A note on the existence of positive solution for a non-autonomous Schrödinger-Poisson system *

Marcelo F. Furtado, Liliane A. Maia

Universidade de Brasília - Departamento de Matemática

70910-900, Brasília - DF Brazil

e-mail: mfurtado@unb.br, lilimaia@unb.br

Everaldo S. Medeiros

Universidade Federal da Paraíba-Departamento de Matemática

58051-900, João Pessoa - PB Brazil

e-mail: everaldo@mat.ufpb.br

Abstract

We consider the system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \qquad (S)$$

where $3 < p < 5$ and the potentials $K(x), a(x)$ and $V(x)$ has finite limits as $|x| \to +\infty$. By imposing some conditions on the decay rate of the potentials we obtain the existence of a nonzero weak solution. In the proof we apply variational methods.

1 Introduction

In this note we are concerned with the existence of a positive solution for the nonlinear system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \qquad (S)$$

where $3 < p < 5$ and the potentials $K(x), a(x)$ and $V(x)$ satisfy some basic assumptions.

*The three authors were partially supported by CNPq/Brazil. The first two authors were partially supported by PROEX/CAPES, UnB.

As quoted in the paper [4], this system arises in many interesting physical context. According to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and the Maxwell's equations. In particular, if one is looking for electrostatic-type solutions, it is natural to solve (S). In many papers the potential V has been supposed constant or radial (see for instance [1, 2, 8] and references therein). Here, motivated by the recent results by G. Cerami and G. Vaira [6] we will assume the following hypotheses:

(H_1) there exist c_K, $\alpha > 0$ such that

$$0 \leq K(x) \leq c_K e^{-\alpha|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3;$$

(H_2) a, $V \in C(\mathbb{R}^3, \mathbb{R})$ are positive continuous functions such that

$$\lim_{|x|\to+\infty} V(x) = V_\infty > 0, \qquad \lim_{|x|\to+\infty} a(x) = a_\infty > 0. \tag{1.1}$$

Furthermore, it is necessary to have some control on the asymptotic behavior of the potentials V and a. So, we also assume that

(H_3) there exist c_V, c_a, γ, $\theta > 0$ such that, for each $x \in \mathbb{R}^3$, there hold

$$V(x) \leq V_\infty + c_V e^{-\gamma|x|}, \qquad a(x) \geq a_\infty + c_a e^{-\theta|x|}, \tag{1.2}$$

with $\theta < \min\{\gamma, \alpha\} \leq \max\{\gamma, \alpha\} < 2\sqrt{V_\infty}$.

Our main result can be stated as follows:

Theorem 1.1 *If $(H_1) - (H_3)$ hold, then the system (S) has a positive ground state solution.*

For the proof, we use an approach similar to that of [6]. It consists in applying the Mountain Pass Theorem together with some sort of Splitting Lemma. This former result enables us to overcome the lack of compactness of the Sobolev embeddings caused by the fact the the problem is set in whole space $\mathbb{R}^N$. Hence, we need to perform a careful investigation of the behavior of the Palais-Smale sequences for the energy functional associated with system (S). Actually, we identify the levels in which the Palais-Smale condition can fail, giving a representation theorem for such sequences, and showing that the only obstacle to prove compactness are the solutions of the *limit problem*

$$-\Delta w + V_\infty w = a_\infty |w|^{p-1} w, \;\; x \in \mathbb{R}^3. \tag{P_∞}$$

In [6] the authors considered the same problem with $V \equiv 1$ and some integrability conditions on the function $a(x) - a_\infty$. By assuming that the L^2-norm of the weight K is smaller than a number related with the least energy level of two limit problems, they obtained the existence of a positive ground state solution. On the other hand, in [10] G. Vaira supposed that $V \equiv 1$, $a(x) \to a_\infty$, $K(x) \to K_\infty$

as $|x| \to +\infty$, with $a_\infty, K_\infty > 0$. Under some integrability conditions on $a(x) - a_\infty$ and $K(x) - K_\infty$, and some other mild conditions on the potentials, she also obtained a positive solution. Our Theorem 1.1 complements (and is not comparable with) the existence results of [6, 10].

We finally point out that a slight modification of our approach allows us to drop condition (H_3) by the following one (see Remark 3.3):

$(\widetilde{H_3})$ there exist c_V, c_a, γ, $\theta > 0$ such that, for each $x \in \mathbb{R}^3$, there hold

$$V(x) \leq V_\infty - c_V e^{-\gamma|x|}, \qquad a(x) \geq a_\infty - c_a e^{-\theta|x|},$$

with $\gamma < \min\{\theta, \alpha\} \leq \max\{\theta, \alpha\} < 2\sqrt{V_\infty}$.

The paper is organized as follows: in the next section we present the variational setting of the problem and state the compactness lemma that we shall use. In Section 3 we prove the main theorem.

2 The variational setting

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^3} u(x)\mathrm{d}x$. For each $u \in W^{1,2}(\mathbb{R}^3)$ we define

$$\|u\| := \left(\int (|\nabla u|^2 + V(x)u^2)\right)^{1/2}.$$

It follows from (H_2) that $\|\cdot\|$ is a norm which is equivalent to the usual one of $W^{1,2}(\mathbb{R}^3)$. For any $A \subset \mathbb{R}^3$ and $u \in L^p(A)$ we denote $\|u\|_{L^p(A)} := (\int_A |u|^p \mathrm{d}x)^{1/p}$. If $A = \mathbb{R}^3$ we write only $\|u\|_p$. Moreover, in what follows, without any loss of generality, we assume that $a_\infty = 1$.

Since $K \in L^2(\mathbb{R}^3)$, a straightforward application of the Lax-Milgram theorem implies that, for any given $u \in W^{1,2}(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int \nabla\phi_u \cdot \nabla v = \int K(x)u^2 v, \quad \text{for all } v \in D^{1,2}(\mathbb{R}^3).$$

Actually, the function ϕ_u weakly solves $-\Delta\phi = K(x)u^2$ and we can construct the application $\phi : W^{1,2}(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ which associates to each $u \in W^{1,2}(\mathbb{R}^3)$ the function $\phi(u)$ as above. From simplicity we write only ϕ_u to denote $\phi(u)$. We collect below some properties of the map ϕ (see [6, Lemma 2.1]).

Lemma 2.1 *The following hold:*

1. *ϕ is continuous and maps bounded sets into bounded sets;*
2. *$\phi_{tu} = t^2\phi_u$, for any $u \in W^{1,2}(\mathbb{R}^3)$, $t > 0$;*
3. *if $u_n \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^3)$ then $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $D^{1,2}(\mathbb{R}^3)$.*

We shall use the following technical result.

Lemma 2.2 *If $(u_n) \subset W^{1,2}(\mathbb{R}^3)$ is such that $u_n \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^3)$, then*

$$\lim_{n\to\infty} \int K(x)\phi_{u_n} u_n^2 = \int K(x)\phi_u u^2$$

and

$$\lim_{n\to\infty} \int K(x)\phi_{u_n} u_n\varphi = \int K(x)\phi_u u\varphi, \quad \text{for all } \varphi \in W^{1,2}(\mathbb{R}^3).$$

Proof. We have that

$$\int K(x)(\phi_{u_n}u_n^2 - \phi_u u^2) = \int K(x)\phi_{u_n}(u_n^2 - u^2) + \int K(x)u^2(\phi_{u_n} - \phi_u).$$

It follows from Lemma 2.1 that $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $D^{1,2}(\mathbb{R}^3)$, and therefore the last term above goes to zero. Hence, in order to prove the first statement of the lemma, it suffices to check that

$$\lim_{n\to+\infty} \int K(x)\phi_{u_n}(u_n^2 - u^2) = 0. \tag{2.1}$$

By using Hölder and Sobolev inequality we get

$$\begin{aligned} \left| \int K(x)\phi_{u_n}(u_n^2 - u^2) \right| &\le \|\phi_{u_n}\|_6 \left(\int K(x)^{\frac{6}{5}} |u_n^2 - u^2|^{\frac{6}{5}} \right)^{5/6} \\ &\le S\|u_n\|_{D^{1,2}} \left(\int K(x)^{\frac{6}{5}} |u_n^2 - u^2|^{\frac{6}{5}} \right)^{5/6}, \end{aligned} \tag{2.2}$$

where S is related with the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

For any given $\rho > 0$, we can use Hölder inequality twice to obtain

$$\int_{\mathbb{R}^3\setminus B_\rho(0)} K(x)^{\frac{6}{5}} |u_n^2 - u^2|^{\frac{6}{5}} \mathrm{d}x \le \|K\|_{L^2(\mathbb{R}^3\setminus B_\rho(0))}^{6/5} \left(\int |u_n^2 - u^2|^3 \right)^{2/5}.$$

Hölder inequality and the boundedness of (u_n) in $L^6(\mathbb{R}^3)$ provide $c_1 > 0$ such that

$$\left(\int |u_n^2 - u^2|^3 \right)^{2/5} \le \|u_n - u\|_6^{6/5} \|u_n + u\|_6^{6/5} \le c_1. \tag{2.3}$$

Moreover, since the condition (H_1) implies $K \in L^2(\mathbb{R}^3)$, we can choose $\rho > 0$ large in such a way that $\|K\|_{L^2(\mathbb{R}^3\setminus B_\rho(0))} < \varepsilon$. Thus, we infer from the above inequalities that

$$\int_{\mathbb{R}^3\setminus B_\rho(0)} K(x)^{\frac{6}{5}} |u_n^2 - u^2|^{\frac{6}{5}} \mathrm{d}x \le c_1\varepsilon. \tag{2.4}$$

For any $M > 0$ we define the set $\Omega_M := \{x \in B_\rho(0) : K(x) \ge M\}$. Since $K \in L^2(\mathbb{R}^3)$, the Lebesgue measure of Ω_M goes to zero as $M \to \infty$. So, for some $M > 0$ sufficiently large, we have that

$$\left(\int_{\Omega_M} K(x)^2 \mathrm{d}x \right)^{3/5} \le \varepsilon.$$

Then we can use Hölder inequality and (2.3) again to get

$$
\begin{aligned}
\int_{B_\rho(0)} K(x)^{\frac{6}{5}}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x = &\int_{\Omega_M} K(x)^{\frac{6}{5}}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x \\
&+\int_{B_\rho(0)\setminus\Omega_M} K(x)^{\frac{6}{5}}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x \\
&\leq c_2\varepsilon + M^{\frac{6}{5}}\int_{B_\rho(0)\setminus\Omega_M}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x.
\end{aligned} \tag{2.5}
$$

On the other hand

$$
\int_{B_\rho(0)\setminus\Omega_M}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x \leq \|u_n+u\|^{6/5}_{L^{12/5}(B_\rho(0))}\|u_n-u\|^{6/5}_{L^{12/5}(B_\rho(0))}.
$$

Since $u_n \to u$ strongly in $L^{\frac{12}{5}}(B_\rho(0))$, we obtain

$$
\lim_{n\to\infty}\int_{B_\rho(0)\setminus\Omega_M}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x = 0,
$$

and therefore it follows from (2.5) that

$$
\int_{B_\rho(0)} K(x)^{\frac{6}{5}}|u_n^2-u^2|^{\frac{6}{5}}\mathrm{d}x \leq c_2\varepsilon + o_n(1),
$$

where $o_n(1)$ stands for a quantity approaching zero as $n\to\infty$. The above expression, (2.4) and (2.2) imply (2.1) and the proof of the first statement of the lemma is concluded. The second one can be proved in the same way. We omit the details. □

The main interest in function ϕ comes from the fact that it enables us dealing with system (P) as a single equation. Actually, it can be proved that $(u,\phi)\in W^{1,2}(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ is a solution of (P) if, and only if, $u\in W^{1,2}(\mathbb{R}^3)$ is a non-negative critical point of the C^1-functional $I: W^{1,2}(\mathbb{R}^3)\to\mathbb{R}$ given by

$$
I(u) := \frac{1}{2}\|u\|^2 + \int K(x)\phi_u(x)u^2 - \frac{1}{p+1}\int a(x)(u^+)^{p+1},
$$

where $u^+(x) := \max\{u(x),0\}$. Since we intend to apply critical point theory to find such critical points, we need to prove some kind of compactness properties for the functional I. In this setting, the limit problem (P_∞) plays an important role. We observe that weak solutions of (P_∞) are precisely the critical points of the functional

$$
I_\infty(w) := \frac{1}{2}\int(|\nabla w|^2 + V_\infty w^2) - \frac{1}{p+1}\int(w^+)^{p+1}, \quad w\in W^{1,2}(\mathbb{R}^3).
$$

Let $\mathcal{N}_\infty$ be the Nehari manifold of I_∞, that is

$$
\mathcal{N}_\infty := \{w\in W^{1,2}(\mathbb{R}^3)\setminus\{0\} : I'_\infty(w)w = 0\}
$$

and consider the related minimization problem

$$c_\infty := \inf_{w \in \mathcal{N}_\infty} I_\infty(w).$$

The proof of the next result can be found in Berestycki-Lions [5].

Proposition 2.3 *Problem (P_∞) has a positive and radially symmetrical solution $\omega \in W^{1,2}(\mathbb{R}^3)$ such that $I_\infty(\omega) = c_\infty$. Moreover, for any $0 < \delta < \sqrt{V_\infty}$, there exists a constant $C = C(\delta) > 0$ such that*

$$\omega(x) \le C e^{-\delta|x|}, \quad \text{for all } x \in \mathbb{R}^3. \tag{2.6}$$

In order to prove that the functional I satisfies a local Palais-Smale condition we shall use the following version of a result due to Struwe [9] (see also [3]).

Lemma 2.4 *Let $(u_n) \subset W^{1,2}(\mathbb{R}^3)$ be such that*

$$I(u_n) \to c, \quad I'(u_n) \to 0$$

and $u_n \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^3)$. Then $I'(u) = 0$ and we have either

(a) *$u_n \to u$ strongly in $W^{1,2}(\mathbb{R}^3)$, or*

(b) *there exists $k \in \mathbb{N}, (y_n^j) \in \mathbb{R}^3$ with $|y_n^j| \to \infty$, $j = 1, \ldots, k$, and nontrivial solutions $w^1, \ldots, w^k \in W^{1,2}(\mathbb{R}^3)$ of the problem (P_∞), such that*

$$I(u_n) \to I(u) + \sum_{j=1}^{k} I_\infty(w^j) \tag{2.7}$$

and

$$\left\| u_n - u - \sum_{j=1}^{k} w^j(\cdot - y_n^j) \right\| \to 0.$$

Proof. To prove this result one can use Lemma 2.2 and similar arguments to that of [6]. Hence we omit the details. □

Corollary 2.5 *If $(u_n) \subset W^{1,2}(\mathbb{R}^3)$ is such that $I(u_n) \to c < c_\infty$ and $I'(u_n) \to 0$, then (u_n) has a convergent subsequence.*

Proof. Let $(u_n) \subset W^{1,2}(\mathbb{R}^3)$ be as in the previous statement. Since $p > 3$ by a standard argument it follows that (u_n) is bounded. Hence, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $W^{1,2}(\mathbb{R}^3)$. By Lemma 2.4 we have $I'(u_0) = 0$ and therefore

$$I(u_0) = I(u_0) - \frac{1}{2} I'(u_0)u_0 = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int a(x)(u_0^+)^{p+1} \ge 0.$$

If $u_n \not\to u_0$ in $W^{1,2}(\mathbb{R}^3)$, we can invoke Lemma 2.4 again to obtain $k \in \mathbb{N}$ and nontrivial solutions $w^1, \ldots, w^k$ of (P_∞) satisfying

$$\lim_{n\to\infty} I(u_n) = c = I(u_0) + \sum_{j=1}^{k} I_\infty(w^j) \ge k c_\infty \ge c_\infty,$$

contrary to the hypothesis. Hence $u_n \to u_0$ strongly in $W^{1,2}(\mathbb{R}^3)$. □

3 The proof of Theorem 1.1

We devote this section to the proof of our main theorem. The idea is looking for critical points of the functional I by considering the following minimization problem

$$c_0 := \inf_{u \in \mathcal{N}} I(u),$$

where $\mathcal{N}$ is the Nehari manifold of I, namely

$$\mathcal{N} := \left\{ u \in W^{1,2}(\mathbb{R}^3) \setminus \{0\} : I'(u)u = 0 \right\}.$$

From now on we denote by ω a positive ground state solution of the problem (P_∞). For $x_n := (0, \cdots, n)$ we also set

$$\omega_n(x) := \omega(x - x_n).$$

Since $p > 3$ we can easily check that, for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $t_n \omega_n \in \mathcal{N}$. Moreover, the following holds

Lemma 3.1 *The sequence* (t_n) *satisfies* $\lim\limits_{n \to +\infty} t_n = 1$.

Proof. Since $I'(t_n\omega_n)(t_n\omega_n) = 0$, we can use item 2 of Lemma 2.1 to get

$$0 = t_n^2 \int (|\nabla \omega_n|^2 + V(x)\omega_n^2) + t_n^4 \int K(x)\phi_{\omega_n}\omega_n^2 - t_n^{p+1} \int a(x)\omega_n^{p+1}. \tag{3.1}$$

By using (1.1), a change o variables and Lebesgue Theorem we get

$$\lim_{n \to \infty} \int V(x)\omega_n^2 = \lim_{n \to \infty} \int V(x + x_n)\omega^2 = \int V_\infty \omega^2$$

and

$$\lim_{n \to \infty} \int a(x)\omega_n^{p+1} = \lim_{n \to \infty} \int a(x + x_n)\omega^{p+1} = \int \omega^{p+1}.$$

Moreover, by item 1 of Lemma 2.1, we also have that

$$\left| \int K(x)\phi_{\omega_n}(x)\omega_n^2 \right| \leq \|K\|_2 \|\phi_{\omega_n}\|_6 \|\omega\|_6 \leq c_1,$$

for some $c_1 > 0$.

We claim that (t_n) is bounded. Indeed, if this is not the case, we can divide equation (3.1) by t_n^{p+1}, take the limit as $n \to \infty$ and use $p + 1 > 4$ and the above statements to conclude that $\int \omega^{p+1} = 0$, which is a contradiction. Hence (t_n) is bounded. Moreover, for some $\bar{t} > 0$, there holds $t_n \geq \bar{t} > 0$. Otherwise, since $\|t_n\omega_n\|_{W^{1,2}(\mathbb{R}^3)} = t_n\|\omega\|_{W^{1,2}(\mathbb{R}^3)}$, we would have $\mathrm{dist}(\mathcal{N}, 0) = 0$, which is impossible.

The above reasoning shows that, up to a subsequence, $t_n \to t_0 > 0$. We claim that

$$\lim_{n \to \infty} \int K(x)\phi_{\omega_n}(x)\omega_n^2 = 0. \tag{3.2}$$

Assuming the claim and taking the limit in (3.1) we obtain

$$0 = t_0^2 \int (|\nabla \omega|^2 + V_\infty \omega^2) - t_0^{p+1} \int \omega^{p+1} = I'_\infty(t_0\omega)(t_0\omega).$$

Since $\omega \in \mathcal{N}_\infty$ we conclude that $t_0 = 1$.

It remains to prove the claim. First notice that, by item 1 of Lemma 2.1, we have that $\|\phi_{\omega_n}\|_6 \le c_2$, for some $c_2 > 0$. Given $\varepsilon > 0$ we choose $\rho > 0$ such that $\|K\|_{L^2(\mathbb{R}^3 \backslash B_\rho(0))} < \varepsilon$. Thus,

$$\left| \int_{\mathbb{R}^3 \backslash B_\rho(0)} K(x) \phi_{\omega_n}(x) \omega_n^2 \mathrm{d}x \right| \le \|K\|^2_{L^2(\mathbb{R}^3 \backslash B_\rho(0))} \|\phi_{\omega_n}\|_6 \|\omega\|_6^2 \le c_2 \|\omega\|_6 \varepsilon. \quad (3.3)$$

On the other hand, Hölder's inequality and a change of variables provide

$$\left| \int_{B_\rho(0)} K(x) \phi_{\omega_n}(x) \omega_n^2 \mathrm{d}x \right| \le \|K\|_2 \|\phi_{\omega_n}\|_6 \left(\int_{B_\rho(x_n)} \omega^6 \mathrm{d}x \right)^{1/3} = o_n(1),$$

since $\omega \in L^6(\mathbb{R}^3)$ and $|x_n| \to \infty$, as $n \to \infty$. The above inequality and (3.3) establishes (3.2). The proof is finished. □

The following result contains the core estimate for the proof of our main theorem.

Proposition 3.2 *If $(H_1) - (H_3)$ hold, then $0 < c_0 < c_\infty$.*

Proof. Let ω, ω_n and $t_n > 0$ be as in the beginning of this section. Since $t_n \omega_n \in \mathcal{N}$, a straightforward calculation provides

$$\begin{aligned} c_0 \le I(t_n\omega_n) &= I_\infty(t_n\omega) + \frac{t_n^2}{2} A_n + \frac{t_n^4}{4} D_n + \frac{t_n^{p+1}}{p+1} E_n \\ &\le c_\infty + \frac{t_n^2}{2} A_n + \frac{t_n^4}{4} D_n + \frac{t_n^{p+1}}{p+1} E_n, \end{aligned} \quad (3.4)$$

where

$$A_n := \int (V(x) - V_\infty) \omega_n^2, \qquad D_n := \int K(x) \phi_{\omega_n}(x) \omega_n^2$$

and

$$E_n := \int (1 - a(x)) \omega_n^{p+1}.$$

Now we need to estimate the decay rate of each of the above terms. It follows from the first estimate in (1.2) that

$$A_n = \int (V(x) - V_\infty) \omega_n^2 \le c_V \int e^{-\gamma |x|} \omega_n^2 = c_V \int e^{-\gamma |x + x_n|} \omega^2.$$

Since $|x + x_n| \ge |x_n| - |x| = n - |x|$, we obtain

$$A_n \le c_V e^{-\gamma n} \int e^{\gamma |x|} \omega^2 = C_V e^{-\gamma n}, \quad (3.5)$$

with $C_V > 0$, where we have used in the last equality the exponential decay of ω given in Proposition 2.3 and that $\gamma < 2\sqrt{V_\infty}$, which implies that $\int e^{-\gamma|x|}\omega^2 < \infty$. In order to estimate D_n we use Hölder's inequality, $\alpha < 2\sqrt{V_\infty}$ and argue as above to get

$$\begin{aligned} D_n = \int K(x)\phi_{\omega_n}(x)\omega_n^2 &\leq \|\phi_{\omega_n}\|_6 \left(\int K(x)^{\frac{6}{5}}\omega_n^{\frac{12}{5}}\right)^{5/6} \\ &\leq c_1 \left(\int e^{-\frac{6\alpha}{5}|x+x_n|}\omega^{\frac{12}{5}}\right)^{5/6} \\ &\leq C_K e^{-\alpha n}, \end{aligned} \tag{3.6}$$

with $C_K > 0$. We now use the second inequality in (1.2) to estimate E_n as follows

$$E_n = \int (1-a(x))\omega_n^{p+1} \leq -c_a \int e^{-\theta|x|}\omega_n^{p+1} = -c_a \int e^{-\theta|x+x_n|}\omega^{p+1}.$$

Since $|x + x_n| \leq n + |x|$, we obtain $C_a > 0$ such that

$$E_n \leq -c_a e^{-\theta n} \int e^{-\theta|x|}\omega^{p+1} = -C_a e^{-\theta n}. \tag{3.7}$$

By replacing (3.5)-(3.7) in (3.4) we obtain,

$$\begin{aligned} c_0 &\leq c_\infty + e^{-\theta n}\left(\frac{t_n^2}{2}C_V e^{(\theta-\gamma)n} + \frac{t_n^4}{4}C_K e^{(\theta-\alpha)n} - \frac{t_n^{p+1}}{p+1}C_a\right) \\ &= c_\infty + e^{-\theta n}(o_n(1) - C_a), \end{aligned}$$

where we have used in the last equality that $t_n \to 1$ and $\theta < \min\{\alpha, \gamma\}$. Since $C_a > 0$ we can take n large enough to conclude that $c_0 < c_\infty$. The proposition is proved. □

We are now ready to obtain the ground state solution of (S).

Proof of Theorem 1.1. Let $(u_n) \subset \mathcal{N}$ be such that $I(u_n) \to c_0$. Since $\mathcal{N}$ is a C^1 regular manifold and is closed (see [6, Lemma 3.1]), we can use Ekeland's Variational Principle to obtain that

$$I(u_n) \to c_0 \quad \text{and} \quad I'(u_n) \to 0.$$

Proposition 3.2 and Corollary 2.5 imply that the sequence (u_n) strongly converges to a function $u_0 \in W^{1,2}(\mathbb{R}^3)$ such that $I(u_0) = c_0 > 0$ and $I'(u_0) = 0$. Setting $u_0^-(x) := \max\{-u_0(x), 0\}$, we can use $0 = I'(u_0)u_0{}^- = -\|u_0^-\|$ to conclude that $u_0 \geq 0$ a.e. in $\mathbb{R}^3$. It follows from elliptic regularity and the strong maximum principle that $u > 0$ in $\mathbb{R}^3$. The theorem is proved. □

Remark 3.3 *A simple inspection of the proof of Proposition 3.2 shows that we can drop the condition* (H_3) *by the hypotheses* $(\widetilde{H_3})$ *stated in the introduction.*

Indeed, with this dual condition what happens is that term A_n of the proof of the proposition becomes negative while the term E_n is positive. The choices of the numbers α, γ and θ guarantee that the desired inequality also holds in this setting.

References

[1] A. Ambrosetti, D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem.* Commun. Contemp. Math. **10** (2008), 391-404.

[2] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008) 90-108.

[3] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rat. Mech. Anal. **99** (1987), 283-300.

[4] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrdinger-Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998) 283-293.

[5] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 313–345.

[6] G. Cerami and G. Vaira, *Positive solutions for some non-autonomous Schrödinger-Poisson systems*, J. Differential Equations **248** (2010), 521-543.

[7] M. F. Furtado, L. Maia and E. S. Medeiros, *Positive and nodal solutions for a nonlinear Schrödinger equation with indefinite potential*, Adv. Non. Studies **8** (2008), 353–373.

[8] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term.* J. Funct. Anal. **237** (2006), 655-674.

[9] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), no. 4, 511–517.

[10] G. Vaira, *Ground states for Schrödinger-Poisson type systems*, Ric. Mat. **60** (2011), 263-297.