1} E_s(\gamma') \quad (2.111)$$

The most tricky part is to obtain the term proportional to $|\underline{\gamma'} \setminus \overline{\tilde{D}'(\gamma')}|$, where \tilde{D}' is the bad region associated with the new control field. The problem is that the original estimate is only in the volume of $\underline{\Gamma}^l$ outside the bad region, while the new estimate involves the new bad region, which is smaller than the image of the bad region under \mathcal{L}^{-1} since the harmless part, \mathcal{D} , has been excluded in the definition of the \tilde{N}' . In fact, the geometric

constraints in the definition of \mathcal{D} were essentially made in order to get the desired estimate.

Final shape-up. The hard part of the RG transformation is now done. However, not all of the properties of the original model are yet shared by the renormalized quantities; in particular, the renormalized weak field \tilde{S}' is not centred and it may have become too large. Both defaults are, however, easily rectified. We define

$$S'_y \equiv \tilde{S}'_y \mathbb{1}_{|\tilde{S}'_y| < \delta} - \mathbb{E} \left(\tilde{S}'_y \mathbb{1}_{|\tilde{S}'_y| < \delta} \right) \quad (2.112)$$

What is left, i.e. the large part of the small field, is taken account of through the redefined control field. We define the final renormalized control field by

$$N'_y \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}} N_x + |\tilde{S}'_y| \mathbb{1}_{|\tilde{S}'_y| > \delta} \quad (2.113)$$

Given N' , we may now define $D' \equiv D(N')$ as (2.74). Then let T_3 (given N') be the map from \mathcal{S}_{n-1} to $\mathcal{S}_{n-i}(D')$ defined through

$$T_3(\Gamma) = (\sigma(\Gamma), \underline{\Gamma} \cup D'(\Gamma)) \quad (2.114)$$

We define the contour energies

$$\begin{aligned} \epsilon'(\gamma'') &\equiv \inf_{\substack{\gamma' : \sigma(\gamma') = \sigma(\gamma'') \\ \gamma' \subset \gamma''}} \bar{\epsilon}(\gamma') + \sum_{y \in \gamma''} \tilde{S}'_y \sigma_y(\gamma'') \mathbb{1}_{|\tilde{S}'_y| \geq \frac{1}{16L}} \\ &+ \sum_{\substack{y \in \gamma'' \\ d(y, \Lambda_{n-1}^c) = 1}} \mathbb{E} \left[S'_y \mathbb{1}_{|\tilde{S}'_y| < \frac{1}{16L}} \right] \sigma_y(\gamma'') \end{aligned} \quad (2.115)$$

Notice that the terms in the second line of (2.115) are a boundary term that is due to the fact that the renormalized fields have mean zero if they are at least at a distance 2 from the boundary.

The final form of the renormalization group map is then given through the following:

Lemma 2.3.8 *For any $\Gamma' \in \mathcal{S}_{N'}(N')$ we have*

$$T_3 T_2 T_1 E(\Gamma') = L^{d-1-\alpha} (\epsilon'(\Gamma') + (S', V(\Gamma'))) \quad (2.116)$$

where ϵ' is an N' -bounded contour energy of level $k+1$.

Proof. The form of the renormalized energy follows from the construction. The N' boundedness of ϵ' is essentially a consequence of Lemma 2.3.5. The only problem are the boundary terms in the second line of (2.114). But these can again be compensated by giving away a small fraction of the interaction energy. \square

This concludes the construction of the entire RG transformation. We may summarize the results of the previous three subsections in the following:

Proposition 2.3.9 *Let $\mathbb{T}^{(N)} \equiv T_3 T_2 T_1 : \mathcal{S}_n(D(N)) \rightarrow \mathcal{S}_{n-1}(D(N'))$ with T_1 , T_2 and T_3 defined above; let N' and S' and ϵ' be defined as above and define $H'_{n-1} \equiv L^{-(d-1-\alpha)}(\mathbb{T}^{(N)} H_n)$ through*

$$H'_{n-1}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \quad (2.117)$$

If H_n is an N -bounded energy function of level k , then H'_{n-1} is an N' -bounded energy function of level $k+1$.

This proposition allows us to control the flow of the RG transformation on the energies through its action on the random fields S and N . What is now left to do is to study the evolution of the probability distributions of these random fields under the RG map.

Probabilistic estimates. Our task is now to control the action of the RG transformation on the random fields S and N , i.e. given the probability distribution of these random fields, we must compute the distribution of the renormalized random fields S' and N' as defined through Eqs. (2.104), (2.105), (2.112), and (2.113). Of course, we only compute certain bounds on these distributions.

Let us begin with the small fields. In the k -th level of iteration, the distributions of the random fields are governed by a parameter Σ_k^2 (essentially the variance of S_x^k) that decreases exponentially fast to zero with k . We will set

$$\Sigma_k^2 \equiv L^{-(d-2-\eta)k} \Sigma^2 \quad (2.118)$$

where η may be chosen as $\eta \equiv 4\alpha$. We denote by $S^{(k)}$ the small random field obtained from S after k iterations of the RG map \mathbb{T} . (Where the action of \mathbb{T} on S is defined through (2.112) and (2.104)).

Proposition 2.3.10 *Let $d \geq 3$. Assume that the initial S satisfy assumptions (i), (iii), and (v) (with Σ^2 sufficiently small). Then, for all $k \in \mathbb{N}$ and for all $\epsilon \geq 0$,*

$$\mathbb{P} \left[S_y^{(k)} \geq \epsilon \right] \leq \exp \left(-\frac{\epsilon^2}{2\Sigma_k^2} \right) \quad (2.119)$$

with Σ_k defined through (2.118), and $S^{(k)}$ satisfy assumptions (i), (iii), (iv), and (v).

Proof. The renormalized small fields are sums of the old ones, where by assumption the old random variables are independent if their distance is

larger than 1. This allows us to represent the sum $\sum_{x \in \mathcal{L}_y} S_x$ as a sum of 2^d sums of independent random variables. Now note that successive use of Hölder's inequality implies that

$$\mathbb{E} e^{t \sum_{i=1}^k Z_i} \leq \prod_{i=1}^k (\mathbb{E} e^{t k Z_i})^{1/k} \quad (2.120)$$

Thus the estimates of the Laplace transforms of S' can be reduced to that of i.i.d. random variables satisfying Gaussian tail estimates. The proposition follows thus from standard computations using the exponential Chebyshev inequality. Details can be found in [12] or [10]. \square

Next we turn to the distribution of the control fields. We denote by $N_x^{(k)}$ the fields obtained after k iterations of the RG transformation from a starting field $N^{(0)}$, where the iterative steps are defined by equations (2.105) and (2.113). We denote by $D^{(k)}$ and $\mathcal{D}^{(k)}$ the bad regions and harmless bad regions in the k -th RG step. What we need to prove for the control fields are two types of results: First, they must be large only with very small probabilities; second, and more important, they must be equal to zero with larger and larger probability, as k increases. This second fact implies that the 'bad regions' become smaller and smaller in each iteration of the RG group. The proof of this second fact must take into account the absorption of parts of the bad regions, the \mathcal{D} , in each step. What is happening is that once a large field has been scaled down sufficiently, it will drop to zero, since it finds itself in the region \mathcal{D} . Due to the complications arising from interactions between neighbouring blocks, this is not quite true, as the field really drops to zero only if the fields at neighbouring sites are small, too. This is being taken into account by considering an upper bound on the control field that is essentially the sum of the original N over small blocks. We define

$$\begin{aligned} \bar{N}_y^{(0)} &= N_y^{(0)} \\ \bar{N}_y^{(k+1)} &= L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}\{y\} \cap D^{(k+1)} \setminus \mathcal{D}^{(k)}} \bar{N}_x^{(k)} + \left| \tilde{S}_y^{(k+1)} \right| \mathbb{I}_{|\tilde{S}_y^{(k+1)}| > \delta} \end{aligned} \quad (2.121)$$

The fields \bar{N} bound the original N from above, but also, in an appropriate sense, from below. Namely:

Lemma 2.3.11 *The fields $\bar{N}^{(k)}$ defined in (2.121) satisfy*

$$\bar{N}_x^{(k)} \geq N_x^{(k)} \quad (2.122)$$

Moreover, if M is an arbitrary subset of \mathbb{Z}^d and if $K \subset \mathbb{Z}^d$ denotes the union of the connected components of $D^{(k)}$ that intersect M , then

$$\sum_{x \in M} \bar{N}_x^{(k)} \leq C_1^k \sum_{x \in \overline{M} \cap K} N_x^{(k)} \quad (2.123)$$

Proof. The lower bound (2.122) is obvious. The upper bound is proven by induction. Assume (2.123) for k . To show that then it then also holds for $k + 1$, we need to show is that

$$\begin{aligned} \sum_{y \in M} \sum_{x \in \mathcal{L}\{y\} \cap D^{(k+1)} \setminus \mathcal{D}^{(k)}} \bar{N}_x^{(k)} &\leq C_1^k \sum_{y \in \overline{M} \cap K} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}^{(k)}} N_x^{(k)} \\ &= C_1^{k+1} \sum_{x \in \mathcal{L}(\overline{M} \cap K) \setminus \mathcal{D}^{(k)}} N_x^{(k)} \end{aligned} \quad (2.124)$$

Now, quite obviously,

$$\sum_{y \in M} \sum_{x \in \mathcal{L}\{y\} \cap D^{(k+1)} \setminus \mathcal{D}^{(k)}} \bar{N}_x^{(k)} \leq |\overline{\{0\}}| \sum_{x \in \mathcal{L}(\overline{M} \cup K) \setminus \mathcal{D}^{(k)}} \bar{N}_x^{(k)} \quad (2.125)$$

where $|\overline{\{0\}}| = 3^d$ takes into account the maximal possible over-counting due to the double sum over y and x . The restriction of the sum over x to the image of K is justified, since all other x must either lie in $\mathcal{D}^{(k)}$, or give a zero contribution. Using the induction hypothesis and the definition of $\bar{N}^{(k)}$, a simple calculation shows now that

$$\sum_{x \in \mathcal{L}(\overline{M} \cup K) \setminus \mathcal{D}^{(k)}} \bar{N}_x^{(k)} \leq (1 + C_1^k) \sum_{x \in \mathcal{L}(\overline{M} \cup K) \setminus \mathcal{D}^{(k)}} N_x^{(k)} \quad (2.126)$$

from which we get (2.124) if C_1 is chosen $2 \cdot 3^d$. \square

Remark 2.3.6 The bound (2.123) is relevant in the estimates for the finite temperature case only.

The main properties of the control fields are given by the following:

Proposition 2.3.12 Let $f_d(z) \equiv z^2 \mathbb{1}_{z \geq 1} + z^{\frac{d-2}{d-1}} \mathbb{1}_{u < 1}$. Then

$$\mathbb{P} \left[L^{-(d-3/2)k} \Sigma > \bar{N}_y^{(k)} > 0 \right] = 0 \quad (2.127)$$

and, for $z \geq L^{-(d-3/2)k} \Sigma$

$$\mathbb{P} \left[\bar{N}_y^{(k)} \geq z \right] \leq \exp \left(- \frac{f_d(z)}{4(16L)^{d/d-1} \Sigma_k^2} \right) \quad (2.128)$$

Proof. The proof of this proposition will be by induction over k . Note that it is trivially verified for $k = 0$. Thus we assume (2.127) and (2.128) for k .

Let us first show that (2.127) holds for $k + 1$. The event under consideration cannot occur if $|\tilde{S}_y^{(k+1)}| > \delta$. Therefore, unless $\bar{N}_y^{(k+1)} = 0$, the site y must lie within $\tilde{D}^{(k+1)}$. But this implies that

$$\mathcal{L}y \cap \left(D^{(k)} \setminus \mathcal{D}^{(k)} \right) \neq \emptyset \quad (2.129)$$

and hence there must exist a $L^{\frac{1}{2}}$ -connected component $D_i \subset D^{(k)}$ intersecting $\mathcal{L}y$ that violates one of the conditions of ‘smallness’ from Definition 2.3.4. Assume first that *only* condition (iii) is violated. In this case, D_i is so small that it is contained in $\mathcal{L}\bar{y}$ and therefore contributes a term larger than $L^{-(d-2)(k+1)+\alpha}\Sigma^2$ to \bar{N}_y , and since $\Sigma^2 \sim 1/L$, this already exceeds $L^{-(d-3/2)(k+1)}\Sigma$. Thus, either condition (i) or (ii) must be violated. In both cases, this implies that the number of sites in D_i exceeds $L^{(1-\alpha)/2}$. Any site in $D_i(h)$ contributes at least the minimal non-zero value of $N_x^{(k)}$, which by inductive assumption is $L^{-(d-3/2)k}\Sigma$. Therefore

$$\begin{aligned} \bar{N}_y^{(k+1)} &\geq L^{-(d-1-\alpha)} \sum_{x \in D_i \cap \mathcal{L}y} L^{-(d-3/2)k}\Sigma \\ &\geq L^{(1-\alpha)/2} L^{-(d-1-\alpha)} L^{-(d-3/2)k}\Sigma \\ &\geq L^{-(d-3/2)(k+1)}\Sigma \end{aligned} \quad (2.130)$$

But this proves (2.127).

To complete the proof of (2.128) we need a property of the function f_d . Before stating it, let us point out that it is crucial to have the function $f_d(z)$, rather than simply z^2 ; namely, our goal is to show that $\bar{N}_x^{(k)}$ is non-zero with very small probability, which is true if $f_d(L^{-(d-3/2)k}\Sigma) \frac{\delta^2}{\Sigma_k^2}$ is large and grows with k . This is true if f_d , for small values of its argument, cannot decay too fast!

Lemma 2.3.13 *The function f_d defined in Proposition 2.3.12 satisfies*

$$\sum_{x \in \mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}) \geq L^{d-2-3\alpha} f_d \left(L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}\bar{y}} \bar{N}_x^{(k)} \right) \quad (2.131)$$

Proof. See [10]. □

We are now ready to prove (2.128) for $k + 1$. Obviously,

$$\begin{aligned} \mathbb{P} \left[\bar{N}_y^{(k+1)} \geq z \right] &\leq \mathbb{P} \left[L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}\bar{y} \setminus \mathcal{D}} \bar{N}_x^{(k)} \geq z/2 \right] \\ &\quad + \mathbb{P} \left[|\tilde{S}_y^{(k)}| \mathbb{1}_{|\tilde{S}_y^{(k)}| > \delta} > z/2 \right] \end{aligned} \quad (2.132)$$

Let us consider the first term in (2.132). By Lemma 2.3.13,

$$\begin{aligned}
& P \left[L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}\bar{y} \setminus \mathcal{D}} \bar{N}_x^{(k)} \geq z/2 \right] \\
&= \mathbb{P} \left[f_d \left(L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}\bar{y} \setminus \mathcal{D}} \bar{N}_x^{(k)} \right) \geq f_d(z/2) \right] \\
&\leq \mathbb{P} \left[\sum_{x \in \mathcal{L}\bar{y}} f_d \left(\bar{N}_x^{(k)} \right) \geq L^{d-2-3\alpha} f_d(z) \right] \tag{2.133}
\end{aligned}$$

The variables $f_d(N_x^{(k)})$ are essentially exponentially distributed in their tails. We can bound their Laplace transform by

$$\begin{aligned}
\mathbb{E} \left(e^{t f_d(N_x^{(k)})} \right) &\leq \mathbb{P} \left[N_x^{(k)} = 0 \right] + t \int_{m_k}^{\infty} e^{t f} e^{-f \alpha_k} df \\
&\leq 1 + t \frac{e^{(t-\alpha_k) f_0}}{\alpha_k - t} \tag{2.134}
\end{aligned}$$

where we have set $m_k \equiv f_d(L^{-(d-3/2)k} \Sigma)$ (the minimal non-zero value $f_d(N^{(k)})$ can take) and $\alpha_k \equiv 1/4(16L)^d \Sigma_k^2$. We will bound the Laplace transform *uniformly* for all $t \leq t^* \equiv (1-\epsilon)\alpha_k$, for some small $\epsilon > 0$. Since $\gamma_k \gg (1-\epsilon)\alpha_k f_0$ (check!), we get in this range of parameters

$$\mathbb{E} \left(e^{t f_d(N_x^{(k)})} \right) \leq 1 + \frac{1-\epsilon}{\epsilon} e^{-\epsilon m_k \alpha_k} \tag{2.135}$$

Using the independence of well-separated $\bar{N}_x^{(k)}$, we find

$$\begin{aligned}
& \mathbb{P} \left[\sum_{x \in \mathcal{L}\bar{y}} f_d \left(\bar{N}_x^{(k)} \right) \geq L^{d-2-3\alpha} f_d(z/2) \right] \\
&\leq e^{-L^{d-2-3\alpha} f_d(z/2) \frac{t^*}{5^d}} \mathbb{E} \left(e^{\frac{t^*}{5^d} \sum_{x \in \mathcal{L}\bar{y}} f_d \left(\bar{N}_x^{(k)} \right)} \right) \\
&\leq e^{-L^{d-2-3\alpha} f_d(z) \frac{\alpha_k(1-\epsilon)}{5^d}} \left[\mathbb{E} \left(e^{t^* f_d \left(\bar{N}_x^{(k)} \right)} \right) \right]^{\frac{L^d}{5^d}} \\
&\leq e^{-L^{d-2-3\alpha} f_d(z/2) \frac{\alpha_k(1-\epsilon)}{5^d}} \left[1 + \frac{1-\epsilon}{\epsilon} e^{-\epsilon m_k \alpha_k} \right]^{\frac{L^d}{5^d}} \tag{2.136}
\end{aligned}$$

The last factor in (2.136) is close to one and may be absorbed in a constant in the exponent, as we only want a bound for $z \geq L^{-(d-3/2)(k+1)} \Sigma$. Moreover, $f_d(z/2) \geq f(z)/2$, and, for L large enough,

$$\begin{aligned}
L^{d-2-3\alpha} f_d(z/2) \frac{\alpha_k(1-\epsilon)}{5^d} &\gg L^{-d-2-\eta} \alpha_k \\
&= \frac{1}{4(16L)^{d/d-1} \Sigma_{k+1}^2} \tag{2.137}
\end{aligned}$$

This gives a bound of the desired form for the first term in (2.132). The

bound on the second term follows easily from the estimates of Proposition 2.3.10. The proof of Proposition 2.3.12 is now finished. \square

Control of the ground-states. From our construction it follows that in a ground-state configuration, the spin at $x \in \mathbb{Z}^d$ will take on the value $+1$, if in no iteration of the renormalization group x will fall into the bas set D . But Proposition 2.3.12 implies that this is quite likely to be the case. More precisely, we get the:

Corollary 2.3.14 *Let $d \geq 3$, Σ^2 small enough. Then, there exists a constant c' (of order unity) such that for any $x \in \mathbb{Z}^d$*

$$\mathbb{P} \left[\exists_{k \geq 0} : N_{\mathcal{L}^{-k}x}^{(k)} \neq 0 \right] \leq \exp \left(-\frac{\delta^2}{c' \Sigma^{2-\frac{d-2}{d-1}}} \right) \quad (2.138)$$

Proof. The supremum in (2.138) is bounded from above by $\bar{N}_x^{(k)}$. Moreover, $\bar{N}_x^{(k)}$ is either zero or larger than $L^{-(d-3/2)k} \Sigma$. Therefore

$$\begin{aligned} \mathbb{P} [\exists_{k \geq 0} : N_{\mathcal{L}^{-k}x} \neq 0] &\leq \mathbb{P} \left[\exists_{k \geq 0} \bar{N}_{\mathcal{L}^{-k}x}^{(k)} \neq 0 \right] \\ &\leq \sum_{k=0}^{\infty} \exp \left(-L^{\left(\frac{d-2}{2(d-1)} - \eta\right)k} \frac{\delta^2}{a \Sigma^{2-\frac{d-2}{d-1}}} \right) \end{aligned} \quad (2.139)$$

which gives (2.138) for a suitable constant c' . \square

Let us denote by $D^{(k)}$, $\mathcal{D}^{(k)}$ the bad regions and ‘harmless’ bad regions in the k -th level. Set further

$$\Delta^{(k)} \equiv \bigcup_{i=0}^k \mathcal{L}^i \overline{\text{int } \mathcal{D}^{(i)}} \quad (2.140)$$

One may keep in mind that the sets $D^{(k)}$ depend in principle on the finite volume in which we are working; however, this dependence is quite weak and only occurs near the boundary. We therefore suppress this dependence in our notations.

In this terminology Corollary 2.3.14 states that even $\Delta^{(\infty)}$ is a very sparse set. This statement has an immediate implication for the ground-states, via the following

Proposition 2.3.15 *Let $\Lambda_n \equiv \mathcal{L}^n 0$, and let $\mathcal{G}_{\Lambda_n}^{(0)}$ be defined through (2.84). Then for any $\Gamma^* \in \mathcal{G}_{\Lambda_n}^{(0)}$,*

$$\underline{\Gamma^*} \subset \Delta^{(n-1)} \cup \mathcal{L}^n D^{(n)} \quad (2.141)$$

Proof. Let γ_i^* denote the maximal weakly connected components of Γ^* . It is clear that for all these components $\sigma_{\text{dint } \underline{\gamma_i^*}} = +1$. Let $\tilde{\gamma}_i^*$ denote the

‘outer’ connected component of γ_i^* , i.e. the connected component of γ_i^* is the connected component of γ_i^* with the property that the interior of its support contains all the supports of the connected components of γ_i^* . If $\tilde{\gamma}_i^*$ is ‘small’ (in the sense of Definition 2.3.5, since it occurs in a ground-state, by Lemma 2.3.2, it is ‘flat’ (i.e. $\sigma_x(\tilde{\gamma}_i^*) \equiv 0$) and its support is contained in $\overline{\mathcal{D}}$ (in the first step this set is even empty). Then all the other connected components of γ_i^* are also small, so that γ_i^* is flat and its support is contained in $\overline{\mathcal{D}}$. Thus $\underline{\Gamma}^* \subset \text{int } \underline{\Gamma}^{*,l} \cup \overline{\mathcal{D}}^{(0)}$. On the other hand, $\underline{\Gamma}^{*,l} \subset \mathcal{L}(\mathbb{T}\underline{\Gamma}^*)$; again the support of the small components of $\mathbb{T}\underline{\Gamma}^*$ will be contained, by the same argument, in the closure of the small parts of the new bad regions, and so $\underline{\mathbb{T}\Gamma}^{*,s} \subset \overline{\mathcal{D}}^{(1)}$, while $\underline{\mathbb{T}\Gamma}^{*,l} \subset \mathcal{L}(\text{int } (\mathbb{T}^2\underline{\Gamma}^{*,l}))$. This may be iterated as long as the renormalized contours still have non-empty supports; in the worst case, after n steps, we are left with $\mathbb{T}^n \underline{\Gamma}^*$, whose support consist at most of the single point 0, and this only if 0 is in the n -th level bad set $D^{(n)}$. But this proves the proposition. \square

The task of the next section will be to carry over these results to the finite-temperature case and the Gibbs measures.

2.3.3 The Gibbs states at finite temperature

In this section we repeat the construction and analysis of the renormalization maps for the finite temperature Gibbs measures. The steps will follow closely those of the previous section and we will be able to make use of many of the results obtained there. The probabilistic analysis will mostly carry over. The difficulties here lie in the technicalities of the various expansions that we will have to use.

Set-up and inductive assumptions. Just as in Section 2.3.2 an object of crucial importance will be the control field N_x . Given such a field, the bad region $D \equiv D(N)$ is defined exactly as in (2.77).

Analogously to Definition 2.3.2 we now define an N -bounded contour measure:

Definition 2.3.7 An N -bounded contour measure is a probability measure on $\mathcal{S}_n(D)$ of the form

$$\mu(\Gamma) = \frac{1}{Z} e^{-\beta(S, V(\Gamma))} \sum_{\Lambda_n \supset G \supset \Gamma} \rho(\Gamma, G) \quad (2.142)$$

where

- (i) S is a non-local small random field, that is, a map that assigns to each

connected (non-empty) set $C \subset \Lambda_n$ and sign \pm a real number S_C^\pm such that

$$|S_C^\pm| \leq e^{-\tilde{b}|C|}, \text{ if } |C| > 1 \quad (2.143)$$

and for sets made of a single point x ,

$$|S_x| \leq \delta \quad (2.144)$$

(ii) $\rho(\Gamma, G)$ are positive activities factorizing over connected components of G , i.e. if (G_1, \dots, G_l) are the connected components of G and if Γ_i denotes the contour made from those connected components of Γ whose supports are contained in G_i , then

$$\rho(\Gamma, G) = \prod_{i=1}^l \rho(\Gamma_i, G_i) \quad (2.145)$$

where it is understood that $\rho(\Gamma, G) = 0$ if $\underline{\Gamma} = \emptyset$. They satisfy the upper bound

$$0 \leq \rho(\Gamma, G) \leq e^{-\beta E_s(\Gamma) - \tilde{b}|G \setminus \overline{D(\Gamma)}| + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) + A|G \cap D(\Gamma)|} \quad (2.146)$$

Let $C \subset D$ be connected and $\gamma = (C, \sigma_x(\Gamma) \equiv s)$ be a connected component of a contour $\Gamma \subset \mathcal{S}_n(D)$. Then

$$\rho(\gamma, C) \geq e^{-\beta(N, V(\gamma) \cap C)} \quad (2.147)$$

Z is the partition function that turns μ into a probability measure.

Here β and \tilde{b} are parameters ('temperatures') that will be renormalized in the course of the iterations. In the k -th level, they will be shown to behave as $\beta^{(k)} = L^{(d-1-\alpha)k}$ and $\tilde{b}^{(k)} = L^{(1-\alpha)k}$. B and A are further k -dependent constants. B will actually be chosen close to 1, i.e. with $B = 1$ in level $k = 0$ we can show that in all levels $1 \leq B \leq 2$. A is close to zero, in fact $A \sim e^{-\tilde{b}^{(k)}}$. These constants are in fact quite irrelevant, but cannot be completely avoided for technical reasons. We have suppressed the dependence of μ and ρ on their parameters to lighten the notation.

The probabilistic assumptions are completely analogous to those in Section 2.3.2 and we will not restate them; all quantities depending on sets C are supposed to be measurable w.r.t. $\mathcal{B}_{\overline{C}}$.

The definition of a proper RG transformation will now be adopted to this set-up.

Definition 2.3.8 For a given control field N , a proper renormalization group transformation, $\mathbb{T}^{(N)}$, is a map from $\mathcal{S}_n(D(N))$ into $\mathcal{S}_{n-1}(D(N'))$, such that, if μ is an N -bounded contour measure on $\mathcal{S}_n(D(N))$ with 'temperatures' β and \tilde{b} and small field S (of level k), then $\mu'_{\Lambda_{n-1}} \equiv \mathbb{T}^{(N)}\mu_{\Lambda_n}$ is

an N' -bounded contour measure on $\mathcal{S}_{n-1}(D(N'))$ for some control field N' , with temperatures β' and \tilde{b}' and small field S' (of level $k+1$).

Absorption of small contours. The construction of the map T_1 on the level of contours proceeds now exactly as before, i.e. Definition 2.3.4 defines the harmless large field region, Definition 2.3.5 the ‘small’ contours, and Definition 2.3.6 the map T_1 . What we have to do is to control the induced action of T_1 on the contour measures. Let us for convenience denote by $\hat{\mu} \equiv Z\mu$ the non-normalized measures; this only simplifies notations since T_1 leaves the partition functions invariant (i.e. $T_1\mu = \frac{1}{Z}T_1\hat{\mu}$).

We have, for any $\Gamma^l \in \mathcal{S}_N^l(D)$,

$$\begin{aligned} (T_1\hat{\mu})(\Gamma^l) &\equiv \sum_{\Gamma: T_1(\Gamma)=\Gamma^l} \hat{\mu}(\Gamma) \\ &= \sum_{\Gamma: T_1(\Gamma)=\Gamma^l} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \Gamma} \rho(\Gamma, G) \end{aligned} \quad (2.148)$$

Now we write

$$(S, V(\Gamma)) = (S, V(\Gamma^l)) + [(S, V(\Gamma)) - (S, V(\Gamma^l))] \quad (2.149)$$

Here the first term is what we would like to have; the second reads explicitly

$$\begin{aligned} [(S, V(\Gamma)) - (S, V(\Gamma^l))] &= \left[\sum_x S_x \sigma_x(\Gamma) - S_x \sigma_x(\Gamma^l) \right] \\ &\quad + \sum_{\pm} \left[\sum_{\substack{C \subset V_{\pm}(\Gamma) \\ C \cap \text{int } \underline{\Gamma}^s \neq \emptyset}} S_C^{\pm} - \sum_{\substack{C \subset V_{\pm}(\Gamma^l) \\ C \cap \text{int } \underline{\Gamma}^s \neq \emptyset}} S_C^{\pm} \right] \\ &\equiv \delta S_{loc}(\Gamma, \Gamma^l) + \delta S_{nl}(\Gamma, \Gamma^l) \end{aligned} \quad (2.150)$$

where we used the suggestive notation $\underline{\Gamma}^s \equiv \Gamma \setminus \Gamma^l$. Note that all sets C are assumed to have a volume of at least 2 and to be connected. The conditions on C to intersect $\underline{\Gamma}^s$ just make manifest that otherwise the two contributions cancel. Thus all these unwanted terms are attached to the supports of the ‘small’ components of Γ . Thus, the local piece, δS_{loc} , poses no particular problem. The non-local piece, however, may join up ‘small’ and ‘large’ components, which spoils the factorization properties of ρ . To overcome this difficulty, we apply a cluster expansion. It is useful to introduce the notation

$$\tilde{\sigma}_{\Gamma, \Gamma^l}(C) \equiv \sum_{\pm} S_C^{\pm} (\mathbb{1}_{C \subset V_{\pm}(\Gamma)} - \mathbb{1}_{C \subset V_{\pm}(\Gamma^l)}) \quad (2.151)$$

so that

$$\delta S_{nl}(\Gamma, \Gamma^l) = \sum_{C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} \tilde{\sigma}_{\Gamma, \Gamma^l}(C) \quad (2.152)$$

Unfortunately the $\tilde{\sigma}_{\Gamma, \Gamma^l}(C)$ have arbitrary signs. Therefore, expanding $\exp(-\beta \delta S_{nl})$ directly would produce a polymer system with possibly negative activities. However, by assumption,

$$|\tilde{\sigma}_{\Gamma, \Gamma^l}(C)| \leq 2 \max_{\pm} |S_C^{\pm}| \leq 2e^{-\tilde{b}|C|} \equiv f(C) \quad (2.153)$$

Therefore, $\tilde{\sigma}_{\Gamma, \Gamma^l}(C) - f(C) \leq 0$ and setting

$$F(\text{int } \underline{\Gamma}^s) \equiv \sum_{C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} f(C) \quad (2.154)$$

we get

$$e^{-\beta \delta S_{nl}(\Gamma, \Gamma^l)} = e^{-\beta F(\text{int } \underline{\Gamma}^s)} e^{\beta \sum_{C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} (f(C) - \tilde{\sigma}_{\Gamma, \Gamma^l}(C))} \quad (2.155)$$

where the second exponential could be expanded in a sum over positive activities. The first exponential does not factor over connected components. However, it is dominated by such a term, and the remainder may be added to the Σ -terms. This follows from the next Lemma.

Lemma 2.3.16 *Let $A \subset \mathbb{Z}^d$ and let (A_1, \dots, A_l) be its connected components. Let $F(A)$ be as defined in (2.155) and set*

$$\delta F(A) \equiv F(A) - \sum_{i=1}^l F(A_i) \quad (2.156)$$

Then

$$\delta F(A) = - \sum_{C \cap A \neq \emptyset} k(A, C) f(C) \quad (2.157)$$

where

$$0 \leq k(A, C) f(C) \leq e^{-\tilde{b}(1-\kappa)|C|} \quad (2.158)$$

for $\kappa = \tilde{b}^{-1}$

Proof. The sum $\sum_{i=1}^l F(A_i)$ counts all C that intersect k connected components of A exactly k times, whereas in $F(A)$ such a C appears only once. Thus, (2.157) holds with $k(A, C) = \#\{A_i : A_i \cap C \neq \emptyset\} - 1$. Furthermore, if C intersects k components, then certainly $|C| \geq k$, from which the upper bound in (2.158) follows. \square

Now we can write the non-local terms in their final form:

Lemma 2.3.17 *Let $\delta S_{nl}(\Gamma, \Gamma^l)$ be defined in (2.150). Then*

$$\begin{aligned} e^{-\beta \delta S_{nl}(\Gamma, \Gamma^l)} &= r(\underline{\Gamma}^s) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{C_1, \dots, C_l \\ C_i \cap \text{int } \underline{\Gamma}^s \neq \emptyset \\ C_i \neq C_j}} \prod_{i=1}^l \phi_{\Gamma, \Gamma^l}(C_i) \\ &\equiv r(\underline{\Gamma}^s) \sum_{C: C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} \phi_{\Gamma, \Gamma^l}(C) \end{aligned} \quad (2.159)$$

where $\phi_{\Gamma, \Gamma^l}(C)$ satisfies

$$0 \leq \phi_{\Gamma, \Gamma^l}(C) \leq e^{-\tilde{b}|C|/2} \quad (2.160)$$

$r(\underline{\Gamma}^s)$ is a non-random positive activity factoring over connected components of $\text{int } \underline{\Gamma}^s$; for a weakly connected component γ^s ,

$$1 \geq r(\underline{\gamma}^s) \equiv e^{-\beta F(\text{int } \underline{\gamma}^s)} \geq e^{-\beta |\text{int } \underline{\gamma}^s| e^{-a\tilde{b}}} \quad (2.161)$$

with some constant $0 < a < 1$.

Proof. Define for $|C| \geq 2$

$$\sigma_{\Gamma, \Gamma^l}(C) \equiv \tilde{\sigma}_{\Gamma, \Gamma^l}(C) - f(C)(k(\text{int } \underline{\Gamma}^s, C) + 1) \quad (2.162)$$

Then we may write

$$\begin{aligned} e^{-\beta \sum_{C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} \sigma_{\Gamma, \Gamma^l}(C)} &= \prod_{C \cap \text{int } \underline{\Gamma}^s \neq \emptyset} \left(e^{-\beta \sigma_{\Gamma, \Gamma^l}(C)} - 1 + 1 \right) \\ &= \sum_{l=0}^{\infty} \sum_{\substack{C_1, \dots, C_l \\ C_i \cap \text{int } \underline{\Gamma}^s \neq \emptyset \\ C_i \neq C_j}} \prod_{i=1}^l \left(e^{-\beta \sigma_{\Gamma, \Gamma^l}(C_i)} - 1 \right) \end{aligned} \quad (2.163)$$

which gives (2.159). But since $|\sigma_{\Gamma, \Gamma^l}(C)| \leq 2e^{-\tilde{b}(1-\kappa)|C|}$ by (2.158) and the assumption on S_C , (2.160) follows if only $2\beta \leq e^{\tilde{b}(1-2\kappa)/2}$. Given the behaviour of β and \tilde{b} as given in the remark after Definition 2.3.8, if this relation holds for the initial values of the parameters, then it will continue to hold after the application of the renormalization group map for the new values of the parameters. The initial choice will be $\tilde{b} = \beta/L$, and with this relation we must only choose β large enough, e.g. $\beta \geq L(\ln L)^2$ will do.

The properties of $r(\underline{\Gamma}^s)$ follow from Lemma 2.3.16. These activities depend only on the geometry of the support of Γ^s and are otherwise non-random. \square

Next we write

$$\begin{aligned}
& (T_1 \hat{\mu})(\Gamma^l) \tag{2.164} \\
&= e^{-\beta(S, V(\Gamma^l))} \sum_{\Gamma: T_1(\Gamma)=\Gamma^l} r(\underline{\Gamma}^s) \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) e^{-\beta \delta S_{loc}(\Gamma, \Gamma^l)} \sum_{\mathcal{C}: \mathcal{C} \cap \text{int } \underline{\Gamma}^s \neq \emptyset} \phi_{\Gamma, \Gamma^l}(\mathcal{C}) \\
&= e^{-\beta(S, V(\Gamma^l))} \sum_{\Gamma: T_1(\Gamma)=\Gamma^l} \sum_{K \supset \underline{\Gamma}} \sum_{\underline{\Gamma} \subset G \subset K} \sum_{\substack{c \cap \text{int } \underline{\Gamma}^s \neq \emptyset \\ c \cup G = K}} r(\underline{\Gamma}^s) \rho(\Gamma, G) e^{-\beta \delta S_{loc}(\Gamma, \Gamma^l)} \phi_{\Gamma, \Gamma^l}(\mathcal{C})
\end{aligned}$$

We decompose the set K into its connected components and call K_1 the union of those components that contain components of $\underline{\Gamma}^l$. We set $K_2 = K \setminus K_1$. Everything factorizes over these two sets, including the sum over Γ (the possible small contours that can be inserted into Γ^l are independent from each other in these sets). We make this explicit by writing

$$\begin{aligned}
(T_1 \hat{\mu})(\Gamma^l) &= e^{-\beta(S, V(\Gamma^l))} \sum_{K_1 \supset \underline{\Gamma}^l} \sum_{\Gamma_1: T_1(\Gamma_1)=\Gamma^l} \sum_{\underline{\Gamma}_1 \subset G_1 \subset K_1} \sum_{\substack{c_1 \subset K_1 \\ c_1 \cap \text{int } \underline{\Gamma}_1^s \neq \emptyset \\ c_1 \cup G_1 = K_1}} \\
&\quad \times r(\underline{\Gamma}_1^s) \rho(\Gamma_1, G_1) e^{-\beta \delta S_{loc}(\Gamma_1, \Gamma^l)} \phi_{\Gamma_1, \Gamma^l}(\mathcal{C}_1) \\
&\quad \times \sum_{K_2: K_2 \cap \overline{K_1} = \emptyset} \sum_{\Gamma_2: T_1(\Gamma_2)=\Gamma^l} \sum_{\underline{\Gamma}_2 \subset G_2 \subset K_2} \sum_{\substack{c_2 \subset K_2 \\ c_2 \cap \text{int } \underline{\Gamma}_2^s \neq \emptyset \\ c_2 \cup G_2 = K_2}} \\
&\quad \times r(\underline{\Gamma}_2^s) \rho(\Gamma_2, G_2) e^{-\beta \delta S_{loc}(\Gamma_2, \Gamma^l)} \phi_{\Gamma_2, \Gamma^l}(\mathcal{C}_2) \\
&\equiv e^{-\beta(S, V(\Gamma^l))} \sum_{K_1 \supset \underline{\Gamma}^l} \hat{\rho}(\Gamma^l, K_1) \sum_{K_2: K_2 \cap \overline{K_1} = \emptyset} \tilde{\rho}(\Gamma^l, K_2) \tag{2.165}
\end{aligned}$$

Here, the contours Γ_1 and Γ_2 are understood to have small components with supports only within the sets K_1 and K_2 , respectively. Also, the set K_2 must contain $D(\Gamma^l) \cap K_1^c$. The final form of (2.165) is almost the original one, except for the sum over K_2 . This latter will give rise to an additional *non-local* field term, as we will now explain.

The sum over K_2 can be factored over the connected components of K_1^c . In these components, $\tilde{\rho}$ depends on Γ^l only through the (constant) value of the spin $\sigma(\Gamma^l)$ in this component. Let Y denote such a connected component. We have

Lemma 2.3.18 *Let $\tilde{\rho}$ be defined in (2.165). Then*

$$\sum_{\mathcal{D} \cap Y \subset K \subset Y} \tilde{\rho}(\Gamma^l, K) = e^{-\beta \sum_{\mathcal{C} \subset Y} \bar{\psi}_{\mathcal{C}} - \beta \sum_{\mathcal{C} \subset Y, \bar{\sigma}_{\mathcal{C}} \neq \emptyset} \psi_{\mathcal{C}}^s(Y)} \prod_i \tilde{\rho}'(\mathcal{B}_i^Y) \tag{2.166}$$

where \mathcal{B}_i^Y denote the connected components of the set $\mathcal{B}^Y \equiv \overline{\mathcal{D}} \cap Y = \mathcal{B}(\Gamma^l) \cap Y$

Y in Y . The sum over C is over connected sets such that $C \setminus \bar{D} \neq \emptyset$. The fields $\bar{\psi}_C$ are independent of Y and Γ^l . Moreover, there exists a strictly positive constant $1 > g > 0$, such that

$$|\bar{\psi}_C| \leq e^{-g\bar{b}|C \setminus \bar{D}|} \quad (2.167)$$

and

$$|\psi_C^s(Y)| \leq e^{-g\bar{b}|C \setminus \bar{D}|} \quad (2.168)$$

and a constant $C_1 > 0$ such that

$$\left| \frac{1}{\beta} \ln(\bar{\rho}'(\mathcal{B}_i^Y)) \right| \leq B \sum_{x \in \mathcal{D}_i} N_x + \frac{C_1}{\beta} |\mathcal{B}_i^Y| \quad (2.169)$$

Proof. Naturally, the form (2.166) will be obtained through a Mayer-expansion, considering the connected components of K as polymers subjected to a hard-core interaction. A complication arises from the fact that these polymers must contain the set $\mathcal{D} \cap Y$. Thus we define the set $\mathcal{G}(Y)$ of permissible polymers through

$$\mathcal{G}(Y) = \left\{ K \subset Y, \text{conn.}, K \cap \mathcal{B}(\Gamma^l) = \cup_{\mathcal{B}_i^Y \cap K \neq \emptyset} \mathcal{B}_i^Y \right\} \quad (2.170)$$

That is, any polymer in this set will *contain* all the connected components of $\bar{D} \cap Y$ it intersects. For such polymers we define the activities

$$\begin{aligned} \bar{\rho}'(K) &= \sum_{\substack{\tilde{K}: K \cap \mathcal{D} \subset \tilde{K} \\ \tilde{K} \setminus \mathcal{B}^Y = K \setminus \mathcal{B}^Y}} \sum_{\Gamma: T_1(\Gamma) = (\emptyset, \pm)} \quad (2.171) \\ &\times \sum_{\Gamma \subset G \subset \tilde{K}} \sum_{\substack{C \subset \tilde{K} \\ C \cap \text{int } \Gamma^s \neq \emptyset \vee C = \emptyset \\ C \cup G = \tilde{K}}} e^{-\beta \sum_{x \in \Gamma} [S_x \sigma_x(\Gamma) \mp S_x]} r(\Gamma) \rho(\Gamma, G) \phi_\Gamma(\mathcal{C}) \end{aligned}$$

Note that by summing over \tilde{K} we collect all polymers that differ only within \mathcal{B}^Y . Thus we get

$$\begin{aligned} &\sum_{\mathcal{D} \cap Y \subset K \subset Y} \bar{\rho}(\Gamma^l, K) \quad (2.172) \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{K_1, \dots, K_N: K_i \in \mathcal{G}(Y) \\ \cup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)}} \prod_{i=1}^N \bar{\rho}'(K_i) \prod_{1 \leq i < j \leq N} \mathbb{1}_{\overline{K_i} \cap K_j = \emptyset} \end{aligned}$$

Next we have to extract the contributions of those polymers that can occur in the ground-states. We set

$$\bar{\rho}(K) = \frac{\bar{\rho}'(K)}{\prod_{\mathcal{B}_i^Y \subset K} \bar{\rho}'(\mathcal{B}_i^Y)} \quad (2.173)$$

Then $\bar{\rho}(\pm, \mathcal{B}_i^Y) = 1$, i.e. the \mathcal{B}_i^Y play the role of the empty polymer. This

procedure allows us to remove the restriction $\bigcup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)$ in the following way. Set

$$\mathcal{G}'(Y) = \mathcal{G}(Y) \setminus \{\mathcal{B}_i^Y, i \in \mathbb{Z}\} \quad (2.174)$$

Each polymer K_i is either in $\mathcal{G}'(Y)$ or is one of the \mathcal{B}_i^Y . Moreover, once all the $K_i \in \mathcal{G}'(Y)$ are chosen, the hard-core interaction plus the constraint $\bigcup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)$ fix the remaining K_i uniquely up to permutations. Since their activities $\bar{\rho}$ are equal to one, the entire sum over these polymers outside $\mathcal{G}'(Y)$ just contributes a factor 1. Therefore

$$\begin{aligned} & \sum_{\mathcal{D} \cap Y \subset K \subset Y} \tilde{\rho}(\Gamma^l, K) \\ &= \prod_{\mathcal{B}_i^Y \subset Y} \tilde{\rho}'(\mathcal{B}_i^Y) \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{K_1, \dots, K_N \\ K_i \in \mathcal{G}'(Y)}} \prod_{i=1}^N \tilde{\rho}(K_i) \prod_{1 \leq i < j \leq N} \mathbb{1}_{\overline{K_i} \cap K_j = \emptyset} \end{aligned} \quad (2.175)$$

This is now (up to the prefactor) the standard form of a polymer partition function with hard-core interaction (see Section 5.2). It can be exponentiated and yields the estimates of the lemma provided we get the bound

$$\bar{\rho}(\pm, K) \leq e^{-c_2 \bar{b} |K \setminus \overline{\mathcal{D}}|} \quad (2.176)$$

on the activities. We skip the tedious details of these estimates that can be found in [10]. \square

Next we need to control the activities $\hat{\rho}(\Gamma^l, K)$. Our aim is to show that they satisfy bounds similar to the original ρ . This is similar to the proof of Lemma 2.3.18 and we skip the details.

We can now write the expression for $T_1 \hat{\mu}$ in the following pleasant form:

$$\begin{aligned} (T_1 \hat{\mu})(\Gamma^l) &= e^{-\beta(S, V(\Gamma^l))} \sum_{K \supset \Gamma^l} \hat{\rho}'(\Gamma^l, K) \prod_{\pm, i: \overline{\mathcal{D}_i} \subset V_{\pm}(\Gamma^l) \cap \overline{K}} \hat{\rho}'(\overline{\mathcal{D}_i}) \\ &\times \prod_{\pm, i: \mathcal{D}_i \subset V_{\pm}(\Gamma^l) \cap \overline{K}^c} \tilde{\rho}'(\overline{\mathcal{D}_i} \cap \overline{K}^c) \\ &\times \exp \left(-\beta(\bar{\psi}, V(\Gamma^l) \cap \overline{K}^c) - \beta \sum_{\substack{\pm, C: C \subset V_{\pm}(\Gamma^l) \cap \overline{K}^c \\ \overline{C} \cap \overline{K} \neq \emptyset}} \psi_C^s(K) \right) \end{aligned} \quad (2.177)$$

Here the $\tilde{\rho}'(\overline{\mathcal{D}_i})$ are independent of the contour and K and can be exponentiated to yield a nonlocal field. For the activities we have the following bounds.

Lemma 2.3.19

$$0 \leq \hat{\rho}'(\Gamma^l, K) \leq e^{-\beta E_s(\Gamma^l) - \frac{q\bar{b}}{2} |K \setminus \overline{\mathcal{D}(\Gamma^l)}| + \beta B(N, V(\Gamma^l) \cap \Gamma^l)} 25^{|K|} \quad (2.178)$$

For contours $\Gamma^l = (C, h_x \equiv h)$, with $C \subset D \setminus \mathcal{D}$ connected we have moreover

$$\hat{\rho}'(\Gamma^l, \underline{\Gamma}^l) \geq e^{-\beta B(N, V(\Gamma^l) \cap \underline{\Gamma}^l)} \quad (2.179)$$

Proof. Notice that $\hat{\rho}'(\Gamma^l, \underline{\Gamma}^l) = \hat{\rho}(\Gamma^l, \underline{\Gamma}^l) = \rho(\Gamma^l, \underline{\Gamma}^l)$ so that (2.179) follows from the assumptions on ρ . The upper bound (2.178) is proven in the same way as the upper bound on $\tilde{\rho}$, since small contours can be summed over in each connected component of the complement of $\underline{\Gamma}^l$ in K . \square

The blocking. We now turn to the main step of the RG transformation, the blocking. As before, nothing changes as far as the action of \mathbb{T} on contours is concerned and all we have to do is to study the effect on the contour measures.

First we exponentiate all terms in (2.177) that give rise to the new random fields. We set

$$z_C \equiv \sum_i \mathbb{I}_{C=\overline{\mathcal{D}}_i} \left(-\frac{1}{\beta} \ln (\hat{\rho}'(\overline{\mathcal{D}}_i)) \right) \quad (2.180)$$

Setting now

$$\tilde{S}_C^\pm \equiv S_C^\pm + z_C + \bar{\psi}_C \quad (2.181)$$

and noticing that

$$(\bar{\psi}, V(\Gamma^l) \cap \overline{K}^c) = (\bar{\psi}, V(\Gamma^l)) - \sum_{\substack{\pm, C \subset V_\pm(\Gamma^l) \\ C \cap \overline{K}^c \neq \emptyset}} \bar{\psi}_C \quad (2.182)$$

We have

$$\begin{aligned} (T_1 \hat{\mu})(\Gamma^l) &= e^{-\beta(\tilde{S}, V(\Gamma^l))} \sum_{K \supset \underline{\Gamma}^l} \hat{\rho}'(\Gamma^l, K) \left(\prod_{\substack{\mathcal{D}_i \subset V_\pm(\Gamma^l) \cap \overline{K}^c \\ \overline{\mathcal{D}}_i \not\subset \overline{K}^c}} \frac{\hat{\rho}'(\overline{\mathcal{D}}_i \cap \overline{K}^c)}{\hat{\rho}'(\overline{\mathcal{D}}_i)} \right) \\ &\times \exp \left(\beta \sum_{\substack{\pm, C \subset V_\pm(\Gamma^l) \\ C \cap \overline{K}^c \neq \emptyset}} \bar{\psi}_C - \beta \sum_{\substack{\pm, C: C \subset V_\pm(\Gamma^l) \cap \overline{K}^c \\ \overline{C} \cap \overline{K}^c \neq \emptyset}} \psi_C^s(K) \right) \end{aligned} \quad (2.183)$$

where the random field and the activity-like contributions are almost well separated. We first prepare the field term for blocking. For given $\Gamma' \subset \mathcal{S}_{n-1}(\mathcal{L}^{-1}D)$, we can split the term into three parts:

$$(\tilde{S}, V_\pm(\Gamma^l)) = L^{d-1-\alpha}(\tilde{S}', V_\pm(\Gamma')) + \delta \tilde{S}_{loc}(\Gamma^l, \Gamma') + \delta \tilde{S}_{nl}(\Gamma^l, \Gamma') \quad (2.184)$$

where for single points y

$$\tilde{S}'_y^\pm \equiv L^{-(d-1-\alpha)} \left(\sum_{x \in \mathcal{L}_y} \tilde{S}_x \sigma_x(\Gamma') + \sum_{\substack{C \subset V_\pm(\Gamma'): C \cap \mathcal{L}_y \neq \emptyset \\ d(C) < L/4 \forall C \subset \mathcal{L}_y}} \frac{\tilde{S}_C^\pm}{|\mathcal{L}^{-1}(C)|} \right) \quad (2.185)$$

and for $|C'| > 1$,

$$\tilde{S}'_{C'} \equiv L^{-(d-1-\alpha)} \sum_{\substack{C: \mathcal{L}^{-1}(C)=C' \\ d(C) \geq L/4}} \tilde{S}_C \quad (2.186)$$

Equations (2.185) and (2.186) are the analogues of (2.104) and almost the final definitions of the renormalized ‘small random fields’. Furthermore

$$\begin{aligned} \delta \tilde{S}_{loc}(\Gamma^l, \Gamma') &\equiv \sum_{y \in \Lambda_{n-1}} \left[\sum_{x \in \mathcal{L}y} \left(\tilde{S}_x \sigma_x(\Gamma^l) - \tilde{S}_x \sigma_{\mathcal{L}^{-1}x}(\Gamma') \right) \right. \\ &\quad \left. + \sum_{\substack{\pm, C: C \cap \mathcal{L}y \neq \emptyset \\ d(C) < L/4 \vee C \subset \mathcal{L}y}} \tilde{S}_C^\pm \left[\mathbb{1}_{C \subset V_\sigma(\Gamma^l)} - \frac{\mathbb{1}_{\sigma_y(\Gamma') = \pm}}{|\mathcal{L}^{-1}C|} \right] \right] \end{aligned} \quad (2.187)$$

and

$$\begin{aligned} \delta \tilde{S}_{nl}(\Gamma^l, \Gamma') &\equiv \sum_{\substack{\pm, C: C \subset \Lambda_{n-1} \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \tilde{S}_C^\pm \left[\mathbb{1}_{C \subset V_\pm(\Gamma^l)} - \mathbb{1}_{\mathcal{L}^{-1}C \subset V_\pm(\Gamma')} \right] \\ &\equiv \sum_{\substack{C: C \subset \Lambda_n \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \tilde{s}_{\Gamma^l, \Gamma'}(C) \end{aligned} \quad (2.188)$$

The point here is that the contributions from $\delta \tilde{S}_{loc}$ will factor over the connected components of the blocked K , while the non-local $\delta \tilde{S}_{nl}$ can be expanded and gives only very small contributions, due to the minimal size condition on the C occurring in it.

In a similar way we decompose the exponent on the last line of (2.183). Here it is convenient to slightly enlarge the supports of the ψ and to define

$$\tilde{\psi}_{\Gamma^l, K}(\tilde{C}) \equiv - \sum_{\substack{\pm, C \subset V_\pm(\Gamma^l), \bar{C} = \tilde{C} \\ C \cap \bar{K} \neq \emptyset}} \bar{\psi}_C + \sum_{\substack{\pm, C: C \subset V_\pm(\Gamma^l) \cap \bar{K}^c, \bar{C} = \tilde{C} \\ \bar{C} \cap \bar{K} \neq \emptyset}} \psi_C^s(K) \quad (2.189)$$

This has the advantage that now $\tilde{\psi}_{\Gamma^l, K}(C) = 0$ if $C \cap K = \emptyset$. We then decompose

$$\begin{aligned} &- \sum_{\substack{\pm, C \subset V_\pm(\Gamma^l) \\ C \cap \bar{K} \neq \emptyset}} \bar{\psi}_C + \sum_{\substack{\pm, C: C \subset V_\pm(\Gamma^l) \cap \bar{K}^c \\ \bar{C} \cap \bar{K} \neq \emptyset}} \psi_C^s(K) = \sum_{C: C \cap K \neq \emptyset} \tilde{\psi}_{\Gamma^l, K}(C) \\ &= \sum_{y \in \Lambda_{n-1}} \sum_{\substack{C \cap \mathcal{L}y \neq \emptyset \\ C \cap K \neq \emptyset \\ d(C) < L/4 \vee C \subset \mathcal{L}y}} \frac{\tilde{\psi}_{\Gamma^l, K}(C)}{|\mathcal{L}^{-1}C|} + \sum_{\substack{C \cap K \neq \emptyset \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \tilde{\psi}_{\Gamma^l, K}(C) \\ &\equiv \delta \psi_{loc}(\Gamma^l, K) + \delta \psi_{nl}(\Gamma^l, K) \end{aligned} \quad (2.190)$$

In all of the non-local terms only sets C give a contribution for which $C \cap \mathcal{L}(\mathcal{L}^{-1}K) \neq \emptyset$, $d(C) \geq L/4$ and $|\mathcal{L}^{-1}C| \geq 2$. Moreover, for connected C

with $d(C) > L/4$ we have that $|C| \leq \text{const}|C \setminus \bar{D}|$ and hence (2.167) implies $|\tilde{\psi}_{\Gamma^l, K}(C)| \leq e^{-\text{const}\tilde{b}|C|}$. The inductive hypothesis yields a similar estimate for \tilde{S} so that in fact

$$\cdot \left| \tilde{s}_{\Gamma^l, \Gamma'}(C) + \tilde{\psi}_{\Gamma^l, K}(C) \right| \leq e^{-\text{const}'\tilde{b}|C|} \equiv \tilde{f}(C) \quad (2.191)$$

In analogy to Lemma 2.3.17 we can therefore expand these contributions to get

$$\begin{aligned} e^{-\beta(\delta\tilde{S}_{nl}(\Gamma^l, \Gamma') + \delta\psi_{nl}(\Gamma^l, K))} &= R(K) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{C_1, \dots, C_l: C_i \neq C_j \\ C_i \cap \mathcal{L}(\mathcal{L}^{-1}K) \neq \emptyset \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \prod \Phi_{\Gamma^l, \Gamma', K}(C_i) \\ &\equiv \sum_{\substack{C: C \cap \mathcal{L}(\mathcal{L}^{-1}K) \neq \emptyset \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \Phi_{\Gamma^l, \Gamma', K}(C) \end{aligned} \quad (2.192)$$

where the activities Φ satisfy

$$0 \leq \Phi_{\Gamma^l, \Gamma', K}(C) \leq e^{-g'\tilde{b}|C|} \quad (2.193)$$

and $R(K)$ are non-random activities factoring over connected components of $\mathcal{L}(\mathcal{L}^{-1}K)$, satisfying, for a connected component,

$$1 \geq R(K) \equiv \exp\left(-\sum_{\substack{C \cap \mathcal{L}(\mathcal{L}^{-1}K) \neq \emptyset \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \tilde{f}(C)\right) \geq e^{-|\mathcal{L}(\mathcal{L}^{-1}K)|e^{-\frac{4}{L}\tilde{b}'}} \quad (2.194)$$

Note that in these bounds the terms \bar{D} no longer appear.

With these preparations we can now write down the blocked contour measures in the form

$$(\mathbb{T}T_1\hat{\mu})(\Gamma') = e^{-\beta L^{d-1-\alpha}(\tilde{S}', V(\Gamma'))} \sum_{G' \supset \underline{\Gamma}'} \rho'(\Gamma', G') \quad (2.195)$$

where

$$\begin{aligned} \rho'(\Gamma', G') &\equiv \sum_{G' \supset K' \supset \underline{\Gamma}'} \sum_{C': C' \cup K' = G'} \sum_{\Gamma^l: T_2(\Gamma^l) = \Gamma'} \sum_{\substack{K \supset \underline{\Gamma}^l \\ \mathcal{L}^{-1}K = K'}} \sum_{\substack{C: \mathcal{L}^{-1}C = C' \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1}C| \geq 2}} \\ &\times e^{-\beta(\delta\tilde{S}_{loc}(\Gamma', \Gamma^l) + \delta\psi_{loc}(\Gamma^l, K))} R(K) \hat{\rho}'(\Gamma^l, K) \Phi_{\Gamma^l, \Gamma', K}(C) \\ &\times \prod_{\substack{\mathcal{D}_i \subset V_{\pm}(\Gamma^l) \cap \bar{K}^c \\ \bar{\mathcal{D}}_i \not\subset \bar{K}^c}} \frac{\tilde{\rho}'(\bar{\mathcal{D}}_i \cap \bar{K}^c)}{\tilde{\rho}'(\bar{\mathcal{D}}_i)} \end{aligned} \quad (2.196)$$

Notice that by construction the C occurring in the local fields $\delta\tilde{S}$ and $\delta\psi$ cannot connect disconnected components of G' , and therefore $\rho'(\Gamma', G')$ factorizes over connected components of G' . The main task that is left is to

prove that ρ' yields an N' bounded contour measure for a suitably defined N' . As in Section 2.3.2, we define the preliminary new control field by

$$\tilde{N}'_y \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}} N_x \quad (2.197)$$

where N has been defined already in (2.142). We will now prove the following

Lemma 2.3.20 *Let \tilde{N}' be defined in (2.197) and set $\hat{D}' \equiv D(\tilde{N}')$. Then the activities ρ' defined in (2.196) factor over connected components of G' and for any connected G' ,*

$$\begin{aligned} 0 &\leq \rho'(\Gamma, G') \\ &\leq e^{-c_1 L^{d-1} \beta E_s(\Gamma') - c_2 L \tilde{b} |G' \setminus \overline{\hat{D}'(\Gamma')}| + L^{d-1-\alpha} \beta B(\tilde{N}', V(\Gamma') \cap G') + C_3 L^d |G'|} \end{aligned} \quad (2.198)$$

for some positive constants c_1, c_2, C_3 . For $\Gamma' = (C, h_y \equiv h)$, with $C \subset \hat{D}'$ connected,

$$\rho'(\Gamma', \underline{\Gamma}') \geq e^{-L^{d-1-\alpha} \beta B(\tilde{N}', V(\Gamma')) - e^{-\text{const.} \tilde{b}} |C|} \quad (2.199)$$

Proof. We will skip the cumbersome, but fairly straightforward proofs of these estimates. \square

Final tidying. Just as in Section 2.3.2 we must make some final changes to the definition of the small and control fields and in the definition of the contours to recover the exact form of N' -bounded contour models. We will also take care of the entropy terms that were created in the estimates in Lemma 2.3.20.

The definition of the local small fields (2.112) and the control fields (2.113) remain unchanged. The non-local small fields will be left unaltered, i.e. we simply set $S'_{C'}^\pm \equiv \tilde{S}'_{C'}^\pm$. The centring has no effect on the contour measures, as the effect cancels with the partition functions (which are *not* invariant under this last part of the RG map), except for some boundary effects that can be easily dealt with as in Section 2.3.2. The final result is then the following:

Proposition 2.3.21 *Let $\mathbb{T}^{(N)} \equiv T_3 T_2 T_1 : \mathcal{S}_n(D(N)) \rightarrow \mathcal{S}_{n-1}(D(N'))$ with T_1, T_2 and T_3 defined above; let N' and S' and ρ' be defined as above and let μ be an N -bounded contour measure at temperatures β and \tilde{b} of level k . Then $\mu' \equiv \mathbb{T}\mu$ is an N' -bounded contour measure with temperatures $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{b}' = L^{1-\alpha}\tilde{b}$ of level $k+1$, for suitably chosen $\alpha > 0$.*

Proof. This is again tedious book-keeping and will be skipped. \square

Remark 2.3.7 Let us briefly summarize where we stand now. Equation (2.142) provides a form of contour measures that remains invariant under renormalization. The specific form of the bounds on the activities is not so important, but they have three main features: The term $E_s(\Gamma)$ (in our case) weighs the renormalized configurations; the term $|G \setminus \overline{D(\Gamma)}|$ suppresses ‘bad histories’, i.e. contours that are images of ‘unlikely’ original configurations; and finally, the control field terms allow deviations from ground-states in exceptional regions; the probabilistic estimates must then ensure that such regions become less and less prominent.

Proof. (of the main theorem). From the definition of the renormalized small fields and control fields it is clear that the probabilistic estimates carried out in Section 2.3.2 apply unaltered at small temperatures provided the hypothesis of Proposition 2.3.21 holds, i.e. if the RG program can be carried through. We will now show how these estimates can be used to prove Theorem 2.3.1. The main idea here is that contours are suppressed outside the union of all the bad regions in all hierarchies and that this latter set is, by the estimates on the control fields, very sparse. Moreover, the randomness essentially only produces local deformations that are very weakly correlated over larger distances, and thus finite volume measures with plus and minus boundary conditions (whose existence follows from the FKG inequalities) will remain distinct.

Let us now assume that β is large enough, Σ small enough, and the parameters L , α , and η chosen such that the preceding results are all valid. We denote by $\mu_\Lambda \equiv \mu_{\Lambda, \beta}$ the finite volume measure in Λ with plus boundary conditions.

A key point needed to prove Theorem 2.3.1 is that

Lemma 2.3.22 *Under the assumptions of Theorem 2.3.1,*

$$\mathbb{P} \left[\lim_{M \uparrow \infty} \mu_{L^{M_0, \beta}}^+(\sigma_0 = +1) > 1/2 \right] > 0 \quad (2.200)$$

Given Lemma 2.3.22, Theorem 2.3.1 follows from the monotonicity properties of the Gibbs measures and ergodicity as in Corollary 1.4.2. \square

Proof. (of Lemma 2.3.22) Let us introduce, for any contour $\Gamma \subset \mathcal{S}_M(D)$ the notation γ_0 for the unique weakly connected component of Γ whose interior contains the origin. If no such component exists, γ_0 is understood to be the empty set. Then

$$\begin{aligned}
\mu_{\mathcal{L}^M 0}(\sigma_0 = -1) &= \sum_{G \subset \mathcal{L}^M 0, G \ni 0, \vee G = \emptyset} \mu_{\mathcal{L}^M 0}(\text{int } \underline{\gamma}_0 = G) \\
&\times \mu_{\mathcal{L}^M 0}(\sigma_0 = -1 \mid \text{int } \underline{\gamma}_0 = G) \\
&\leq \sum_{k=1}^M \sum_{\substack{G \subset \mathcal{L}^M 0, G \ni 0 \\ G \subset \mathcal{L}^k 0, G \not\subset \mathcal{L}^{k-1} 0}} \mu_{\mathcal{L}^M 0}(\text{int } \underline{\gamma}_0 = G)
\end{aligned} \tag{2.201}$$

The final estimate in (2.201) can be rewritten in the form

$$\mu_{\mathcal{L}^M 0}(\sigma_0 = -1) \leq \sum_{k=1}^M \alpha_M^{(k-1)} \tag{2.202}$$

where

$$\alpha_M^{(k)} \equiv \mu_{\mathcal{L}^M 0}(\text{int } \underline{\gamma}_0 \not\subset \mathcal{L}^k 0) \tag{2.203}$$

We must prove that $\alpha_M^{(k)}$ decays rapidly with k ; a crude estimate on $s_M^{(k)}$ will then suffice. The estimate on $\alpha_M^{(k)}$ is the Peierls-type estimate we alluded to before. It will tell us that it is indeed unlikely that a connected component with large support encircles the origin. Of course, such an estimate has to be conditioned on the environments. The precise form is:

Lemma 2.3.23 *Let $0 \leq k \leq M-1$ and let denote $F_{l,M} \subset \mathcal{A}$ the event*

$$F_{l,M} \equiv \left\{ d(D^{(l)}, 0) \leq \frac{L}{2} \right\} \tag{2.204}$$

Then there exists a constant $b > 0$ s.t.

$$\left\{ \alpha_M^{(k)} \geq e^{-b\bar{b}^{(k)}} \right\} \subset \bigcup_{l=k}^M F_{l,M} \tag{2.205}$$

The proof of this lemma will be postponed. Assuming Lemma 2.3.23, it is easy to prove Lemma 2.3.22.

Note first that the events $F_{l,M}$ are independent of M (recall that $D^{(k)}$ depends on the finite volume only near the boundary). Therefore,

$$\sum_{k=0}^{\infty} \mathbb{P} \left[\sup_{M \geq k} \alpha_M^{(k)} \geq e^{-b\bar{b}^{(k)}} \right] \leq \sum_{k=0}^{\infty} \mathbb{P}[F_{k,\infty}] + \sum_{k=0}^{\infty} \mathbb{P}[F_{k,k}] \tag{2.206}$$

Moreover, the probabilities of the events $F_{l,M}$ satisfy

$$\mathbb{P}[F_{k,M}] \leq L^d \exp \left(-L \left(\frac{d-2}{2(d-1)} - \eta \right) k \frac{\delta^2}{a\Sigma^{2-\frac{d-2}{d-1}}} \right) \tag{2.207}$$

and are estimated as in Corollary 2.3.14. Since $\sum_{k=1}^{\infty} e^{-b\bar{b}^{(k-1)}} \delta_k < C e^{-b\bar{b}} \delta_1$,

$$\sum_{k=1}^{\infty} \sup_{M \geq k} \alpha_M^{(k-1)} < \infty, \quad \mathbb{P} - \text{a.s.} \tag{2.208}$$

and choosing β large enough, this quantity is in fact smaller than $\frac{1}{2}$ with positive probability. This proves Lemma 2.3.22. \square

Proof. (of Lemma 2.3.23) For simplicity, let us fix $\Lambda \equiv \mathcal{L}^M 0$ and let us write $\mu^{(k)} \equiv \mathbb{T}^k \mu_{\Lambda, \beta}$ for the renormalized measures. The key observation allowing the use of the RG in this estimate is that, if Γ is such that $\gamma_0(\Gamma) \not\subset \mathcal{L}^k 0$, then $\text{int } \mathbb{T}^k(\Gamma) \ni 0$ (simply because a connected component of such a size cannot have become ‘small’ in only $k - 1$ RG steps). But this implies that

$$\mu(\gamma_0 \not\subset \mathcal{L}^k 0) \leq \mu^{(k)}(\text{int } \underline{\Gamma} \ni 0) \quad (2.209)$$

To analyse the right-hand side of this bound, we decompose the event $\text{int } \underline{\Gamma} \ni 0$ according to decomposition of contours in small and large parts: either 0 is contained in the interior of the support of Γ^l , or else it is in the interior of the support of Γ^s and *not* in that of Γ^l . That is

$$\mu^{(k)}(\text{int } \underline{\Gamma} \ni 0) \leq \mu^{(k)}(\text{int } \underline{\Gamma}^l \ni 0) + \mu^{(k)}(\text{int } \underline{\Gamma}^s \ni 0, \text{int } \underline{\Gamma}^l \not\ni 0) \quad (2.210)$$

If $\text{int } \underline{\Gamma}^l \ni 0$, then $\text{int } \underline{\mathbb{T}\Gamma} \ni 0$, which allows us to push the estimation of the first term in (2.210) into the next hierarchy; the second term concerns an event that is sufficiently ‘local’ to be estimated, as we will see. Iterating this procedure, we arrive at the bound

$$\mu(\gamma_0 \not\subset \mathcal{L}^k 0) \leq \sum_{l=k}^{M-1} \mu^{(l)}(\text{int } \underline{\Gamma}^s \ni 0, \text{int } \underline{\Gamma}^l \not\ni 0) + \mu^{(M)}(\underline{\Gamma} \ni 0) \quad (2.211)$$

The last term in (2.211) concerns a single-site measure and will be very easy to estimate. To bound the other terms, we have to deal with the non-locality of the contour measures. To do so, we introduce the non-normalized measure

$$\nu(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{1}_{G \ni 0} \quad (2.212)$$

For all G contributing to ν (i.e. containing the origin) we write $G_0 \equiv G_0(G)$ for the connected component of G that contains the origin. We then define further

$$\nu_s(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{1}_{G \ni 0} \mathbb{1}_{G_0 \cap \underline{\Gamma}^l = \emptyset} \quad (2.213)$$

and

$$\nu_l(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{1}_{G \ni 0} \mathbb{1}_{G_0 \cap \underline{\Gamma}^l \neq \emptyset} \quad (2.214)$$

Of course, $\nu = \nu_s + \nu_l$. Let us further set

$$m_s \equiv \frac{1}{Z} \sum_{\Gamma} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \Gamma} \rho(\Gamma, G) \mathbb{I}_{\text{int } \underline{\Gamma^s} \ni 0} \mathbb{I}_{\text{int } \underline{\Gamma^l} \not\ni 0} \mathbb{I}_{g_0 \cap \underline{\Gamma^l} = \emptyset} \quad (2.215)$$

and

$$m_l \equiv \frac{1}{Z} \sum_{\Gamma} e^{-\beta(S, V(\Gamma))} \sum_{G \supset \Gamma} \rho(\Gamma, G) \mathbb{I}_{\text{int } \underline{\Gamma^s} \ni 0} \mathbb{I}_{\text{int } \underline{\Gamma^l} \not\ni 0} \mathbb{I}_{g_0 \cap \underline{\Gamma^l} \neq \emptyset} \quad (2.216)$$

where $g_0 \equiv g_0(G, \Gamma)$ denotes the connected component of G that contains the maximal connected component of Γ_s whose interior contains the origin. (Note that, in general, $g_0 \neq G_0$). The point here is that

$$\mu(\text{int } \underline{\Gamma^s} \ni 0, \text{int } \underline{\Gamma^l} \not\ni 0) = m_s + m_l \quad (2.217)$$

We will shortly see that we can easily estimate m_s . On the other hand, the estimation of m_l can be pushed to the next RG level. Namely,

$$\sum_{\Gamma: T(\Gamma) = \Gamma'} \nu_l(\Gamma) \leq \nu'(\Gamma') \quad (2.218)$$

and

$$m_l \leq \nu'(\mathcal{S}) \quad (2.219)$$

To see why (2.218) holds, consider just the first two steps of the RG procedure. The point is that the G_0 contributing to ν_l , as they contain the support of a large component of Γ , are never summed over in the first RG step. In the second step (the blocking) they contribute to terms in which G' is such that $\mathcal{L}G' \supset G \ni 0$, and in particular $G' \ni 0$. Therefore

$$Z \sum_{\Gamma: T_2 T_1(\Gamma) = \tilde{\Gamma}'} \nu_l(\Gamma) \leq e^{-\beta'(S', V(\Gamma'))} \sum_{G' \supset \tilde{\Gamma}'} \rho'(\tilde{\Gamma}', G') \mathbb{I}_{G' \ni 0} \quad (2.220)$$

In the third step, finally, the number of terms on the right can only be increased, while the constant produced by centring the small fields cancels against the corresponding change of the partition function. This then yields (2.218).

Equation (2.219) is understood in much the same way. The set γ_0 is not summed away in the first step. But g_0 contains a small connected component γ_0 whose interior contains the origin. By the geometric smallness of these components, $\mathcal{L}^{-1}\underline{\Gamma}_0 = \{0\}$ and so $\mathcal{L}^{-1}G_0 \ni 0$, implying (2.219).

Iterating these two relations, we get, in analogy to (2.211),

$$\nu^{(l+1)}(\mathbb{I}) \leq \sum_{j=l+1}^{M-1} \nu_s^{(j)}(\mathbb{I}) + \nu^{(M)}(\mathbb{I}) \quad (2.221)$$

where the superscripts refer to the RG level. Combining all this, we get

$$\begin{aligned}
& \mu_{\mathcal{L}^{M_0}}(\gamma_0 \not\subset \mathcal{L}^k 0) \tag{2.222} \\
& \leq \sum_{l=k}^{M-1} \left[m_{s, \mathcal{L}^{M_0}}^{(l)} + \sum_{j=l+1}^{M-1} \nu_{s, \mathcal{L}^{M_0}}^{(j)}(\mathbb{1}) + \nu_{\mathcal{L}^{M_0}}^{(M)}(\mathbb{1}) \right] + \mu_{\mathcal{L}^{M_0}}^{(M)}(\underline{\Gamma} \ni 0) \\
& = \sum_{l=k}^{M-1} m_{s, \mathcal{L}^{M_0}}^{(l)} + \sum_{j=k+1}^{M-1} (j-k) \nu_{s, \mathcal{L}^{M_0}}^{(j)}(\mathbb{1}) + (M-k) \nu_{\mathcal{L}^{M_0}}^{(M)}(\mathbb{1}) \\
& \quad + \mu_{\mathcal{L}^{M_0}}^{(M)}(\underline{\Gamma} \ni 0)
\end{aligned}$$

All the terms appearing in this final bound can be estimated without recourse to further renormalization. The result is:

Lemma 2.3.24 *Let $F_{l,M} \subset \mathcal{A}$ be defined as in Lemma 2.3.23. Then there exists a positive constant $\bar{b} > 0$ such that*

$$\begin{aligned}
& \left\{ m_{s, \mathcal{L}^{M_0}}^{(l)} \geq e^{-\bar{b}\bar{b}^{(l)}} \right\} \subset F_{l,M} \\
& \left\{ \nu_{s, \mathcal{L}^{M_0}}^{(l)}(1) \geq e^{-\bar{b}\bar{b}^{(l)}} \right\} \subset F_{l,M} \tag{2.223}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \nu_{\mathcal{L}^{M_0}}^{(M)}(1) \geq e^{-\bar{b}\bar{b}^{(M)}} \right\} \subset F_{M,M} \\
& \left\{ \mu_{\mathcal{L}^{M_0}}^{(M)}(\underline{\Gamma} \ni 0) \geq e^{-\bar{b}\bar{b}^{(M)}} \right\} \subset F_{M,M} \tag{2.224}
\end{aligned}$$

Proof. Relations (2.3.24) are easy to verify as they refer to systems with a single lattice site. The proof of the two relations (2.223) is similar. We explain only for the first one. We suppress the index l in our notation.

$$\begin{aligned}
m_s &= \frac{1}{Z} \sum_{\substack{\gamma_0 \text{ small} \\ \text{int } \underline{\gamma_0} \ni 0}} \sum_{\substack{G_0 \supset \gamma_0 \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s, \Gamma_0^s \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} \rho(\Gamma_0^s, G_0) \\
& \times \sum_{G \cap G_0 = \emptyset} \sum_{\substack{\Gamma: \underline{\Gamma} \subset G \\ \text{int } \underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma \cup \Gamma_0^s))} \tag{2.225}
\end{aligned}$$

Note that the second line almost reconstitutes a partition function outside the region G_0 , except for the constraint on the support of Γ and the fact that the field term is not the correct one. This latter problem can be repaired by noting that

$$(S, V(\Gamma \cup \Gamma_0^s)) = (S, V(\Gamma) \setminus G_0) + \sum_{\pm} \sum_{\substack{C \subset V_{\pm}(\Gamma \cup \Gamma_0^s) \\ C \cap G_0 \neq \emptyset}} S_C^{\pm} \tag{2.226}$$

The second term on the right consists of a local term (i.e. involving only

C consisting of a single site x) that depends only on Γ_0^s , and the non-local one, which as in the previous instances is very small, namely

$$\left| \sum_{\pm} \sum_{\substack{C \subset V_{\pm}(\Gamma \cup \Gamma_0^s), |C| \geq 2 \\ C \cap G_0 \neq \emptyset}} S_C^{\pm} \right| \leq \text{Const} |G_0| e^{-\bar{b}} \quad (2.227)$$

Thus we get the upper bound

$$\begin{aligned} m_s &\leq \sum_{\substack{\gamma_0 \text{ small} \\ \text{int } \gamma_0 \ni 0}} \sum_{\substack{G_0 \supset \gamma_0 \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s, \Gamma_0^s \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} \rho(\Gamma_0^s, G_0) e^{-\beta(S_{loc, V}(\Gamma_0^s) \cap G_0)} e^{\text{Const} |G_0| e^{-\bar{b}}} \\ &\times \frac{1}{Z} \sum_{G \cap \overline{G_0} = \emptyset} \sum_{\substack{\Gamma: \Gamma \subset G \\ \text{int } \Gamma \ni G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \setminus G_0)} \end{aligned} \quad (2.228)$$

The last line has the desired form. A slight problem is that the contours contributing to the denominator are not (in general) allowed to have empty support in G_0 , as the support of any Γ must contain $D(\Gamma)$. However, G_0 is necessarily such that $D \cap G_0 \subset \mathcal{D}$, since otherwise G_0 would have to contain support from large contours. Thus, for given G_0 , we may bound the partition function from below by summing only over contours that within G_0 have $\sigma_x(\Gamma) \equiv +1$, and the support of those in G_0 is exactly given by $\mathcal{D} \cap G_0$. Treating the small-field term as above gives the lower bound on the partition function

$$\begin{aligned} Z &\geq \prod_{i: \mathcal{D}_i \subset G_0} \rho(\mathcal{D}_i, \mathcal{D}_i) e^{-\beta \sum_{x \in G_0} S_x} e^{-\text{Const} |G_0| e^{-\bar{b}}} \\ &\times \sum_{G \cap \overline{G_0} = \emptyset} \sum_{\substack{\Gamma: \Gamma \subset G \\ \text{int } \Gamma \ni G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \setminus G_0)} \end{aligned} \quad (2.229)$$

and so

$$m_s \leq \frac{1}{Z} \sum_{\substack{\gamma_0 \text{ small} \\ \text{int } \gamma_0 \ni 0}} \sum_{\substack{G_0 \supset \gamma_0 \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s, \Gamma_0^s \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} e^{2 \text{Const} |G_0| e^{-\bar{b}}} \quad (2.230)$$

$$\times e^{-\beta(S_{loc, V}(\Gamma_0^s) \cap G_0) + \beta \sum_{x \in G_0} S_x \sigma_x} \frac{\rho(\Gamma_0^s, G_0)}{\prod_{i: \mathcal{D}_i \subset G_0} \rho(\mathcal{D}_i, \mathcal{D}_i)} \quad (2.231)$$

Here the ρ 's appearing in the denominator are exactly those for which we have lower bounds. Note that for this reason we could not deal directly with expressions in which G_0 is allowed to contain large components of Γ . The estimation of the sums in (2.232) is now performed as in the absorption of small contours. Γ_0^s with non-constant spins give essentially no contribution, and due to the separatedness of the components \mathcal{D}_i , and the smallness of the total control field on one such component, the main contribution comes from the term where Γ_0^s has support in only one component \mathcal{D}_i . If there is

such a component that surrounds 0, this could give a contribution of order one. But on $F_{l,M}$ this is excluded, so that G_0 cannot be contained in \mathcal{D} and therefore

$$m_{s,\mathcal{L}^{M_0}}^{(l)} \leq \text{Const } e^{-b\tilde{b}^{(l)}} \quad (2.232)$$

as claimed. \square

From Lemma 2.3.24 and the bound (2.222), Lemma 2.3.23 follows immediately. \square

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