Phase Transitions in Ferromagnetic Ising Models with spatially dependent magnetic fields

Rodrigo Bissacot
Applied Math Department
rodrigo.bissacot@gmail.com
Institut of Mathematics and Statistics - IME USP - University of São Paulo

Marzio Cassandro
GSSI, Via. F. Crispi 7, 00167 L'Aquila, Italy
marzio.cassandro@gmail.com

Leandro Cioletti
Mathematics Department
Universidade de Brasília, 70910-900 Brasília - DF, Brasil
leandro.mat@gmail.com

Errico Presutti
GSSI, Via. F. Crispi 7, 00167 L'Aquila, Italy
errico.presutti@gmail.com

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Abstract

In this paper we study the nearest neighbor Ising model with ferromagnetic interactions in the presence of a space dependent magnetic field which vanishes as $|x|^{-\alpha}$, $\alpha > 0$, as $|x| \to \infty$. We prove that in dimensions $d \geq 2$ for all β large enough if $\alpha > 1$ there is a phase transition while if $\alpha < 1$ there is a unique DLR state.

1 Introduction

The Ising Model is one of the most studied subjects in Statistical Physics and will complete a century in a few years[1]. The literature about ferromagnetic Ising models on $\mathbb{Z}^d$, $d \geq 2$, is mainly focused on cases where the external magnetic field is constant. We will study ferromagnetic nearest neighbor hamiltonians of the form

$$H_\Lambda^w(\sigma) = -J \sum_{|x-y|=1, x,y\in\Lambda} \sigma(x)\sigma(y) - \sum_{x\in\Lambda} h(x)\sigma(x) - J \sum_{|x-y|=1, x\in\Lambda, y\notin\Lambda} \sigma(x)w(y) \qquad (1)$$

where Λ is any finite subset of $\mathbb{Z}^d$, $\sigma \in \{-1,1\}^\Lambda$ is a spin configuration in Λ, $w \in \{-1,1\}^{\Lambda^c}$ a boundary condition and $J > 0$ the interaction strength.

[1] Wilhelm Lenz introduced the model in 1920.

When the magnetic field $h(\cdot)$ is constant, that is $h(x) = h$ for all $x \in \mathbb{Z}^d$ and $h = 0$, then the classical Peierls' argument guarantees the existence of a phase transition. If instead $h \neq 0$ at all temperatures there is a unique DLR measure, as it follows from the Lee-Yang Theory and GHS inequalities. The absence of phase transitions comes from the differentiability of the free energy with respect to the parameter h.

Alternating signs fields on the lattice $\mathbb{Z}^2$ are considered in [15], constant fields on semi-infinite lattices are studied in [2, 11]. The magnetic field in all these models has some spatial symmetry. The challenging case of i.i.d. random magnetic fields on $\mathbb{Z}^d$ with zero mean has been studied in [1, 4, 6, 7, 8] and the case with positive mean in [10]. Some deterministic and not spatially symmetric fields have been considered in [3].

In this paper we study the hamiltonian (1) in $\mathbb{Z}^d$, $d \geq 2$, with a non negative, space dependent magnetic field $h(\cdot)$ of the form

$$h(x) = \begin{cases} \frac{h^*}{|x|^\alpha} & x \neq 0 \\ h^* & x = 0 \end{cases}, \quad \alpha > 0, h^* > 0 \tag{2}$$

where if $x = (x_1, \dots, x_d)$ then $|x| = \sum_{i=1}^d |x_i|$. Calling $Z^w_{\beta,h(\cdot),\Lambda}$ the corresponding partition function one can easily check that (along van Hove sequences)

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{\log Z^w_{\beta,h(\cdot),\Lambda}}{\beta|\Lambda|} = p_\beta$$

independently of the boundary conditions w. The limit p_β is equal to the thermodynamic pressure without magnetic fields (i.e. $h^* = 0$). This indicates that the presence of $h(\cdot)$ does not change the thermodynamics thus suggesting that a phase transition may occur for β large, just as when the magnetic field is absent. However surface effects are relevant in the analysis of phase transitions and indeed we shall prove in Theorem 5 that when $\alpha < 1$ there is a unique DLR measure, while when $\alpha > 1$ there is a phase transition for β large enough, see Theorem 1.

The existence of phase transitions at $\alpha > 1$ is based on the validity of the Peierls bounds for contours. The proof of uniqueness when $\alpha < 1$ at low temperatures is more involved and it is based on an iterative scheme introduced in [5]. For $\alpha = 1$ we have partial results but not a complete characterization.

2 Existence of phase transitions

In this section we shall prove:

Theorem 1. *Let $h(\cdot)$ be as in (2) with $\alpha > 1$. Then for β large enough there is a phase transition, namely the plus and minus Gibbs measures $\mu^\pm_{\beta,h(\cdot),\Lambda}$ converge weakly as $\Lambda \to \mathbb{Z}^d$ to mutually distinct DLR measures.*

As we shall see the result extends to $\alpha = 1$ under the additional assumption that h^* is small enough and to non negative magnetic fields which are "local perturbations" of (2) (by this we mean that the L^1 norm of the difference is finite). We shall first prove the theorem under a stronger assumption on the magnetic field, see (3) below, which allows to reproduce the Peierls' argument. We need some geometric notation that will be used extensively throughout the paper.

Definition 1. *Two sites* x *and* y *in* $\mathbb{Z}^d$ *are connected iff they are nearest neighbors. Given a finite set* K *in* $\mathbb{Z}^d$ *we call* $\bar{K}$ *its complement,* $\delta_{\rm out}(K)$ *the sites* $y \in \bar{K}$ *which are connected to sites* $x \in K$ *and* $\delta_{\rm in}(K)$ *those in* K *connected to sites in* $\bar{K}$*.* $|\partial K|$ *denotes the number of connected pairs* x, y *with* $x \in \delta_{\rm in}(K)$ *and* $y \in \delta_{\rm out}(K)$*.*

Lemma 2. *Let* $h(\cdot)$ *be any non negative magnetic field such that*

$$J|\partial\Delta| > 2\sum_{x\in\Delta} h(x) \tag{3}$$

for all finite regions $\Delta \subset \mathbb{Z}^d$*. Then for all* β *large enough there is a phase transition.*

Proof. We shall use (3) to prove the validity of the Peierls bounds, see (4) below. Then for all β large enough the weak limits of the Gibbs measures with plus and minus boundary conditions are distinct DLR measures $\mu^{\pm}_{\beta,h(\cdot)}$. We thus have a phase transition hence the lemma. We shall use later that $\mu^{\pm}_{\beta,h(\cdot)}$ have trivial σ-algebra at infinity so that they have disjoint support, see for instance the Georgii book, [12].

Proof of the Peierls bounds. Contours are geometric objects in the dual lattice $\mathbb{Z}^d_*$, namely call C_x, $x \in \mathbb{Z}^d$, the closed unit cube in $\mathbb{R}^d$ with center x, then $\mathbb{Z}^d_*$ is the union over all n.n. pairs x, y of the faces $C_x \cap C_y$. Given a spin configuration σ its contours γ are the maximal connected (in the sense of non void intersection) components of the union of all faces $C_x \cap C_y$ with $\sigma(x) \neq \sigma(y)$.

Let γ be a contour and $I(\gamma)$ the interior of γ, i.e. the points which are connected to ∞ only via paths which cross γ. Suppose γ is a minus contour i.e. $\sigma(y) = -1$ on $\delta_{\rm out}(I(\gamma))$. Denote by $Z^-_{I(\gamma);h(\cdot)}(\sigma_{I(\gamma)}(x) = 1, x \in \delta_{\rm in}(I(\gamma))$ the partition function in $I(\gamma)$ with magnetic field $h(\cdot)$, minus boundary conditions and with the constraint that $\sigma_{I(\gamma)}(x) = 1$ for all $x \in \delta_{\rm in}(I(\gamma)$. Then

$$\begin{aligned}
&Z^-_{I(\gamma);h(\cdot)}(\sigma_{I(\gamma)}(x) = 1, x \in \delta_{\rm in}(I(\gamma)))\\
&\qquad \le e^{\beta\sum_{x\in I(\gamma)} h_x}\ Z^-_{I(\gamma);h\equiv 0}(\sigma_{I(\gamma)}(x) = 1, x \in \delta_{\rm in}(I(\gamma)))\\
&\qquad \le e^{-2\beta J|\partial I(\gamma)|} e^{\beta\sum_{x\in I(\gamma)} h_x} Z^-_{I(\gamma);h\equiv 0}(\sigma_{I(\gamma)}(x) = -1, x \in \delta_{\rm in}(I(\gamma)))\\
&\qquad \le e^{-2\beta J|\partial I(\gamma)|} e^{2\beta\sum_{x\in I(\gamma)} h_x} Z^-_{I(\gamma);h(\cdot)}(\sigma_{I(\gamma)}(x) = -1, x \in \delta_{\rm in}(I(\gamma))).
\end{aligned}$$

Thus by (3) the weight of the contour γ is bounded by

$$\frac{Z^-_{I(\gamma);h(\cdot)}(\sigma_{I(\gamma)}(x) = 1, x \in \delta_{\rm in}(I(\gamma)))}{Z^-_{I(\gamma);h(\cdot)}(\sigma_{I(\gamma)}(x) = -1, x \in \delta_{\rm in}(I(\gamma)))} \le e^{-\beta J|\partial I(\gamma)|}. \tag{4}$$

Same bound holds for the plus contours. □

The proof of Theorem 1 will be obtained by reducing to magnetic fields for which (3) is satisfied, a task that will be achieved via a few lemmas where we shall extensively use the Isoperimetric Inequality (see [14] for a proof): for any finite $\Delta \subset \mathbb{Z}^d$ ($d \ge 2$)

$$|\Delta|^{\frac{d-1}{d}} \le \frac{|\partial\Delta|}{2d}.$$

Lemma 3. *Let $h(\cdot)$ be as in* (2) *with $\alpha > 1$. Then there is $C \equiv C(h^*, \alpha, d, J) > 0$ so that* (3) *holds for all finite regions Δ such that $|\Delta| > C$.*

Proof. Since $h(x)$ is a non increasing function of $|x|$, calling $B(0,R) := \{x : |x| \le R\}$ we have

$$\sum_{x\in\Delta} h(x) \le \sum_{x\in B(0,R)} h(x), \qquad \text{for } R \text{ such that } |B(0,R)| \ge |\Delta|$$

We claim that the condition $|B(0,R)| \ge |\Delta|$ is satisfied if

$$R = \text{ smallest integer } \ge c|\partial\Delta|^{\frac{1}{d-1}} \tag{5}$$

with c large enough. In fact, recalling that $|\partial B(0,n)| = 2d \cdot n^{d-1}$, we have $|B(0,R)| \ge aR^d$, $a > 0$ small enough, hence using the isoperimetric inequality

$$|B(0,R)| \ge aR^d \ge ac^d|\partial\Delta|^{\frac{d}{d-1}} \ge ac^d(2d)^{\frac{d}{d-1}}|\Delta| \ge |\Delta|$$

for c large enough.

Thus the lemma will be proved once we show that

$$\lim_{R\to\infty} \frac{1}{R^{d-1}} \sum_{|x|\le R} h(x) = 0.$$

Recalling that $|\partial B(0,n)| = 2d \cdot n^{d-1}$ this is implied by

$$\lim_{R\to\infty} \sum_{n=1}^{R} \frac{n^{d-1}}{R^{d-1}} \frac{1}{n^\alpha} = 0$$

whose validity follows from the Lebesgue dominated convergence theorem. The lemma is thus proved. □

Observe that when $\alpha = 1$ and h^* is small enough then (3) holds again for all finite regions Δ large enough. The proof is analogous except at the end as we only have

$$\limsup_{R\to\infty} \frac{1}{R^{d-1}} \sum_{|x|\le R} \frac{1}{|x|^\alpha} \le c$$

Lemma 4. *Let $h(\cdot)$ be as in* (2) *with $\alpha > 1$, then there is R so that* (3) *holds for all finite Δ when the magnetic field is $\hat{h}$:*

$$\hat{h}(x) = \begin{cases} 0 & \text{if } |x| \le R \\ h(x) & \text{if } |x| > R \end{cases}$$

Proof. Suppose $|\Delta| > C$, C the constant in Lemma 3, then

$$2\sum_{x\in\Delta} \hat{h}(x) \le 2\sum_{x\in\Delta} h(x) \le J|\partial\Delta|$$

Suppose next $|\Delta| \le C$, then by the Isoperimetric Inequality,

$$\sum_{x\in\Delta} \hat{h}(x) = \sum_{x\in\Delta;|x|>R} \hat{h}(x)$$
$$\le \frac{h^*|\Delta|}{R^\alpha} \le \frac{h^*|\partial\Delta|^{\frac{d}{d-1}}}{R^\alpha (2d)^{\frac{d}{d-1}}} \le \frac{h^* C^{\frac{1}{d-1}}|\partial\Delta|}{R^\alpha (2d)^{\frac{d}{d-1}}}$$

which is $\le J|\partial\Delta|$ for R sufficiently large. □

Proof of Theorem 1. Let $h(\cdot)$ be as in (2) with $\alpha > 1$. By Lemma 2 and 4 for β large enough there is a phase transition for the system with magnetic field $\hat{h}(\cdot)$, let $\mu^{\pm}_{\beta,\hat{h}(\cdot)}$ the corresponding DLR measures obtained as limit of the Gibbs measures with plus respectively minus boundary conditions. Call $\phi(x) := h(x) - \hat{h}(x) = \mathbf{1}_{|x|<R} h(x)$ and define the probability measures

$$d\nu^{\pm}_{\beta,h(\cdot)}(\sigma) := C_{\pm} e^{\beta\sum\phi(x)\sigma(x)} d\mu^{\pm}_{\beta,\hat{h}(\cdot)}(\sigma) \tag{6}$$

($C_{\pm}$ the normalization constants). We shall first check that they are DLR measures with magnetic field $h(\cdot)$. To have lighter notation we drop super and subscripts writing just ν, μ and C. We need to show that for any finite cube Λ large enough (we need below that $\Lambda \supset B(0,R)$) the ν conditional probability given $\sigma_{\bar{\Lambda}}$ is the Gibbs measure with magnetic field $h(\cdot)$. By the DLR property for μ we have

$$d\nu(\sigma) = C e^{\beta\sum\phi(x)\sigma(x)} \frac{e^{-\beta\hat{H}(\sigma_\Lambda|\sigma_{\bar{\Lambda}})}}{\hat{Z}_\Lambda(\sigma_{\bar{\Lambda}})} d\mu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}})$$

where $d\mu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}})$ is the marginal of μ on the spin configurations in $\bar{\Lambda}$. We then have

$$d\nu(\sigma) = C \frac{e^{-\beta H(\sigma_\Lambda|\sigma_{\bar{\Lambda}})}}{Z_\Lambda(\sigma_{\bar{\Lambda}})} \frac{Z_\Lambda(\sigma_{\bar{\Lambda}})}{\hat{Z}_\Lambda(\sigma_{\bar{\Lambda}})} d\mu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}})$$

By integrating over σ_Λ we get

$$d\nu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}}) = C \frac{Z_\Lambda(\sigma_{\bar{\Lambda}})}{\hat{Z}_\Lambda(\sigma_{\bar{\Lambda}})} d\mu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}})$$

hence

$$d\nu(\sigma) = \frac{e^{-\beta H(\sigma_\Lambda|\sigma_{\bar{\Lambda}})}}{Z_\Lambda(\sigma_{\bar{\Lambda}})} d\nu_{\bar{\Lambda}}(\sigma_{\bar{\Lambda}})$$

which proves the DLR property. Thus $d\nu^{\pm}_{\beta,h(\cdot)}(\sigma)$ are DLR measures with magnetic field $h(\cdot)$ and are absolutely continuous w.r.t. $\mu^{\pm}_{\beta,\hat{h}(\cdot)}$. Hence they also have disjoint supports and are therefore distinct. Theorem 1 is proved. □

3 Restricted ensembles and contour partition functions

We fix hereafter $h(x)$ as in (2) and we shall prove that

Theorem 5. *Let $h(\cdot)$ as in (2), then for any β large enough there is a unique DLR measure.*

In this section we shall prove some crucial estimates which will be used in the next section to prove Theorem 5 but which have an interest in their own right. Observe that when $h(\cdot)$ is given by (2) the condition (3) may fail for some Δ for instance a large ball centered at the origin.

With this in mind we classify the contours γ by saying that γ is "slim" if

$$J|\partial I(\gamma)| > 2 \sum_{x \in I(\gamma)} h(x) \tag{7}$$

see the proof of Lemma 2 for notation. We call "fat" the contours which do not satisfy (7). Following Pirogov-Sinai we then introduce plus-minus restricted ensembles where spin configurations are restricted in such a way that there are only slim contours. We thus define for any bounded region Λ the plus-minus restricted partition functions

$$Z_\Lambda^{\pm,\text{slim}} := \sum_{\sigma_\Lambda:\text{all contours are slim}} e^{-\beta H(\sigma_\Lambda | \pm \mathbf{1}_{\Lambda^c})}. \tag{8}$$

Obviously the pressures in the plus and minus ensembles are equal but the Pirogov-Sinai theory requires for the existence of a phase transition finer conditions on the finite volume corrections to the pressure namely that the latter differs from the limit pressure by a surface term. In our case the correction is larger than a surface term because $\alpha < 1$ as shown by the following:

Theorem 6. *For any β large enough there are positive constants c_1 and c_2 so that*

$$Z_\Lambda^{-,\text{slim}} \le c_1 e^{-\beta c_2 \sum_{x\in\Lambda} h(x)} Z_\Lambda^{+,\text{slim}}. \tag{9}$$

Proof. By repeating the proof of Theorem 1 and denoting by $E_\Lambda^{-,\text{slim}}$ the expectation w.r.t. the Gibbs measure in the minus restricted ensemble, we have for any $x \in \Lambda$:

$$E_\Lambda^{-,\text{slim}}(\sigma(x)) \le -1 + 2 \sum_{\gamma: I(\gamma) \ni 0} e^{-\beta J |\partial I(\gamma)|} = -m^*, \quad m^* > 0 \tag{10}$$

for β large enough. Then

$$\mu_{\beta,h(\cdot),\Lambda}^{-,\text{slim}} \Big[\frac{\sum_{x\in\Lambda} h(x)\sigma_\Lambda(x)}{\sum_{x\in\Lambda} h(x)} \le -\frac{m^*}{2} \Big] \ge \frac{m^*}{2 - m^*} \tag{11}$$

To prove (11) let X be a random variable with values in $[-1, 1]$ and P its law. Suppose that $E(X) \le -m^*$ and call $p := P[X \ge -m^*/2]$, then

$$-m^* \ge -1(1-p) - \frac{m^*}{2} p, \quad (1 - \frac{m^*}{2}) p \le (1 - m^*), \quad (1-p) \ge \frac{m^*}{2 - m^*}$$

hence (11).

Calling $Z_\Lambda^{-,\mathrm{slim}}(A)$ the partition function with the constraint A, we can rewrite (11) as:

$$\begin{aligned} Z_\Lambda^{-,\mathrm{slim}} &\le \frac{2-m^*}{m^*} Z_\Lambda^{-,\mathrm{slim}}\Big(\frac{\sum_{x\in\Lambda} h(x)\sigma_\Lambda(x)}{\sum_{x\in\Lambda} h(x)} \le -\frac{m^*}{2}\Big) \\ &\le \frac{2-m^*}{m^*} e^{-\beta\frac{m^*}{2}\sum_{x\in\Lambda} h(x)} Z_{\Lambda,h\equiv 0}^{-,\mathrm{slim}} \\ &= \frac{2-m^*}{m^*} e^{-\beta\frac{m^*}{2}\sum_{x\in\Lambda} h(x)} Z_{\Lambda,h\equiv 0}^{+,\mathrm{slim}}. \end{aligned}$$

By repeating the previous argument we get

$$Z_{\Lambda,h\equiv 0}^{+,\mathrm{slim}} \le \frac{2-m^*}{m^*} e^{-\beta\frac{m^*}{2}\sum_{x\in\Lambda} h(x)} Z_\Lambda^{+,\mathrm{slim}}$$

where $Z_\Lambda^{+,\mathrm{slim}}$ is the partition function with the contribution of the magnetic field $h(\cdot)$. This concludes the proof of the theorem. □

In the next section we shall use a corollary of Theorem 6 that we state after introducing some notation. The geometry is as follows:

Λ is a cube with center the origin, Δ a subset of Λ and K a subset of Δ which is union of disjoint connected set K_i where for each i the complement $\bar{K}_i$ of K_i has a unique maximally connected component (i.e. there are no "holes" in K_i). We also suppose that each K_i is fat, i.e.

$$J|\partial K_i| \le 2\sum_{x\in K_i} h(x)$$

and that $\delta_{\mathrm{out}}K \subset \Delta$, see Definition 1.

With Λ, Δ and K as above we denote by $\mathcal{X}_{\Lambda,\Delta,K,M}$, $M \subset \delta_{\mathrm{out}}\Delta$, the set of all configuration σ_Λ which have the following properties.

- $\sigma_\Lambda = -1$ on $\delta_{\mathrm{in}}\Delta$, $\sigma_\Lambda = -1$ on $M \subset \delta_{\mathrm{out}}\Delta$ and $\sigma_\Lambda = +1$ on $\delta_{\mathrm{out}}\Delta \setminus M$.
- $\sigma_\Lambda = -1$ on δK and $\sigma_\Lambda = +1$ on $\delta_{\mathrm{in}}K$.
- σ_Λ has only slim contours in $\Delta \setminus K$

We denote by $Z_\Lambda^\omega(\mathcal{X}_{\Lambda,\Delta,K,M})$ the partition function in Λ with constraint $\mathcal{X}_{\Lambda,\Delta,K,M}$ and boundary conditions ω. Then:

Corollary 1. *Under the same assumptions of Theorem 6*

$$Z_\Lambda^\omega(\mathcal{X}_{\Lambda,\Delta,K,M}) \le c_1 e^{-\beta c_2\sum_{x\in\Delta\setminus K} h(x)} e^{-2\beta J|\partial K|} e^{-2\beta J|\partial\Delta| + 4\beta J|M|} Z_\Lambda^\omega \tag{12}$$

In the applications of the next section the connected components of Δ should intersect some given set and this will enable to control the sum over Δ via the bound $e^{-2\beta J|\partial\Delta|}$.

The sum over K is instead controlled as follows. We introduce the fat-contours partition function on the whole $\mathbb{Z}^d$ as

$$Z^{\mathrm{fat}} := \sum_{n=0}^{\infty} \sum_{\gamma_1,..,\gamma_n}^{*} e^{-\beta J\sum|\partial I(\gamma_i)|} \tag{13}$$

where the sum $*$ refers to a sum over only fat contours such that $I(\gamma_i) \cap I(\gamma_j) = \emptyset$ for all $i \neq j$.

Theorem 7. *For any β large enough there is a positive constant c_3 so that*

$$Z^{\text{fat}} \leq c_3 \tag{14}$$

Proof. We order the points of $\mathbb{Z}^d$ in a way which respects the distance from the origin and given a contour γ we denote by $X(\gamma)$ the minimal point in γ with the given order. By the definition of fat contours and supposing $X(\gamma) \neq 0$,

$$J|\partial I(\gamma)| \leq 2 \sum_{x \in I(\gamma)} h(x) \leq \frac{2h^*}{|X(\gamma)|^\alpha} |I(\gamma)| \leq \frac{2h^* C_p}{|X(\gamma)|^\alpha} |\partial I(\gamma)|^{\frac{d}{d-1}}$$

where C_p is the isoperimetric constant. Hence

$$|\partial I(\gamma)| \geq (\frac{J}{2C_p h^*})^{d-1} |X(\gamma)|^{\alpha(d-1)}, \quad X(\gamma) \neq 0 \tag{15}$$

We write

$$\begin{aligned} Z^{\text{fat}} &= \sum_n \sum_{x_1,..,x_n} \sum^*_{\gamma_1,..,\gamma_n} \prod_{i=1}^n \mathbf{1}_{X(\gamma_i)=x_i} e^{-\beta J|\partial I(\gamma_i)|} \\ &\leq \prod_{x \in \mathbb{Z}^d} \Big(1 + \sum_{\gamma \text{ fat}: X(\gamma)=x} e^{-\beta J|\partial I(\gamma)|}\Big) \\ &= (1 + \sum_{\gamma \text{ fat}: X(\gamma)=0} e^{-\beta J|\partial I(\gamma)|}\Big) \prod_{x \neq 0} \Big(1 + \sum_{\gamma \text{ fat}: X(\gamma)=x} e^{-\beta J|\partial I(\gamma)|}\Big) \end{aligned}$$

which using (15) proves (14). □

Before moving to the next section with the proof of Theorem 5 we point out that by the Dobrushin's Uniqueness Theorem there is a unique DLR state also at high temperatures and since the system is ferromagnetic, uniqueness may be expected to hold at all temperatures. However the proof of such a statement when the external field is zero does not seem to extend easily to our case, see [9] and [13].

4 Uniqueness at low temperatures

In this section we prove Theorem 5. For any positive integer n we denote by Λ_n the cube with center the origin and side $2n+1$. We fix a positive integer L, eventually $L \to \infty$, and arbitrarily the spins outside Λ_L, denoting by μ_L the Gibbs measure on $\{-1,1\}^{\Lambda_L}$ with the given boundary conditions and external magnetic field as in (2).

Definitions.

- Given σ_{Λ_L}, $\Delta \subset \Lambda_L$, $B : B \cap \Delta = \emptyset$ we say that $x \in \Delta$ is $-$ connected in Δ to B if there is $X \subset \Delta$ such that: $x \in X$, X is connected to B and $\sigma_\Lambda \equiv -1$ on X.

- Let $\mathfrak{C}_L$ be the random set of sites $x \in \Lambda_L$ which are $-$ connected in Λ_L to $\Lambda_{L+1} \setminus \Lambda_L$ and let $\mathfrak{M}_k = \mathfrak{C}_L \cap \Lambda_{k+1} \setminus \Lambda_k$, $k < L$; $\mathfrak{M}_L = \Lambda_{L+1} \setminus \Lambda_L$.
- Given $k \le L$ and $M \subset \Lambda_{k+1} \setminus \Lambda_k$ we define $\mathfrak{C}_{k,M}(\sigma_{\Lambda_L})$ as the set of all $x \in \Lambda_k$ which are $-$ connected in Λ_k to M. In particular $\mathfrak{C}_{L,M} = \mathfrak{C}_L$ if $M = \Lambda_{L+1} \setminus \Lambda_L$.

Suppose $\mathfrak{C}_L = C$ then the spins in

$\delta_{\text{out}}(C \cup \bar{\Lambda}_L)$ are all equal to $+1$. Moreover if we change the configuration σ_Λ leaving unchanged the spins in $C' := (C \cup \bar{\Lambda}_L) \cup \delta_{\text{out}}(C \cup \bar{\Lambda}_L)$ we still have $\mathfrak{C}_L = C$.

Thus the spins in $\Lambda_L \setminus C'$ are distributed with Gibbs measure with plus boundary conditions. We shall prove that there exists $b^* < 1$ so that

$$\lim_{L \to \infty} \mu_L \Big[\mathfrak{C}_L \cap \Lambda_{L(1-b^*)} = \emptyset \Big] = 1 \tag{16}$$

which then proves that μ_L converges weakly to the plus DLR measure, which is the weak limit of Gibbs measures with plus boundary conditions. Thus any DLR measure is equal to the plus DLR measure and Theorem 5 is proved. We are therefore reduced to the proof of (16) which uses an iterative argument introduced in [5].

It readily follows from the definitions that for $k < L$:

$$\mathfrak{C}_L \cap \Lambda_k = \mathfrak{C}_{k,\mathfrak{M}_k}, \quad \mathfrak{M}_k = \mathfrak{C}_L \cap (\Lambda_{k+1} \setminus \Lambda_k). \tag{17}$$

The next property will be used to establish a connection with Corollary 1, it is therefore crucial in the proof of Theorem 5. We claim that:

$$\sigma_{\Lambda_L}(x) = 1 \ \text{ for all } x \text{ in } \delta_{\text{out}}(\mathfrak{C}_{k,\mathfrak{M}_k}) \setminus \mathfrak{M}_k \tag{18}$$

Proof: By definition $\sigma_{\Lambda_L}(x) = 1$ for all x as in (18) which are in Λ_k. It remains to consider all x as in (18) which are in $\Lambda_{k+1} \setminus \Lambda_k$. We argue by contradiction supposing $\sigma_{\Lambda_L}(x) = -1$. In such a case there is a path with all minuses which starts at x and ends in $\mathfrak{M}_k$. Since $\mathfrak{M}_k \subset \mathfrak{C}_L$ and $\mathfrak{C}_L$ is connected, then $x \in \mathfrak{C}_L$ which implies (since $x \in \Lambda_{k+1} \setminus \Lambda_k$) that $x \in \mathfrak{M}_k$, hence the contradiction. (18) is proved.

Before proceeding we need some extra notation:

Notation. We decompose $\mathfrak{C}_{k,M}$ into maximally connected components, each one of them is a connected set whose complement has an unbounded maximally connected component and maybe several maximally connected finite components. The latter are distinguished into fat and slim and we call $\bar{\mathfrak{C}}^{\text{fat}}_{k,M}$ and $\bar{\mathfrak{C}}^{\text{slim}}_{k,M}$ the union of all the fat, respectively slim ones.

It then follows directly from (18) that

$$\{\mathfrak{M}_k = M\} \cap \{\bar{\mathfrak{C}}^{\text{fat}}_{k,M} = K\} \cap \{\mathfrak{C}_{k,M} \cup \bar{\mathfrak{C}}^{\text{slim}}_{k,M} = \Delta\} \subset \mathcal{X}_{\Lambda,\Delta,K,M} \tag{19}$$

$\mathcal{X}_{\Lambda,\Delta,K,M}$ the set considered in Corollary 1.

We are now ready for the proof of Theorem 5. The basic point is that if $|\mathfrak{M}_{k_0}|$ is small for some k_0 then (with large probability) there is $k > k_0$ with $|\mathfrak{M}_k|$ even smaller. Iterating the argument we will then find a k where $|\mathfrak{M}_k| = 0$. The heuristic idea behind the proof of such properties is the following.

Suppose that $|\mathfrak{M}_{k_0}| = L^a$, $a > 0$, k_0 a fraction of L. Let $0 < a' < a$, fix a constant $b < 1$ suitably small and distinguish two cases:

$$|\mathfrak{M}_k| \le L^{a'} \quad \text{for some } k \in [k_0 - bL, k_0)$$

and the complement where

$$|\mathfrak{M}_k| > L^{a'} \quad \text{for all } k \in [k_0 - bL, k_0) \tag{20}$$

We argue that the event (20) has vanishing probability as $L \to \infty$. To this end we use (19) (with $k = k_0$) and Corollary 1 observing that (with the above notation) $|\Delta| \ge |\mathfrak{C}_{k_0,M}| \ge bLL^{a'}$, by (20). In (12) we then have a dangerous term $e^{4\beta J L^a}$ (which comes from $M = \mathfrak{M}_{k_0}$, $|M| \le L^a$), while the contribution of the magnetic field is bounded by $e^{-\beta c_2 (h^* L^{-\alpha}) bLL^{a'}}$. If

$$L^{1+a'-\alpha} > L^a$$

the magnetic field wins against the dangerous term. We need a lengthy counting argument to sum over all possible values of Δ, K and M which will be given in the end of the section and which will prove that with probability going to 1 as $L \to \infty$ we can reduce to the case $|\mathfrak{M}_k| \le L^{a'}$ for some $k \in [k_0 - bL, k_0)$.

We can satisfy the previous inequality with $a' = a - \frac{1-\alpha}{2}$ and then iterate the argument to prove that after finitely many steps we get $\mathfrak{M}_k = \emptyset$ and thus conclude the proof.

With this in mind we introduce the sequence a_n, $n \ge 0$, by setting

$$a_0 = d - 1, \;\; a_{n+1} = a_n - \frac{1-\alpha}{2} \tag{21}$$

and call n^* the largest integer such that $a_{n^*} \ge 0$. Let $s_0 = L$ and define recursively s_n for $1 \le n \le n^*$ by setting

$$s_n \text{ the largest } k \text{ not larger than } s_{n-1} \text{ such that } |\mathfrak{M}_k| \le L^{a_n} \tag{22}$$

and, if there is no k as in (22), we then set $s_n = 0$ and stop the sequence. Observe that if $|\mathfrak{M}_{s_{n-1}}| = 0$ then $s_n = s_{n-1}$. If not stopped earlier we define s_{n^*+1} as

$$s_{n^*+1} \text{ is the largest } k \text{ not larger than } s_{n^*} \text{ such that } |\mathfrak{M}_k| = 0 \tag{23}$$

setting $s_{n^*+1} = 0$ if k does not exist.

Let $b > 0$ be such that

$$bn^* < \frac{1}{100} \tag{24}$$

Then $\mathfrak{C}_L \cap \Lambda_{L(1-b^*)} = \emptyset$ in the set

$$\mathcal{G} := \bigcap_{1 \le n \le n^*+1} \{s_{n-1} - s_n \le bL\} \tag{25}$$

provided $b^* > 1/2$ so that (16) will follow once we prove that

$$\lim_{L \to \infty} \mu_L \Big[\mathcal{G} \Big] = 1. \tag{26}$$

We shall prove that for any $1 \le p \le n^* + 1$

$$\lim_{L \to \infty} \mu_L \Big[s_{p+1} < s_p - bL \; ; \; s_p \ge L - pbL \Big] = 0 \tag{27}$$

which yields (26).

We write $\mu_L\big[s_{p+1} < s_p - bL\ ;\ s_p \ge L - pbL\big]$ as the ratio of two partition functions, the one in the denominator is the full partition function $Z^\omega_{\Lambda_L}$, ω the boundary conditions outside Λ_L, while the one in the numerator will be simply called Z and it will be the object of our analysis. We decompose the configurations according to the value k of s_p and M of $\mathfrak{M}_k$. If $|M| = 0$ we do not have to prove anything so that in the sequel we tacitly suppose $|M| > 0$. We have

$$Z \le \sum_{L\ge k\ge L-pbL} \sum_{M\subset\Lambda_{k+1}\setminus\Lambda_k:|M|\le L^{a_p}} \sum_{C_{k,M}\subset\Lambda_k:|C_{k,M}|\ge bL^{1+a_{p+1}}} Z_{\Lambda_L}\Big(\mathfrak{M}_k = M; \mathfrak{C}_{k,M} = C_{k,M}\Big) \tag{28}$$

The sets $K = \bar{\mathfrak{C}}^{\rm fat}_{k,M}$ and $\bar{C}^{\rm slim}_{k,M} = \bar{\mathfrak{C}}^{\rm slim}_{k,M}$ are uniquely determined by $C_{k,M}$ and we can rewrite (28) as

$$Z \le \sum_{L\ge k\ge L-pbL} \sum_{M\subset\Lambda_{k+1}\setminus\Lambda_k:|M|\le L^{a_p}} \sum_{K,\Delta:|\Delta|\ge bL^{1+a_{p+1}}} Z_{\Lambda_L}\Big(\mathfrak{M}_k = M; \bar{\mathfrak{C}}^{\rm fat}_{k,M} = K; \mathfrak{C}_{k,M}\cup\bar{C}^{\rm slim}_{k,M} = \Delta\Big) \tag{29}$$

observing that $\Delta \subset \Lambda_k$ is the union of a finite number of disjoint connected sets (without "holes", see Section 3), say Δ_1,..,Δ_n, each one connected to M. K is the union of fat connected sets without holes each one contained in Δ. When we add a $*$ to the sum over K and Δ we mean that the sum is over sets with such a restriction. We then get from (29) after using (19) and (12)

$$Z \le \sum_{L\ge k\ge L-pbL} \sum_{M\subset\Lambda_{k+1}\setminus\Lambda_k:|M|\le L^{a_p}} \sum^{*}_{K,\Delta:|\Delta\setminus K|\ge bL^{1+a_{p+1}}} c_1 e^{-\beta c_2 h^* L^{-\alpha} bL^{1+a_{p+1}}} e^{-2\beta J|\partial K|} e^{-2\beta J|\partial\Delta|+4\beta J|M|} Z^\omega_{\Lambda_L} \tag{30}$$

where $Z^\omega_{\Lambda_L}$ is the full partition function. We next specify the maximal connected components of Δ, called $\Delta_1, .., \Delta_n$, and use Theorem 7 and (14) to perform the sum over K then getting

$$\frac{Z}{Z^\omega_{\Lambda_L}} \le \sum_{L\ge k\ge L-pbL} \sum_{M\subset\Lambda_{k+1}\setminus\Lambda_k:|M|\le L^{a_p}} \sum_{n\ge 1} \sum^{*}_{\Delta_1,..,\Delta_n} c_1 e^{-\beta c_2 h^* L^{-\alpha} bL^{1+a_{p+1}}} c_3^n e^{-2\beta J|\partial\Delta|+4\beta J L^{a_p}} \tag{31}$$

where the $*$ recalls that $\Delta_1, .., \Delta_n$ are mutually disjoint connected sets without holes each one connected to M, this implies that the sum is over $n \le |M| \le L^{a_p}$. Each Δ_i is then in one to one correspondence with $\delta_{\rm out}(\Delta_i)$, which is a $*$connected set which intersects M.

Thus we can bound the $*$ sum by summing over $n \le |M|$ disjoint $*$connected sets which intersect M. Hence

$$\sum_{n\ge 1} \sum^{*}_{\Delta_1,..,\Delta_n} e^{-2\beta J|\partial\Delta|} \le \sum_{n=1}^{|M|} \frac{M!}{n!(M-n)!} e^{-\beta c_4 n} \le \Big(1 + e^{-\beta c_4}\Big)^{|M|} \tag{32}$$

where c_4 is such that

$$e^{-\beta c_4} \geq \sum_{D \ni 0, D* \text{connected}} e^{-2\beta J|D|} \tag{33}$$

(33) holds for β large enough, see for instance Lemma 3.1.2.4 in [16].

Then recalling (31)

$$\frac{Z}{Z^{\omega}_{\Lambda_L}} \leq \sum_{L \geq k \geq L-pbL} \sum_{M \subset \Lambda_{k+1} \setminus \Lambda_k : |M| \leq L^{a_p}} c_1 e^{-\beta c_2 h^* L^{-\alpha} b L^{1+a_{p+1}}} c_3^{L^{a_p}} e^{4\beta J L^{a_p}} \Big(1 + e^{-\beta c_4}\Big)^{L^{a_p}} \tag{34}$$

We can now perform the sum over M which using the Stirling formula is bounded by $e^{c_5 L^{a_p} \log L}$, c_5 a suitable constant and thus get

$$\frac{Z}{Z^{\omega}_{\Lambda_L}} \leq L e^{c_5 L^{a_p} \log L} c_1 e^{-\beta c_2 h^* L^{-\alpha} b L^{1+a_{p+1}}} c_3^{L^{a_p}} e^{4\beta J L^{a_p}} \Big(1 + e^{-\beta c_4}\Big)^{L^{a_p}} \tag{35}$$

which recalling the definition of a_n proves that

$$\mu_L \Big[s_{p+1} < s_p - bL \; ; \; s_p \geq L - pbL \Big] \leq c_6 e^{-\beta \frac{c_2}{2} b L^{1+a_{p+1}-\alpha}} \tag{36}$$

thus proving (27) and hence (26).

5 Concluding remarks

We have proved that when the magnetic field is given by (2) for all β large enough there is a phase transition when $\alpha > 1$ while, if $\alpha < 1$, there is a unique DLR state. It seems plausible that uniqueness extends to all β but we do not have a proof. Using the random cluster representation uniqueness is related to the absence of percolation (see [9]), perhaps this can be useful to deal with this question. When $\alpha = 1$ and h^* small enough the proof of Section 2 applies and we thus have a phase transition. However, our proof of uniqueness does not extend to the case $\alpha = 1$ no matter how large is h^* and a different approach should be used maybe related to an extension of Minlos-Sinai or the Wulff shape problem.

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