# Multiplicity of self-similar solutions for a critical equation


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**Abstract**

We consider the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u) + \beta |u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

with $\beta > 0$, $f$ superlinear and $2^* := 2N/(N-2)$ for $N \geq 3$. We prove that, for each $k \in \mathbb{N}$, there exists $\beta^* = \beta^*(k) > 0$ such that the equation has at least $k$ pairs of solutions provided $\beta \in (0, \beta^*)$. In the proof we use variational methods for the (even) functional associated to the equation.



*Key words:* critical problems; symmetric functionals; self-similar solutions.



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## 1 Introduction

In this paper we consider the equation

$$(P) \qquad -\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u) + \beta |u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $2^* := 2N/(N-2)$ for $N \geq 3$ and $f$ is a superlinear function with subcritical growth. The number $\beta$ is a positive parameter whose smallness provides the existence of multiple solutions for the equation. According to the function space in which we seek solutions, they are forced to have a rapid decay at infinity.

Before presenting our main result let us make some comments about known results and motivations for our study. We first recall that $(P)$ is closely related to the study of self-similar solutions for the heat equation as quoted in the works of Haraux and Weissler [7], and Escobedo and Kavian [6] (see also [3,8]). In this direction, equations like $(P)$ arise naturally when one seek for solutions of the form $\omega(t,x) := t^{-1/(p-2)}u(t^{-1/2}x)$ for the evolution equation

$$\omega_t - \Delta \omega = |\omega|^{p-2}\omega, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N.$$

More precisely, $\omega(t,x)$ satisfies the previous equation if and only if $u : \mathbb{R}^N \to \mathbb{R}$ is a solution of $(P)$ with $f(u) = 1/(p-2)u$, $\beta = 1$ and $2^*$ replaced by the power $p$.

In what follows we summarize some known results about a related equation, namely

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda |u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

Although many of the following results can be stated in a more general setting we present only the simplest cases for simplicity. We first consider the subcritical case with $\lambda = 0$ and $2 < p < 2^*$. In this case there hold

1. the equation has s a positive solution
2. the equation possesees infinitely many solutions (possibly changing sign)

Concerning the critical case $p = 2^*$ with $q = 2$ we have that

3. the only solution for $\lambda \leq N/4$ is $u \equiv 0$
4. if $N = 4$ the problem has a positive solution if, and only if, $\lambda \in (N/4, N/2)$
5. if $N = 3$ there is a positive solution for $\lambda \in (1, 3/2)$ and there is no positive solution for $\lambda \geq 3/2$
6. if $N \geq 4$ and $\lambda \geq N/2$ then the problem has a (nodal) nontrivial solution

Even for $p = 2^*$ but now considering $2 < q < 2^*$, the following happen

7. if $N \geq 4$ the problem has a positive solution for any $\lambda > 0$
8. if $N = 3$ and $2 < q \leq 4$ the problem has a positive solution for large values of $\lambda$; there is no restriction on $\lambda$ if $4 < q < 2^*$

The results 1, 2, 4 and 5 above are proved by Escobedo and Kavian in [6]. As far we know, they were the first authors to deal with this class of equations by using variational methods. It is worthwhile to mention that the results in the subcritical case (items 1 and 2) are proved for a more general class of nonlinearities, namely for superlinear functions of the Ambrosetti-Rabinowitz type. All the existence and nonexistence results in [6] are extended to a more general class of problems in [3]. In particular, the nonexistence result quoted in item 3 above can be found in this last paper. Finally, items 6, 7 and 8 are proved in [8], where the authors also deal with a more general equation (like in [3]).

The results for the critical case above are clearly related to analogous one for the Brezis and Nirenberg [2] problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. There are several papers concerning the solvability of the above equation. The question of the number of solutions was studied, among others, by Silva and Xavier in [11]. There the authors consider a variant of the above problem and obtained several solutions depending on a small parameter on the critical term. In this paper we obtain an analogous result for the equation $(P)$ with small values of the parameter $\beta$.

In what follows we present our main assumptions on the nonlinearity $f$.

$(f_0)$ $f \in C(\mathbb{R}, \mathbb{R})$ is odd

$(f_1)$ there exists $a_1$, $a_2 \in \mathbb{R}$ and $p \in (2, 2^*)$ such that

$$|f(s)| \leq a_1 + a_2|s|^{p-1}$$

$(f_2)$ $\lim_{s \to 0} \dfrac{f(s)}{s} = l \in \mathbb{R}$

$(f_3)$ there exists $\theta > 2$ such that

$$0 < \theta F(s) := \theta \int_0^s f(\tau) \mathrm{d}\tau \leq f(s)s$$

The main result of this paper can be stated as follows.

**Theorem 1.1** *For any given $k \in \mathbb{N}$ there exists $\beta^* = \beta^*(k) > 0$ such that the equation $(P)$ has at leat $k$ pairs of solutions provided $\beta \in (0, \beta^*)$.*

According to condition $(f_0)$ the functional associated to the equation in $(P)$ is even. Thus, we can apply a version of the Symmetric Mountain Pass Theorem to get our solutions. Since the equation has critical growth and the domain is unbounded, the main problem here is the handling of the Palais-Smale sequences. We try to adapt the argument presented in [11]. However, since there the authors worked in a bounded domain, some convergence arguments do not apply. We overcome this new difficulty by using ideias introduced by Bianchi, Chabrowski and Szulkin [5,4], namely some kind of concentration-compactness principle at infinity (see also [12] for a related application).

We would like to emphasize that our result can be proved with many other assumptions on the subcritical term. For instance, we may suppose that $f$ satisfies conditions analogous to that presented in [11], which are different (and not comparable) to the usual Ambrosetti-Rabinowitz condition. We also could consider $p$-laplace type operator like in [9] with some minor modifications in our arguments. However, we prefer to consider the simplest case in order to emphasize the new ideas related with the proof of the Palais-Smale condition.

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$.

The paper is organized as follows. In the next section we present the variational framework to deal with $(P)$ and state a compactness result for the associated functional. By assuming this result we prove the main theorem. In the final Section 3 we prove the compactness result.

## 2 Proof of the main theorem

We start this section by noticing that the equation in $(P)$ can be rewritten in a divergence form. Indeed, if we set

$$K(x) := \exp(|x|^2/4), \qquad x \in \mathbb{R}^N,$$

a straightforward calculation shows that the equation in $(P)$ is equivalent to

$$-\mathrm{div}(K(x)\nabla u) = K(x)f(u) + \beta K(x)|u|^{2^*-2}u, \qquad x \in \mathbb{R}^N. \tag{2.1}$$

We shall denote by $X$ the Hilbert space obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int K(x)|\nabla u|^2\right)^{\frac{1}{2}}$$

which is induced by the inner product

$$(u,v) := \int K(x)(\nabla u \cdot \nabla v).$$

For each $q \in [2, 2^*]$ we denote by $L^q_K$ the following space

$$L^q_K := \left\{ u \text{ measurable in } \mathbb{R}^N \ : \ \|u\|_q := \left( \int K(x)|u|^q \right)^{1/q} < \infty \right\}.$$

Due to the rapid decay at infinity of the functions belonging to $X$ we have the following embedding result proved in [6].

**Proposition 2.1** *The embedding $X \hookrightarrow L^q_K$ is continuous for all $q \in [2, 2^*]$ and it is compact for all $q \in [2, 2^*)$.*

By using the above result we can prove that is well defined the functional $I_\beta : X \to \mathbb{R}$ given by

$$I_\beta(u) := \frac{1}{2} \int K(x)|\nabla u|^2 - \int K(x)F(u) - \frac{\beta}{2^*} \int K(x)|u|^{2^*},$$

where $F(s) := \int_0^s f(\tau)\mathrm{d}\tau$. Standard calculations and Proposition 2.1 show that $I_\beta \in C^1(X, \mathbb{R})$ and the derivative of $I_\beta$ at the point $u$ is given by

$$I'_\beta(u)v = \int K(x)\nabla u \cdot \nabla v - \int K(x)f(u)v - \beta \int K(x)|u|^{2^*-2}uv,$$

for any $v \in X$. Hence, the critical points of $I_\beta$ are precisely the weak solutions of equation (2.1).

For future reference we remark that the compactness of the embedding $X \hookrightarrow L^2_K$ and standard spectral theory for compact operators shows that the linear problem

$$(LP) \qquad\qquad -\mathrm{div}(K(x)\nabla u) = \lambda K(x)u, \qquad x \in \mathbb{R}^N,$$

has a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$. Moreover, if we denote by $\varphi_n$ a normalized eigenfunction associated to $\lambda_n$, the following variational inequality holds

$$\lambda_{j+1} \int K(x)u^2 \le \int K(x)|\nabla u|^2, \qquad \forall\, u \in \mathrm{span}\{\varphi_1, \ldots, \varphi_j\}^\perp. \tag{2.2}$$

Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$. We say that $I$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) if every sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ possesses a convergent subsequence. In order to prove our main results we shall use the following version of the Symmetric Mountain Pass Theorem (see [1]).

**Theorem 2.2** *Let $E = V \oplus W$ be a real Banach space with $\dim V < \infty$. Suppose $I \in C^1(E, \mathbb{R})$ is an even functional satisfying $I(0) = 0$ and*

($I^1$) *there are constants* $\rho$, $\alpha > 0$ *such that* $I|_{B_\rho(0)\cap W} \geq \alpha$ *;*

($I^2$) *there is a subspace* $\widetilde{V}$ *of* $E$ *with* $\dim V < \dim \widetilde{V} < \infty$ *such that* $\max_{u\in\widetilde{V}} I(u) \leq M$ *for some constant* $M > 0$*;*

($I^3$) *considering* $M > 0$ *given by* ($I_2$), $I$ *satisfies (PS)$_c$ for* $0 < c < M$.

*Then* $I$ *possesses at least* $\dim \widetilde{V} - \dim V$ *pairs of nontrivial critical points.*

As we will see the geometric conditions imposed by the above theorem are easily checked for superlinear functionals like ours. The difficulty in proving our main theorem is related with the compactness condition. So, the next result is the key point of our proof.

**Proposition 2.3** *Suppose* $f$ *satisfies* ($f_1$) *and* ($f_2$)*. Then, for any given* $M > 0$*, there exists* $\beta^* = \beta^*(M) > 0$ *such that, for any* $\beta \in (0, \beta^*)$*, the functional* $I_\beta$ *satisfies the (PS)$_c$ condition for any* $c \leq M$.

We postpone the proof of the above result for the next section. In what follows, we show how it can be used to prove our main theorem.

*Proof of Theorem 1.1.* Since the sequence of eigenvalues $\lambda_j$ of $(LP)$ goes to infinity there exists $j \in \mathbb{N}$ such that $l < \lambda_{j+1}$, where $l \in \mathbb{R}$ is given by ($f_2$). We set

$$V := \text{span}\{\varphi_1, \ldots, \varphi_j\}, \quad W := V^\perp,$$

in such way that $X = V \oplus W$.

Given $\varepsilon > 0$ satisfying $0 < l + \varepsilon < \lambda_{j+1}$, it follows from ($f_1$) and ($f_2$) that, for some $c_1 > 0$, there holds

$$|F(s)| \leq \frac{(l+\varepsilon)}{2}s^2 + c_1|s|^p, \qquad \forall\, s \in \mathbb{R}.$$

Hence, we can use the variational inequality (2.2) and the embedding $X \hookrightarrow L^q_K$ to get, for any $u \in W$,

$$\begin{aligned} I_\beta(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(l+\varepsilon)\int K(x)u^2 - c_1\|u\|_p^p - \frac{\beta}{2^*}\|u\|_{2^*}^{2^*} \\ &\geq \frac{1}{2}\left(1 - \frac{(l+\varepsilon)}{\lambda_{j+1}}\right)\|u\|^2 - c_2\|u\|^p - c_3\|u\|^{2^*}. \end{aligned}$$

Since $(l+\varepsilon) < \lambda_{j+1}$ and $2 < p < 2^*$, a straightforward computations provides $\alpha$, $\rho > 0$ such that $I_\beta(u) \geq \alpha$ whenever $u \in B_\rho(0) \cap W$, and therefore the functional $I_\beta$ verifies the condition ($I^1$) of Theorem 2.2.

Given $k \in \mathbb{N}$ we set $m := k + \dim V$, consider $\{\psi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ a collection of smooth function with disjoint supports and define

$$\widetilde{V} := \text{span}\{\psi_1, \ldots, \psi_m\}.$$

Notice that $\dim \widetilde{V} = m$ and that, for some $R > 0$, the support of any functions belonging to $\widetilde{V}$ is contained in $B_R(0) \subset \mathbb{R}^N$. Moreover, in view of $(f_3)$,

$$F(s) \geq c_4|s|^\theta - c_5, \qquad \forall\, s \in \mathbb{R},$$

with $c_4$, $c_5 > 0$. Then, for any $u \in \widetilde{V}$, we have that

$$I_\beta(u) \leq \frac{1}{2}\|u\|^2 - c_6\|u\|^\theta - c_5|B_R(0)| \to -\infty, \qquad \text{as } \|u\| \to +\infty,$$

where we have used the fact that all the norms in $\widetilde{V}$ are equivalent. Since $\dim \widetilde{V} < \infty$ the above expression provides $M > 0$ such that $\max_{u\in\widetilde{V}} I(u) \leq M$, that is, $I$ satisfies $(I^2)$.

We now apply Proposition 2.3 to to obtain $\beta^* \in \mathbb{R}$ such that $I_\beta$ verifies $(PS)_c$ for any $c \leq M$, provided $\beta \in (0, \beta^*)$. For such $\beta$, it follows from Theorem 2.2 that $I_\beta$ has at least $k = \dim \widetilde{V} - \dim V$ pairs of nontrivial critical points. This concludes the proof of the theorem. $\square$

## 3 The local Palais-Smale condition

We devote this section to the proof of Proposition 2.3. It will be done in several steps. From now on we denote by $\mathcal{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures over $\mathbb{R}^N$ and we shall assume that $(u_n) \subset X$ a $(PS)_c$ sequence at level $c \leq M$ for $I_\beta$, namely

$$I_\beta(u_n) \to c \leq M \qquad \text{and} \qquad I'_\beta(u_n) \to 0.$$

Since $K \geq 0$ and $f$ satisfies $(f_3)$, it is standard to check that $(u_n)$ is bounded in $X$. Hence, we may suppose that, for some $u \in X$, there hold

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } X, \\ u_n \to u \text{ strongly in } L^q_K, \text{ for any } 2 \leq q < 2^*, \\ u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $X$ we can easily check that $I'_\beta(u) = 0$.

In what follows we denote by $S_K$ the best constant for the embedding $X \hookrightarrow L^{2^*}_K$, namely

$$S_K := \inf_{u\in X\setminus\{0\}} \left\{ \int K(x)|\nabla u|^2,\ \int K(x)|u|^{2^*} = 1 \right\}.$$

The above inequality implies that $S_K(\int K(x)|u|^{2^*})^{2/2^*} \leq \int K(x)|\nabla u|^2$, for any $u \in X$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $X$ we can argue along the same lines of the proof the classical concentration-compactness principle due to Lions [10, Lemma I.1], by using this last weighted inequality instead of the usual Sobolev one, to obtain two measures $\mu$, $\nu \in \mathcal{M}(\mathbb{R}^N)$ such that,

$$K(x)|\nabla u_n|^2\,\mathrm{d}x \rightharpoonup \mu \geq K(x)|\nabla u|^2\mathrm{d}x + \sum_{j\in J}\mu_j\delta_{x_j}, \tag{3.2}$$

$$K(x)|u_n|^{2^*}\mathrm{d}x \rightharpoonup \nu = K(x)|u|^{2^*}\,\mathrm{d}x + \sum_{j\in J}\nu_j\delta_{x_j}, \tag{3.3}$$

with the above convergence holding weakly in the sense of measures, $J$ being an at most countable family, $\{x_j\}_{j\in J}$ a family of points in $\mathbb{R}^N$ and $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ families of nonnegative numbers verifying $S_K(\nu_j)^{2/2^*} \leq \mu_j$, for each $j \in J$.

**Lemma 3.1** *For each $j \in J$ we have either $\nu_j = 0$ or $\nu_j \geq (S_K/\beta)^{N/2}$.*

*Proof.* Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ in $B_2(0)^c$. For any fixed $j \in K$, we set $\phi_j^\varepsilon(x) := \phi\left((x - x_j)/\varepsilon\right)$, where $\varepsilon > 0$. We have that

$$\begin{aligned} I'_\beta(u_n)(u_n\phi_j^\varepsilon) = \int K(x)|\nabla u_n|^2\phi_j^\varepsilon + \int K(x)(\nabla u_n \cdot \nabla\phi_j^\varepsilon)u_n \\ - \int K(x)f(u_n)u_n\phi_j^\varepsilon - \beta\int K(x)|u_n|^{2^*}\phi_j^\varepsilon. \end{aligned} \tag{3.4}$$

Since $\phi$ has compact support, the strong convergence in (3.1), the definition of $\phi$ and the Lebesgue Theorem imply that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int K(x)f(u_n)u_n\phi_j^\varepsilon = 0. \tag{3.5}$$

By using Hölder's inequality and a change of variables we get

$$\left|\int K(x)(\nabla u_n \cdot \nabla\phi_j^\varepsilon)u_n\right| \leq \|\nabla\phi\|_{L^\infty(\mathbb{R}^N)}\|u_n\|\left(\int_{B_{2\varepsilon}(x_j)} K(x)|u_n|^2\,\mathrm{d}x\right)^{1/2}.$$

Since $(u_n)$ is bounded in $X$ and $u_n \to u$ strongly in $L^2(B_{2\varepsilon}(x_j))$, we infer from the above inequality that

$$\left|\int K(x)(\nabla u_n \cdot \nabla\phi_j^\varepsilon)u_n\right| \leq c_1\left(\int_{B_{2\varepsilon}(x_j)}\psi(x)^2\right)^{1/2},$$

for some $c_1 > 0$ and $\psi \in L^2(B_{2\varepsilon}(x_j))$ independent of $\varepsilon$. It follows from

Lebesgue Theorem that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int K(x)(\nabla u_n\cdot\nabla\phi_j^\varepsilon)u_n = 0.$$

Since $I'_\beta(u_n)\to 0$ and $(u_n\phi_j^\varepsilon)$ is bounded in $X$, we can take the lim sup as $n\to+\infty$ and the limit as $\varepsilon\to 0$ in the expression (3.4), and use all the above statements to get

$$0=\lim_{\varepsilon\to 0}\left(\int\phi_j^\varepsilon\,\mathrm{d}\mu-\beta\int\phi_j^\varepsilon\,\mathrm{d}\nu\right)\geq\mu_j-\beta\nu_j,$$

and therefore $\mu_j\leq\beta\nu_j$. Recalling that $S_K\nu_j^{2/2^*}\leq\mu_j$ we conclude that, if $\nu_j>0$, then $\nu_j\geq(S_K/\beta)^{N/2}$. The lemma is proved. □

**Lemma 3.2** *If*

$$\beta<\beta^*:=\left(\frac{S_K^{N/2}}{NM}\right)^{2/(N-2)}$$

*then* $\nu_j=0$ *for any* $j\in J$.

*Proof.* It follows from $(f_3)$ and $\theta>2$ that

$$\begin{aligned}c+o_n(1)&=I_\beta(u_n)-\frac{1}{2}I'_\beta(u_n)u_n\\&\geq\int K(x)\left(\frac{1}{\theta}f(u_n)u_n-F(u_n)\right)+\beta\left(\frac{1}{2}-\frac{1}{2^*}\right)\int K(x)|u_n|^{2^*}\\&\geq\frac{\beta}{N}\int K(x)|u_n|^{2^*}.\end{aligned}\tag{3.6}$$

If $\nu_j\neq 0$ for some $j\in J$ then, taking the limit in the above expression and using the last lemma, we obtain

$$M\geq c\geq\frac{\beta}{N}\nu_j\geq\frac{\beta}{N}\left(\frac{S_K}{\beta}\right)^{N/2}>M,$$

which does not make sense. □

In view of the above lemma and (3.2) we conclude that

$$K(x)|u_n|^{2^*}\mathrm{d}x\rightharpoonup K(x)|u|^{2^*}\mathrm{d}x\ \text{ weakly in the sense of measures.}\tag{3.7}$$

Hence, we have the following convergence result, which is the keystone for the proof of the main result of this section.

**Lemma 3.3** *If*

$$\beta < \beta^* := \left( \frac{S_K^{N/2}}{NM} \right)^{2/(N-2)}$$

*then, up to a subsequence,*

$$\lim_{n\to\infty} \int K(x)|u_n|^{2^*} = \int K(x)|u|^{2^*}.$$

*Proof.* The pointwise convergence in (3.1) and Fatou's Lemma provide

$$\int K(x)|u|^{2^*} \leq \liminf_{n\to\infty} \int K(x)|u_n|^{2^*},$$

and therefore it suffices to prove that

$$\limsup_{n\to\infty} \int K(x)|u_n|^{2^*} \leq \int K(x)|u|^{2^*}. \tag{3.8}$$

By assuming that $\beta > 0$ satisfies the hypothesis of the lemma we first prove that, for any $R > 0$, there holds

$$\lim_{n\to\infty} \int_{B_R(0)} K(x)|u_n|^{2^*} \mathrm{d}x = \int_{B_R(0)} K(x)|u|^{2^*} \mathrm{d}x. \tag{3.9}$$

Indeed, for any fixed $k \in \mathbb{N}$ we consider $\varphi_k \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi_k \leq 1$, $\varphi_k \equiv 1$ in $B_{R-(1/k)}(0)$ and $\varphi_k \equiv 0$ in $B_R(0)^c$. It follows from (3.7) that

$$\begin{aligned} \liminf_{n\to\infty} \int_{B_R(0)} K(x)|u_n|^{2^*} \mathrm{d}x &\geq \liminf_{n\to\infty} \int \varphi_k(x) K(x)|u_n|^{2^*} \mathrm{d}x \\ &= \int \varphi_k(x) K(x)|u|^{2^*} \mathrm{d}x. \end{aligned}$$

By taking $k \to \infty$ and using Lebesgue's Theorem we conclude that

$$\liminf_{n\to\infty} \int_{B_R(0)} K(x)|u_n|^{2^*} \mathrm{d}x \geq \int_{B_R(0)} K(x)|u|^{2^*} \mathrm{d}x.$$

A dual argument with $\varphi^k \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi^k \leq 1$, $\varphi^k \equiv 1$ in $B_R(0)$ and $\varphi^k \equiv 0$ in $B_{R+(1/k)}(0)^c$ proves that

$$\limsup_{n\to\infty} \int_{B_R(0)} K(x)|u_n|^{2^*} \mathrm{d}x \leq \int_{B_R(0)} K(x)|u|^{2^*} \mathrm{d}x$$

and therefore (3.9) holds.

In view of the convergence in (3.9) we have that

$$\limsup_{n\to\infty}\int K(x)|u_n|^{2^*} = \limsup_{n\to\infty}\left(\int_{|x|\geq R}+\int_{B_R(0)}\right)$$
$$= \limsup_{n\to\infty}\int_{|x|\geq R}K(x)|u_n|^{2^*}\mathrm{d}x+\int_{B_R(0)}K(x)|u|^{2^*}\mathrm{d}x.$$

Letting $R\to+\infty$ we conclude from Lebesgue's Theorem that

$$\limsup_{n\to\infty}\int K(x)|u_n|^{2^*} = \nu_\infty+\int K(x)|u|^{2^*}, \tag{3.10}$$

where

$$\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R}K(x)|u_n|^{2^*}\mathrm{d}x.$$

**Claim.** we have either $\nu_\infty=0$ or $\nu_\infty\geq (S_K/\beta)^{N/2}$

By assuming the claim let us show that the second alternative above does not occurs. As in (3.6), for any $R>0$ we have that

$$c+o_n(1)\geq \frac{\beta}{N}\int K(x)|u_n|^{2^*}\geq \frac{\beta}{N}\int_{|x|\geq R}K(x)|u_n|^{2^*}\mathrm{d}x$$

and therefore

$$M\geq c\geq \lim_{R\to\infty}\limsup_{n\to\infty}\frac{\beta}{N}\int_{|x|\geq R}K(x)|u_n|^{2^*}\mathrm{d}x=\frac{\beta}{N}\nu_\infty.$$

Since $\beta<(S_K^{N/2}N^{-1}M^{-1})^{2/(N-2)}$ we can argue as in the the proof of Lemma 3.2 to conclude that $\nu_\infty=0$. Thus, (3.8) follows from (3.10) and the lemma is proved.

It remains to prove the claim. In order to achieve this objective we adapt some arguments of [4,5] (see also [13]) by introducing another quantity which measures the loss of mass at infinity, namely

$$\mu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R}K(x)|\nabla u_n|^2\mathrm{d}x$$

Following the same arguments of [13, Lemma 1.40] and the weighted inequality $S_K(\int K(x)|u|^{2^*})^{2/2^*}\leq\int K(x)|\nabla u|^2$ for $u\in X$, we can prove that $S_K\nu_\infty^{2/2^*}\leq \mu_\infty$.

For $R>0$ fixed we consider a cut-off function $\psi_R$ such that $0\leq\psi_R\leq 1$, $\psi_R\equiv 0$ in $B_R(0)$ and $\psi_R\equiv 1$ in $B_{R+1}(0)^c$. Since $(u_n\psi_R)$ is bounded in $X$ we have that $I'_\beta(u_n)(u_n\psi_R)=o_n(1)$, and therefore

$$-A_n+\int K(x)f(u_n)u_n\psi_R = C_n-\beta D_n+o_n(1),$$

where

$$A_n := \int K(x)(\nabla u_n \cdot \nabla \psi_R) u_n, \qquad C_n := \int \psi_R K(x) |\nabla u_n|^2,$$

$$D_n := \int \psi_R K(x) |u_n|^{2^*}.$$

By taking the lim sup as $n \to \infty$ and recalling the strong convergence in (3.1) we obtain

$$\begin{aligned}\limsup_{n\to\infty}(-A_n) + \int K(x) f(u) u \psi_R &= \limsup_{n\to\infty}(C_n - \beta D_n) \\ &\geq \limsup_{n\to\infty} C_n - \beta \limsup_{n\to\infty} D_n.\end{aligned}$$

By Lebesgue Theorem we have that $\lim_{R\to\infty} \int K(x) f(u) u \psi_R = 0$, and therefore the above expression can be rewritten as

$$\limsup_{n\to\infty}(-A_n) \geq \limsup_{n\to\infty} C_n - \beta \limsup_{n\to\infty} D_n + o_R(1), \tag{3.11}$$

with $o_R(1)$ denoting a quantity approaching zero as $R \to \infty$.

By Hölder's inequality,

$$\begin{aligned}-A_n &\leq \|u_n\| \left( \int_{B_{R+1}(0)\setminus B_R(0)} K(x) |u_n|^2 |\nabla \psi_R|^2 \right)^{1/2} \\ &\leq c_1 \left( \int_{B_{R+1}(0)\setminus B_R(0)} K(x) |u_n|^2 \mathrm{d}x \right)^{1/2},\end{aligned}$$

and therefore the local convergence in (3.1) implies that

$$\limsup_{n\to\infty}(-A_n) \leq c_1 \left( \int_{B_{R+1}(0)\setminus B_R(0)} K(x) |u|^2 \mathrm{d}x \right)^{1/2}.$$

We infer from the above inequality and Lebesgue Theorem that

$$\limsup_{R\to\infty} \limsup_{n\to\infty}(-A_n) \leq 0. \tag{3.12}$$

Moreover, since

$$\int_{\mathbb{R}^N \setminus B_{R+1}(0)} K(x) |\nabla u_n|^2 \,\mathrm{d}x \leq C_n \leq \int_{\mathbb{R}^N \setminus B_R(0)} K(x) |\nabla u_n|^2 \,\mathrm{d}x,$$

we have that

$$\lim_{R\to\infty} \limsup_{n\to\infty} C_n = \mu_\infty. \tag{3.13}$$

The same argument implies that

$$\lim_{R\to\infty} \limsup_{n\to\infty} D_n = \nu_\infty. \tag{3.14}$$

Hence, by taking the lim sup as $R \to \infty$ in (3.11) and using (3.12)-(3.14) we conclude that $\mu_\infty \le \beta\nu_\infty$. Since $S_K \nu_\infty^{2/2^*} \le \mu_\infty$ we conclude that, if $\nu_\infty > 0$, then $\nu_\infty \ge (S_K/\beta)^{N/2}$. The claim is proved. □

We are now ready to prove the main result of this section.

*Proof of Proposition 2.3.* Since $I'_\beta(u_n)u_n = o_n(1)$, the strong convergence in (3.1) and Lemma 3.3 imply that

$$\begin{aligned} o_n(1) &= \|u_n\|^2 - \int K(x) f(u_n) u_n - \beta \int K(x) |u_n|^{2^*} \\ &= \|u_n\|^2 - \int K(x) f(u) u - \beta \int K(x) |u|^{2^*} + o_n(1) \\ &= \|u_n\|^2 - \|u\|^2 + I'_\beta(u) u + o_n(1). \end{aligned} \tag{3.15}$$

Since $I'_\beta(u) = 0$ we infer that $\|u_n\| \to \|u\|$. This and the weak convergence of $u_n$ in $X$ implies that $u_n \to u$ strongly in $X$ and proves the proposition. □

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## References


[1] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Func. Anal. **14** (1973), 349-381.

[2] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math. **36** (1983), 437-477.

[3] F. Catrina, M. F. Furtado, M. Montenegro, *Positive solutions for nonlinear elliptic equations with fast increasing weights*, Proc. Royal Soc. Edinburgh **137A** (2007), 1157-1178.

[4] J. Chabrowski, *Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents*, Calc. Var. Partial Differential Equations **3** (1995), 493-512.

[5] G. Bianchi, J. Chabrowski and A. Szulkin, *On symmetric solutions of elliptic equations with a nonlinearity involving critical Sobolev exponent*, Nonlinear Anal. **25** (1995), 41-59.

[6] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*. Nonlinear Anal. **11** (1987), no. 10, 1103-1133.

[7] A. Haraux and F.B. Weissler, *Nonuniqueness for a semilinear initial value problem*. Indiana Univ. Math. J. **31** (1982), no. 2, 167-189.

[8] M.F. Furtado, O.H. Miyagaki e J.P.P. da Silva, *On a class of nonlinear elliptic equations with fast increasing weight and critical growth*, J. Differential Equations **249** (2010), 1035-1055.

[9] H. Ohya, *Existence results for some quasilinear elliptic equations involving critical Sobolev exponents*, Adv. Differential Equations **9** (2004), 1339-1368.

[10] P.L. Lions, *The concentration compactness principle in the calculus of variations. The limit case. I.*, Rev. Mat. Iberoamericana **1** (1985), 145-201.

[11] E.A.B. Silva and M.S. Xavier, *Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 341-358.

[12] Y. Wang, Y. Zhang and Y. Shen, *Multiple solutions for quasilinear Schrödinger equations involving critical exponent*, Applied Math. and Computation **216** (2010), 849-856.

[13] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.