An obstacle problem in a plane domain with two solutions

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Abstract

We prove the existence of two nontrivial solutions for the variational inequality $\int_\Omega \nabla u \nabla (v-u) \geq \int_\Omega f(u)(v-u)$ for every v belonging to some convex set, where $\Omega \subset \mathbb{R}^2$. The function f has critical exponential growth, in the sense that it behaves like $\exp(\alpha_0 s^2)$ as $|s| \to \infty$, for some $\alpha_0 > 0$. We use variational methods for lower semicontinuous functionals.

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1 Introduction

The obstacle problem arises in many branches as for example in the elasticity theory, in control games, minimal surfaces, constrained heating and financial mathematics [6]. In a simple version, one is lead to minimize the functional

$$\mathcal{J}(u) = \int_\Omega |\nabla u|^2 dx$$

in the set

$$\mathcal{K} = \{u \in H^1(\Omega) : u = h \text{ on } \partial\Omega \text{ and } u \geq \zeta \text{ a.e. in } \Omega\},$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, h is a smooth function defined on $\partial\Omega$ and the obstacle ζ is a smooth function defined in Ω such that $\zeta|_{\partial\Omega} < h$. This minimization problem can also be rephrased in terms of variational inequalities similar to (1.2) below, see [9] for an account. Due to the convexity of $\mathcal{J}$ and $\mathcal{K}$, the problem has a unique minimum. It is also possible to apply the maximum principle to show that the minimizer is superharmonic and is $C^{1,1}$. The boundary of the contact set where the minimizer is equal to the obstacle ζ is called free boundary, see [6]. In the present paper we study a problem in plane domains with an exponentially increasing function, we take $h \equiv 0$ and the obstacle ζ is replaced by a function ψ with suitable conditions.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain and

$$\psi \in C^{1,\beta}(\Omega),\ 0 < \beta < 1, \text{ such that } \psi < 0 \text{ on } \partial\Omega \text{ and } \psi^+ \not\equiv 0. \tag{1.1}$$

We consider the set

$$\mathcal{K}_\psi := \{v \in H_0^1(\Omega)\ :\ v \geq \psi \text{ a.e. in } \Omega\}.$$

We are interested in multiplicity of positive solutions for the following obstacle problem

$$\int_\Omega \nabla u \nabla(v - u) \geq \int_\Omega f(u)(v - u), \quad \forall\, v \in \mathcal{K}_\psi, \tag{1.2}$$

where the nonlinearity f has a suitable growth at infinity.

Problems like (1.2) in domains $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ have been studied in [10]. There the authors considered a nonlinearity $f(u) = u^{q-1}$ with $1 < q \leq (N+2)/(N-2)$ and showed the existence of at least 2 solutions when the obstacle is sufficiently small. Inspired by them, results involving the p-Lapacian have been obtained in [8], where the author considers a function $f(u) = \lambda u^{(pN-N+p)/(N-p)}$. He obtained the existence of two solutions, provided $\lambda > 0$ is small enough, such assumption translates into the fact that the obstacle is small, since in their problem the nonlinearity is homogeneous. In this paper we deal with a problem in dimension 2 with f having exponential growth. Our results complement those of [8, 10], see also [5] for an account on physical motivations. One of the difficulties we have to face in the present paper is is the convergence of certain sequences related to the levels of the functional corresponding to problem (1.2), see Lemma 2.1 and Proposition 3.5.

It is well known that the notion of criticality in dimension 2 is different from that of $N \geq 3$. Actually, in dimension 2 the maximal growth is related to the so called Trudinger-Moser inequality, namely

$$\sup_{\|u\|_{H_0^1(\Omega)\leq 1}} \int_\Omega \exp(\alpha u^2) \leq C(\alpha)$$

for all $\alpha \leq 4\pi$. Motivated by this inequality we suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous and has exponential critical growth at infinity, this meaning that the following basic condition holds.

(f_0) there exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{\exp(\alpha s^2)} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases}$$

In order to obtain a positive solution we also impose a condition on the behavior of f near the origin, namely we suppose that

(f_1) $f(s) = o(s)$ as $s \to 0^+$.

By using a minimization argument we prove the following result.

Theorem 1.1 *Suppose that $f \in C([0,\infty),\mathbb{R})$ satisfies $(f_0) - (f_1)$. Then there exists $\delta^* > 0$ such that the problem (1.2) has a positive weak solution in $C^{1,\beta}$, $0 < \beta < 1$, whenever $\int_\Omega |\nabla\psi^+|^2 dx < \delta^*$.*

In our second result we are interested in the existence of multiple positive solutions for (1.2). Since we are intending to apply a version of the Mountain Pass Theorem for non-differentiable functionals due to Szulkin [14], we need to deal with the Palais-Smale sequences of the associated functional. Hence, we suppose that f satisfies the so-called Ambrosetti-Rabinowitz condition

(f_2) there exists $\theta > 2$ such that

$$0 < \theta F(s) := \theta \int_0^s f(t)\mathrm{d}t \leq s f(s), \quad \text{for all } s > 0,$$

as well as

(f_3) there exist $p > 2$ and $\mu = \mu(p) > 0$ such that

$$F(s) \geq \frac{2\mu}{p} s^p, \quad \text{for all } s > 0.$$

If we denote by S_p be the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, that is

$$S_p := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^p \mathrm{d}x\right)^{2/p}}, \tag{1.3}$$

the main result of this paper can be stated as follows.

Theorem 1.2 *Suppose $f \in C([0,\infty),\mathbb{R})$ satisfies $(f_0)-(f_3)$ with*

$$\mu^{2/(p-2)} > \frac{\alpha_0}{\pi}\left(\frac{2\theta}{\theta-2}\right)\left(\frac{p-2}{2p}\right)S_p^{p/(p-2)}. \tag{1.4}$$

Then there exists $\delta^{**} > 0$ such that the problem (1.2) has two positive weak solutions in $C^{1,\beta}$, $0<\beta<1$, whenever $\int_\Omega |\nabla\psi^+|^2\,dx < \delta^{**}$.

We choose $v = u_0^+ \in \mathcal{K}_\psi$ as a test function we conclude that $u_0^- \equiv 0$, and therefore $u_0 \geq 0$. By the conditions on ψ assumed in (1.1), specifically $\psi \not\equiv 0$, we obtain the $C^{1,\beta}$ regularity of u_0 (see [2] and [8]). Applying the maximum principle we conclude that $u_0 > 0$. □

The conditions (f_3) and (1.4) are technical in nature. They are related with the critical growth of f and are important for the correct localization of the Mountain Pass level of the functional related to the problem (1.2). Similar assumptions have already appeared in [1]. A typical example of function satisfying the conditions stated above is $f(s) = As(\exp(s^2)-1)$, with $A>0$ such that estimate (1.4) is satisfied.

There is a parallel between Theorem 1.2 and some known results concerning the problem

$$-\Delta u = \lambda \exp(u) \text{ in } \Omega, \quad u \in H_0^1(\Omega).$$

It has been proved in [7] that, if Ω is a ball, then this problem has exactly two positive solutions provided $\lambda \in (0,\lambda^*)$ for some $0<\lambda^*<\infty$. We also mention a generalization of such a result for convex simply connected domains obtained in [12, 13].

The paper contains two more sections. In section 2 we prove Theorem 1.1 by using minimization arguments. In Section 3, after recalling some abstract results, we prove Theorem 1.2.

2 Proof of Theorem 1.1

Throughout the paper we denote by $\|\cdot\|$ and $\|\cdot\|_r$ the norms in $H_0^1(\Omega)$ and $L^r(\Omega)$, respectively. Also, we write only $\int u$ instead of $\int_\Omega u(x)dx$.

Since we are looking for positive solutions we start by extending f to the whole real line by setting $f(s) := 0$ for each $s \leq 0$. We shall use variational methods to deal with our problem. We recall the so called Trudinger-Moser inequalities (see [15, 11]) which provide

$$\exp(\alpha u^2) \in L^1(\Omega), \quad \forall\, \alpha > 0,\ u \in H_0^1(\Omega), \tag{2.1}$$

and

$$\sup_{\|u\|\leq 1} \int \exp(\alpha u^2) \leq C(\alpha), \quad \forall\, \alpha \leq 4\pi,\ u \in H_0^1(\Omega), \tag{2.2}$$

for some constant $C(\alpha) > 0$.

Let $\alpha_0 > 0$ be given by the growth condition (f_0) and consider $\beta > \alpha_0$. A straightforward calculation provides a constant $C = C(\beta) > 0$ such that

$$F(s) \leq Cs\exp(\beta s^2), \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$

Let $\gamma > 1$ and $\gamma' := \gamma/(\gamma - 1)$ its conjugated exponent. It follows from the above estimate, Hölder's inequality and (2.1) that, for each $u \in H_0^1(\Omega)$, there holds

$$\left|\int F(u)\right| \leq C\int |u|\exp(\beta u^2) \leq C\|u\|_{\gamma'}\left(\int \exp(\beta\gamma u^2)\right)^{1/\gamma} < +\infty, \tag{2.4}$$

and therefore the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\|u\|^2 - \int F(u)$$

is well defined. Moreover, by using (2.1), (2.3) and standard calculations we can prove that $J \in C^1(H_0^1(\Omega), \mathbb{R})$.

For each $\delta > 0$ we define

$$\Sigma_\delta := \{u \in H_0^1(\Omega) : \|u\|^2 \leq \delta,\, u \in \mathcal{K}_\psi\}$$

and

$$m_\delta := \inf_{u \in \Sigma_\delta} J(u).$$

In the next lemma we present a sufficient condition for the solvability of the above minimization problem.

Lemma 2.1 *Let $\alpha_0 > 0$ be given by (f_0). If $0 < \delta < (4\pi/\alpha_0)$ and $\Sigma_\delta \neq \varnothing$, then the (finite) number m_δ is achieved in Σ_δ.*

Proof. Since $0 < \delta < (4\pi/\alpha_0)$ we can choose $\beta > \alpha_0$ and $\gamma > 1$ in such way that $\beta\gamma\delta \leq 4\pi$. It follows from (2.4) that, for every $u \in \Sigma_\delta$, there holds

$$\begin{aligned}\left|\int F(u)\right| &\leq C\|u\|_{\gamma'}\left(\int \exp(\beta\gamma\|u\|^2(u/\|u\|)^2)\right)^{1/\gamma} \\ &\leq C\|u\|_{\gamma'}\left(\int \exp(4\pi(u/\|u\|)^2)\right)^{1/\gamma} \\ &\leq C_1\|u\|,\end{aligned}$$

where we have used, in the last inequality, the embedding $H_0^1(\Omega) \subset L^{\gamma'}(\Omega)$ and (2.2) with $\alpha = 4\pi$. It follows that

$$J(u) \geq -\int F(u) \geq -C_1\|u\| \geq -C_1\sqrt{\delta},$$

for every $u \in \Sigma_\delta$ and therefore the infimum m_δ is finite.

In order to prove that m_δ is achieved we take $(u_n) \subset \Sigma_\delta$ such that $J(u_n) \to m$. Taking a subsequence if necessary, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_n \to u & \text{strongly in } L^s(\Omega) \text{ for } s \geq 2, \\ u_n(x) \to u(x) & \text{a.e. in } \Omega, \\ |u_n(x)| \leq \phi_{\gamma'}(x) & \text{a.e. in } \Omega, \end{cases} \tag{2.5}$$

for some $u \in H_0^1(\Omega)$ and $\phi_{\gamma'} \in L^{\gamma'}(\Omega)$. Since $\mathcal{K}_\psi$ is closed and convex it is weakly closed, and therefore $u \in \mathcal{K}_\psi$. Moreover,

$$\|u\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2 \leq \delta$$

from which it follows that $u \in \Sigma_\delta$. Since the functional J is continuous it suffices to prove that $u_n \to u$ strongly in $H_0^1(\Omega)$.

We start by assuming that

$$\lim_{n\to\infty} \int F(u_n) = \int F(u). \tag{2.6}$$

If this is true we can use the weak convergence of (u_n) to compute

$$\begin{aligned} \frac{1}{2}\|u_n - u\|^2 &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 + o_n(1) \\ &= J(u_n) + \int F(u_n) - J(u) - \int F(u) \\ &= m_\delta - J(u) + o_n(1), \end{aligned}$$

where $o_n(1)$ is a quantity approaching zero as $n \to \infty$. Since $u \in \Sigma_\delta$ we have that $m_\delta \leq J(u)$. Hence

$$\limsup_{n\to\infty} \|u_n - u\|^2 = 2(m - J(u)) \leq 0$$

and therefore $u_n \to u$ strongly in $H_0^1(\Omega)$.

It remains to prove (2.6). Given a measurable set $E \subset \Omega$ we can use (2.5) and the same argument of the beginning of the proof to get

$$\begin{aligned} \left| \int_E F(u_n)\mathrm{d}x \right| &\leq \int_E |u_n| \exp(\beta |u_n|^2)\mathrm{d}x \\ &\leq \left(\int_E |u_n|^{\gamma'}\mathrm{d}x \right)^{1/\gamma'} \left(\int_\Omega \exp(\gamma\beta \|u_n\|^2 \left(u_n/\|u_n\|\right)^2)\mathrm{d}x \right)^{1/\gamma} \\ &\leq c_5 \left(\int_E |\phi_{\gamma'}|^{\gamma'}\mathrm{d}x \right)^{1/\gamma'}. \end{aligned}$$

Given $\varepsilon > 0$, we can use the above inequality, $\phi_{\gamma'} \in L^{\gamma'}(\Omega)$ and $F(u) \in L^1(\Omega)$ to conclude that

$$\max\left\{\left|\int_E F(u_n)\mathrm{d}x\right|, \left|\int_E F(u)\mathrm{d}x\right|\right\} < \varepsilon, \tag{2.7}$$

whenever the set E has small measure, namely $\mathrm{meas}(E) < \nu$. On the other hand, the pointwise convergence in (2.5) implies that $F(u_n(x)) \to F(u(x))$ a.e. in Ω. So, we can apply Egoroff's Theorem to obtain a measurable set $E \subset \Omega$ such that $\mathrm{meas}(E) < \nu$ and $F(u_n(x)) \to F(u(x))$ uniformly for $x \in \Omega \setminus E$. Hence,

$$\lim_{n\to\infty} \int_{\Omega\setminus E} F(u_n)\mathrm{d}x = \int_{\Omega\setminus E} F(u)\mathrm{d}x.$$

Since $\varepsilon > 0$ is arbitrary, the convergence in (2.6) follows from the above one and (2.7). The lemma is proved. □

We are able now to present the proof of Theorem 1.1.

Proof of Theorem 1.1. We fix $q > 2$ and consider $0 < \delta < (4\pi/\alpha_0)$ to be chosen later. Let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Given $0 < \varepsilon < (\lambda_1/2)$ and $q > 2$ we can use (f_1) and (2.3) to obtain $c_1 > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2}s^2 + c_1 s^q \exp(\beta s^2), \quad \text{for all } s \in \mathbb{R}.$$

This inequality, Sobolev embeddings and (2.2) provide

$$\begin{aligned}
\left|\int F(u)\right| &\leq \frac{\varepsilon}{2}\|u\|_2^2 + C\|u\|_{q\gamma'}^q \left(\int \exp\left(\beta\gamma\|u\|^2 \left(u/\|u\|\right)^2\right)\right)^{1/\gamma} \\
&\leq \frac{\varepsilon}{2\lambda_1}\|u\|^2 + c_3\|u\|^q \left(\int \exp\left(4\pi \left(u/\|u\|\right)^2\right)\right)^{1/\gamma} \\
&\leq \frac{1}{4}\|u\|^2 + c_4\|u\|^q,
\end{aligned}$$

whenever $u \in \Sigma_\delta$. Hence,

$$J(u) \geq \frac{1}{4}\|u\|^2 - c_4\|u\|^q, \quad u \in \Sigma_\delta,$$

and therefore

$$\widehat{m_\delta} := \inf\{J(u) : \|u\|^2 = \delta\} \geq \delta\left(\frac{1}{4} - c_4\delta^{(q-2)/2}\right).$$

Since $q > 2$, the term in the parenthesis above tends to $1/4$ as $\delta \to 0^+$. Hence, we can choose $\delta < 4\pi/\alpha_0$ small in such way that $\widehat{m_\delta} > 0$. Since $J(0) = 0$ and J is continuous we can take $\delta^* < \delta$ such that

$$|J(u)| < \widehat{m_\delta} \quad \text{whenever } \|u\|^2 < \delta^*. \tag{2.8}$$

We claim that the theorem holds for this choice of δ^*. Indeed, if $\|\psi^+\|^2 < \delta^*$ then $\psi^+ \in \Sigma_\delta \neq \varnothing$. It follows from Lemma 2.1 that the infimum m_δ is achieved at some point $u_0 \in \Sigma_\delta$. By (2.8) we have that $J(\psi^+) < \widehat{m_\delta}$ and therefore we conclude that $u_0 \in B_{\sqrt{\delta}}(0)$.

In order to prove that u_0 is a weak solution of the problem (1.2) we take $v \in \mathcal{K}_\psi$ and notice that $u_0 + t(v - u_0) \in \mathcal{K}_\psi$ for each $t \in [0,1]$. Since $\|u_0\| < \sqrt{\delta}$ there exists $t_0 > 0$ such that $u_0 + t(v - u_0) \in \Sigma_\delta$ for each $t \in [0, t_0]$. If we consider the real function $g(t) := J(u_0 + t(v - u_0))$, we have that $g(t) \geq g(0)$, for each $t \in [0, t_0]$. Hence

$$0 \leq \lim_{t \to 0^+} \frac{g(t) - g(0)}{t} = J'(u_0)(v - u_0),$$

that is,

$$\int \nabla u_0 \cdot \nabla (v - u_0) \geq \int f(u_0)(v - u_0), \quad \forall\, v \in \mathcal{K}_\psi.$$

We choose $v = u_0^+ \in \mathcal{K}_\psi$ as a test function we conclude that $u_0^- \equiv 0$, and therefore $u_0 \geq 0$. By the conditions on ψ assumed in (1.1), specifically $\psi \not\equiv 0$, we obtain the $C^{1,\beta}$ regularity of u_0 (see [2] and [8]). Applying the maximum principle we conclude that $u_0 > 0$. □

3 Proof of Theorem 1.2

From now on we shall denote by u_0 the solution provided by Theorem 1.1 and consider it as a new obstacle by defining

$$\mathcal{K}_{u_0} := \{v \in H_0^1(\Omega) \ : \ v \geq u_0 \text{ a.e. in } \Omega\}.$$

The same argument of [10, Lemma 2.3] shows that all solutions of the problem

$$\int \nabla u \nabla (v - u) \geq \int f(u)(v - u), \quad \forall\, v \in \mathcal{K}_{u_0}, \tag{3.1}$$

are also a weak solutions of (1.2).

We follow [10, 14] by introducing the variational setting to deal with (3.1). Firstly, we consider the functional $I : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I(u) := J(u) + \Phi(u), \tag{3.2}$$

where

$$\Phi(u) := \begin{cases} 0, & \text{if } u \in \mathcal{K}_{u_0}, \\ +\infty, & \text{otherwise.} \end{cases}$$

It can be proved that each critical point $u \in H_0^1(\Omega)$ of the functional I satisfies (3.1).

The functional I is not regular and it can assume the value $+\infty$ at some points. However, it is a C^1 functional added to a convex proper lower semicontinuous part. Hence, we can use the minimax theory developed by Szulkin in

[14] for this kind of functionals. Such functionals have been also used in [3] and [4] to study eigenvalue problems. We briefly recall the main concepts below.

Let $(X, \langle\cdot,\cdot\rangle_X)$ be a real Hilbert space and $\mathcal{K} \subset X$ a proper closed convex subset. Suppose that a functional $\mathcal{I} : X \to \mathbb{R} \cup \{+\infty\}$ if of the form $\mathcal{I} := \mathcal{J} + \eta$, with $\mathcal{J}$ of class C^1, $\eta \equiv 1$ in K and $\eta \equiv +\infty$ in $X \setminus \{\mathcal{K}\}$. A critical point $u \in X$ of $\mathcal{I}$ is a point $u \in \mathcal{K}$ such that

$$\mathcal{J}'(u)(v-u) \geq 0, \quad \forall\, v \in \mathcal{K}.$$

We say that $(u_n) \subset \mathcal{K}$ is a Palais-Smale sequence at level $d \in \mathbb{R}$ ($(\mathrm{PS})_d$ for short) for $\mathcal{I}$ if $\mathcal{I}(u_n) \to d$ and there exists $(z_n) \subset X$ such that $z_n \to 0$ strongly in X and

$$\mathcal{J}'(u_n)(v-u_n) \geq \langle z_n, v-u_n\rangle_X, \quad \forall\, v \in \mathcal{K}.$$

The functional $\mathcal{I}$ satisfies the $(\mathrm{PS})_d$ condition if each $(\mathrm{PS})_d$ sequence for $\mathcal{I}$ has a convergent subsequence.

The following result is a generalization of the classical Mountain Pass Theorem for this new setting (see [14]).

Theorem 3.1 *Let X and $\mathcal{I}$ as above and $\underline{u} \in X$. Suppose that*

(i) there exists an open set $U \subset X$ such that $\underline{u} \in U$ and

$$\mathcal{I}(\underline{u}) \leq \inf_{u \in U} \mathcal{I}(u) \qquad \text{and} \qquad \mathcal{I}(\underline{u}) < \inf_{u \in \partial U} \mathcal{I}(u);$$

(ii) there exists $e \notin U$ such that $\mathcal{I}(e) \leq \mathcal{I}(\underline{u})$.

Let

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = \underline{u},\, \mathcal{I}(\gamma(1)) < \mathcal{I}(\underline{u})\}$$

and define the minimax level

$$c^* := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)). \tag{3.3}$$

If there exists $c_0 > c^*$ such that $\mathcal{I}$ satisfies the Palais-Smale condition at all levels $d \leq c_0$, then $\mathcal{I}$ possesses a critical point $\overline{u} \in X \cap \mathcal{K}$ such that $\mathcal{I}(\overline{u}) = c^*$.

We intend to apply the above theorem with $X = H_0^1(\Omega)$, $\mathcal{K} = \mathcal{K}_{u_0}$, $\mathcal{I} = I$ and $\underline{u} = u_0$. We start with the geometric conditions.

Lemma 3.2 *The functional I defined in (3.2) satisfies the geometric conditions (i) and (ii) of Theorem 3.1*

Proof. Let $\delta^* > 0$ be given by Theorem 1.1 and set $U := B_{\sqrt{\delta^*}}(0)$. Since $u_0 \in U$, $\mathcal{K}_{u_0} \subset \mathcal{K}_\psi$, $I \equiv +\infty$ outside $\mathcal{K}_{u_0}$ and the solution u_0 was obtained by minimization on $\Sigma_\delta \subset \mathcal{K}_\psi$ we have that

$$I(u_0) \leq \inf_{u \in U} I(u).$$

If $I(u_0) = \inf_{u\in\partial U} I(u)$ then there exists $u_1 \in \mathcal{K}_{u_0}$ such that $I(u_1) = I(u_0)$. Since $\delta^* < \delta$ the same argument employed in the proof of Theorem 1.1 shows that $u_1 \in \mathcal{K}_\psi$ is a second solution of the problem (1.2). Hence, we may suppose that

$$I(u_0) < \inf_{u\in\partial U} I(u),$$

and therefore the condition (i) holds.

In order to verify (ii) we take $\phi \not\equiv 0$ a nonnegative function. Since $u_0 \geq 0$ we can use (f_3) and Cauchy-Schwarz's inequality to obtain, for every $t \geq 0$,

$$\begin{aligned} I(u_0 + t\phi) &= J(u_0 + t\phi) \\ &\leq \frac{1}{2}\|u_0\|^2 + \frac{t^2}{2}\|\phi\|^2 + t\|u_0\|\|\phi\| - \frac{2\mu}{p}\int |u_0 + t\phi|^p \\ &\leq \frac{1}{2}\|u_0\|^2 + \frac{t^2}{2}\|\phi\|^2 + t\|u_0\|\|\phi\| - \frac{2\mu}{p}t^p\int |\phi|^p. \end{aligned}$$

Since $p > 2$ we conclude that $I(u_0 + t\phi) \to -\infty$ as $t \to \infty$. It suffices to set $e := u_0 + t\phi$ with $t > 0$ large enough. □

The proof of the Palais-Smale condition is more involved and it will be done in several steps. Firstly we recall that the solution u_0 of Theorem 1.1 was obtained by minimizing the functional J in the set Σ_δ. By making the number δ^* of that theorem smaller if necessary, we may assume that

$$\|u_0\|^2 < \delta \leq \left(\frac{\theta - 2}{2\theta}\right)\frac{\pi}{\alpha_0}. \tag{3.4}$$

Lemma 3.3 *If f satisfies (f_3) with $\mu > 0$ as in (1.4), then the Mountain Pass level defined in* (3.3) *verifies*

$$c^* < \left(\frac{\theta - 2}{2\theta}\right)\frac{3\pi}{\alpha_0}.$$

Proof. Let $\omega_0 \in H_0^1(\Omega)$ be such that $\omega_0 > 0$ in Ω and ω_0 realizes the infimum in (1.3), that is, $S_p = \|\omega_0\|^2\|\omega_0\|_p^{-2}$. For each $t \geq 0$ we have that $u_0 + t\omega_0 \in \mathcal{K}_{u_0}$. Hence, we can use (f_3) as before to conclude that $I(u_0 + t\omega_0) \to -\infty$ as $t \to +\infty$. Hence, if we set $\omega := t_0\omega_0$ with $t_0 > 0$ large enough, we have that $I(u_0 + \omega) < I(u_0)$, which implies that $\gamma(t) := u_0 + t\omega$ belongs to the class of admissible paths Γ defined in the statement of Theorem 3.1.

It follows from Young's inequality, (f_3) and $u_0 \geq 0$, that

$$\begin{aligned} I(\gamma(t)) &\leq \frac{1}{2}\|u_0\|^2 + t\|u_0\|\|\omega\| + \frac{t^2}{2}\|\omega\|^2 - \int F(u_0 + t\omega) \\ &\leq \|u_0\|^2 + t^2\|\omega\|^2 - t^p\frac{2\mu}{p}\int |\omega|^p \end{aligned}$$

and therefore we can use the definition of c^* given in (3.3) to get

$$c^* \leq \max_{t\in[0,1]} I(\gamma(t)) \leq \delta + 2\max_{t\geq 0}\left\{\frac{t^2}{2}\|\omega\|^2 - \frac{\mu}{p}t^p\int |\omega|^p\right\}. \tag{3.5}$$

Since $p > 2$ and ω also realizes the infimum in (1.3), a simple calculation and the choice of μ in (1.4) provide

$$\begin{aligned}\max_{t\geq 0}\left\{\frac{t^2}{2}\|\omega\|^2 - t^p\frac{\mu}{p}\int|\omega|^p\right\} &= \frac{1}{\mu^{2/(p-2)}}\left(\frac{p-2}{2p}\right)S_p^{p/(p-2)}\\ &\leq \left(\frac{\theta-2}{2\theta}\right)\frac{\pi}{\alpha_0}.\end{aligned}$$

The above inequality, (3.5) and (3.4) imply the desired result. □

Lemma 3.4 *If $(u_n) \subset \mathcal{K}_{u_0}$ is a (PS)$_d$ sequence then*

$$\limsup_{n\to\infty}\|u_n\|^2 \leq \left(\frac{2\theta}{\theta-2}\right)d. \tag{3.6}$$

In particular, (u_n) is bounded in $H_0^1(\Omega)$.

Proof. Let (u_n) be a (PS)$_d$ sequence for I. Then $I(u_n) \to d$ and there exists $(z_n) \in H_0^1(\Omega)$ such that $z_n \to 0$ and

$$\langle u_n, v-u_n\rangle - \int f(u_n)(v-u_n) \geq \langle z_n, (v-u_n)\rangle, \quad \forall\, v \in \mathcal{K}_{u_0}. \tag{3.7}$$

Since $u_n \geq u_0 \geq 0$ and $u_n \in \mathcal{K}_{u_0}$, we can put $v = 2u_n \in \mathcal{K}_{u_0}$ in the above expression to get

$$-\int f(u_n)u_n \geq \langle z_n, u_n\rangle - \|u_n\|^2.$$

This estimate and (f_2) imply that

$$\begin{aligned}d + o_n(1) = I(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{\theta}\int f(u_n)u_n - \int\left(F(u_n) - \frac{1}{\theta}f(u_n)u_n\right)\\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \frac{1}{\theta}\langle z_n, u_n\rangle.\end{aligned}$$

Since $z_n \to 0$ the result follows. □

Proposition 3.5 *The functional I satisfies the (PS)$_d$ condition for each*

$$d \leq c^* + \left(\frac{\theta-2}{2\theta}\right)\frac{\pi}{\alpha_0}. \tag{3.8}$$

Proof. Let (u_n) be a (PS)$_d$ sequence for I. By the previous lemma there exists $u \in H_0^1(\Omega)$ verifying

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\ u_n \to u & \text{strongly in } L^s(\Omega) \text{ for } s \geq 2,\\ u_n(x) \to u(x) & \text{a.e. in } \Omega.\end{cases}$$

We claim that

$$\lim_{n\to\infty}\int f(u_n)(u_n-u)=0. \tag{3.9}$$

If this is true we can prove the proposition as follows. Take $v=u$ in (3.7) to obtain

$$\|u_n\|^2 \le \langle u_n,u\rangle + \int f(u_n)(u_n-u) + \langle z_n, u_n-u\rangle.$$

Since $z_n \to 0$, we can use the above inequality, the weak convergence of (u_n) and (3.9) to get

$$\limsup_{n\to\infty}\|u_n\|^2 \le \|u\|^2.$$

The inequality $\|u\|^2 \le \liminf_{n\to\infty}\|u_n\|^2$ is a consequence of the weak convergence of (u_n). We conclude that $u_n \to u$ strongly in $H_0^1(\Omega)$ and the proposition is proved.

It remains to prove (3.9). First notice that, by (3.6), (3.8) and Lemma 3.3 we obtain

$$\limsup_{n\to\infty}\|u_n\|^2 \le \left(\frac{2\theta}{\theta-2}\right)\left(c^* + \left(\frac{\theta-2}{2\theta}\right)\frac{\pi}{\alpha_0}\right) < \frac{4\pi}{\alpha_0}.$$

Thus, we can choose $\beta > \alpha_0$ and $\gamma > 1$ sufficiently close to α_0 and 1, respectively, in such way that $\beta\gamma\|u_n\|^2 \le 4\pi$, for each $n \in \mathbb{N}$. The growth condition (f_0) provides $C > 0$ such that

$$f(s) \le C\exp(\beta s^2), \quad \text{for all } s \in \mathbb{R}.$$

It follows from Hölder's inequality and (2.2) that

$$\begin{aligned}\left|\int f(u_n)(u_n-u)\right| &\le C\|u_n-u\|_{\gamma'}\left(\int \exp\left(\beta\gamma\|u_n\|^2\left(u_n/\|u_n\|\right)^2\right)\right)^{1/\gamma}\\ &\le c_1\|u_n-u\|_{\gamma'}.\end{aligned}$$

The strong convergence $u_n \to u$ in $L^{\gamma'}(\Omega)$ implies that (3.9) holds and finishes the proof. □

Proof of Theorem 1.2. In view of Lemma 3.2 and the preceding proposition, we can apply Theorem 3.1 with $X = H_0^1(\Omega)$, $\mathcal{K} = \mathcal{K}_{u_0}$, $\mathcal{I} = I$ and $\underline{u} = u_0$, to obtain a (positive) critical point $\overline{u} \neq u_0$. □

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References

[1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problems with critical growth for the N-Laplacian*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. **17** (1990), 393–413.

[2] H. J. Choe, *A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems*, Arch. Rational Mech. Anal. **114** (1991), 383–394.

[3] J. P. Dias, *Variational inequalities and eigenvalue problems for nonlinear maximal monotone operators in a Hilbert space*, Amer. J. Math. **97** (1975), 905–914.

[4] J. P. Dias and J. Hernndez, *A Sturm-Liouville theorem for some odd multivalued maps*, Proc. Amer. Math. Soc. **53** (1975), 72–74.

[5] G. Duvaut and J.L. Lions, Inequalities in Mechanics and Physics. Springer, Berlin, 1976.

[6] A. Friedman, Variational principles and free boundary problems, Pure and Applied Mathematics, New York, John Wiley and Sons (1982).

[7] D. Joseph and T. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.

[8] Y. Jianfu, *Positive solutions of quasilinear elliptic obstacle problems with critical exponents*, Nonlinear Anal. **25** (1995), 1283–1306.

[9] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Pure and Applied Mathematics, 88, New York, Academic Press (1980).

[10] G. Mancini and R. Musina, *A free boundary problem involving limiting Sobolev exponents*, Manuscripta Math. **58** (1987), 77–93.

[11] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[12] T. Suzuki, *Semilinear Elliptic Equations*, Gakkotosho, Tokyo (1994).

[13] T. Suzuki and K. Nagasaki, *On the nonlinear eigenvalue problem* $\Delta u + e^u = 0$, Trans. Amer. Math. Soc., **39** (1988), 591–608.

[14] A. Szulkin, *Minimax principle for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Analysis non Lineaire **3** (1986), 77–109

[15] N. Trudinger, *On embedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.