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Quasilinear Schrödinger Equations with Asymptotically Linear Nonlinearities *

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Abstract

We deal with the existence of nonzero solution for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x,u), \quad x \in \mathbb{R}^N, u \in H^1(\mathbb{R}^N),$$

where V is a positive potential and the nonlinearity $g(x,s)$ behaves like $K_0(x)s$ at the origin and like $K_\infty(x)|s|^p$, $1 \leq p \leq 3$, at infinity. In the proofs we apply minimization methods.

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1 Introduction

In this paper we study the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - l(x,|z|^2)z - \kappa[\Delta\rho(|z|^2)]\rho'(|z|^2)z$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a real constant and l, ρ are real functions. Equations of this type appear naturally in mathematical physics and have been accepted as

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models of several physical phenomena corresponding to various types of nonlinear terms ρ. They include equations in fluid mechanics, theory of Heisenberg ferromagnetism and magnons, dissipative quantum mechanics and matter theory (see [11, 12] and references therein).

Here we focus on the case which models the time evolution of the condensate wave function in superfluid film equation in plasma physics, see [10]. If we look for standing wave solutions $z(t,x) := \exp(-iEt)u(x)$ with $E > 0$, we are lead to consider the following elliptic equation

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

with $V(x) := W(x) - E$ and $g(x,s) := l(x,|s|^2)s$ being the new nonlinear term.

We address the existence of solution for que quasilinear equation

$$(P) \qquad \begin{cases} -\Delta u + V(x)u - \Delta(u^2)u = g(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$ and the potential V satisfy

(V_1) $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$;

(V_2) for any $M > 0$ there holds

$$\text{meas}\Big(\{x \in \mathbb{R}^N : V(x) \leq M\}\Big) < +\infty.$$

and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies the growth condition

(g_1) there exist $a, b \in L^\alpha(\mathbb{R}^N)$, $\alpha > N/2$, such that

$$|g(x,s)| \leq a(x)|s| + b(x)|s|^3, \quad \text{for all } x \in \mathbb{R}^N,\ s \in \mathbb{R}.$$

In our first result we are interested in the case where $g(x,\cdot)$ behaves like s at the origin and like s^3 at infinity. We define $G(x,s) := \int_0^s g(x,\tau)\mathrm{d}\tau$, introduce the set

$$\mathcal{F} := \{w : \mathbb{R}^N \to \mathbb{R} \ : \ w^+ \not\equiv 0, w \in L^\alpha(\mathbb{R}^N) \ \text{ for some } \ \alpha > N/2\},$$

and consider the following asymptotic assumption at the origin and at infinity:

(G_0) there exists $K_0 \in \mathcal{F}$ such that

$$\liminf_{s\to 0} \frac{2G(x,s)}{s^2} = K_0(x), \quad \text{uniformly for a.e. } x \in \mathbb{R}^N;$$

(G_∞) there exists $K_\infty \in \mathcal{F}$ such that

$$\limsup_{|s|\to+\infty} \frac{4G(x,s)}{s^4} = K_\infty(x), \quad \text{uniformly for a.e. } x \in \mathbb{R}^N.$$

In order to state our results we define the space

$$X := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x < \infty\right\}, \tag{1.2}$$

endowed with the norm

$$\|u\|_X := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) \mathrm{d}x \right)^{1/2}.$$

It is well known (see [9]) that, under the conditions $(V_0) - (V_1)$, the space X is a closed subspace of $H^1(\mathbb{R}^N)$. Moreover, the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any $2 \leq q < 2^* := 2N(N-2)$. Hence, for any given $K \in \mathcal{F}$, we can prove that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda K(x)u \text{ in } \mathbb{R}^N, \quad u \in X, \tag{1.3}$$

has a first positive eigenvalue $\lambda_1(K) > 0$. The same occurs with the eigenvalue problem

$$-\Delta u = \mu K(x)u \text{ in } \mathbb{R}^N, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{1.4}$$

We shall denote by $\mu_1(K) > 0$ its first positive eigenvalue.

In our first result we consider the case of resonance at infinity and prove the following result:

Theorem 1.1 *Suppose that V satisfies* $(V_1) - (V_2)$*, and g satisfies* $(g_1), (G_0)$*,* (G_∞) *and*

(g_2) *there exists a nonnegative function* $\Gamma \in L^1(\mathbb{R}^N)$ *such that*

$$g(x,s)s - 4G(x,s) \geq -\Gamma(x), \quad \textit{for all } x \in \mathbb{R}^N,\ s \in \mathbb{R}.$$

Then problem (P) *admits at least one nontrivial solution provided*

$$\lambda_1(K_0) < 1 \leq \mu_1(K_\infty).$$

In our second result we are interested in the case that $\mu_1(K_\infty) > 1$. However, in this setting we impose a slightly different growth condition. So, we define

$$g_\infty(x,s) := g(x,s) - K_\infty(x)s^3, \quad x \in \mathbb{R}^N, \quad s \in \mathbb{R},$$

where $K_\infty \in \mathcal{F}$ comes from (G_∞) and replace the condition (g_1) by the stronger one:

$(\widetilde{g_1})$ there exist $r \in (2,4]$ such that

$$|g_\infty(x,s)| \leq \widetilde{a}(x)|s| + \widetilde{b}(x)|s|^{r-1}, \quad \text{for all } x \in \mathbb{R}^N,\ s \in \mathbb{R},$$

where $\widetilde{a} \in L^{(2^*)'}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$, $\widetilde{b} \in L^{(2\cdot 2^*/r)'}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$ and $\alpha > N/2$,

with s' denoting the conjugated exponent of $s > 1$, namely the unique $s' > 1$ satisfying $1/s + 1/s' = 1$.

With this new condition we can consider the nonresonant case and prove the following result:

Theorem 1.2 *Suppose that V satisfies* $(V_1) - (V_2)$*, and g satisfies* (G_0)*,* (G_∞) *and* $(\widetilde{g_1})$*. If* $\lambda_1(K_0) < 1$*, then the problem* (P) *admits at least one nontrivial solution provided one of the following conditions hold*

(C_1) $\mu_1(K_\infty) > 1$ *and* $r \in (2,4)$*;*

(C_2) $\mu_1(K_\infty) > 1$*,* $r = 4$ *and* $\|\widetilde{b}\|_{L^{(2^*/2)'}(\mathbb{R}^N)}$ *is sufficiently small;*

(C_3) $K_\infty \equiv 0$ *and* $r \in (2,4)$*.*

We stress that conditions (C_1) or (C_2) do not cover the sublinear case. For example, if G behaves at the infinity like the function $|s|^p \ln(1 + \theta|s|)$, for $p \in (2, 4)$ and $\theta \geq 0$, then the limit function K_∞ vanishes. However, even in this case, we have that (C_3) is verified and therefore we can obtain a solution.

In what follows we present some comments on known results for the equation (1.1). The semilinear case $\kappa = 0$ has been studied extensively in recent years with a huge variety of conditions on the potential V and the nonlinearity g (see e.g. [1, 14]). As far we know, the case $\kappa \neq 0$ was firstly considered in [13], where the existence of positive ground state solution was obtained via minimization methods. By using a change of variables the authors in [12] reduced the equation to a semilinear one and an Orlicz space framework was used to prove the existence of a positive solutions via Mountain Pass Theorem. The same method was also used in [2], but the usual Sobolev space $H^1(\mathbb{R}^N)$ was used as the working space. We refer the reader to [4, 15, 16, 5, 8] for more results.

Usually the authors consider the case that $g(x, \cdot)$ is sublinear at the origin and superlinear at infinity. Due to the change of variables introduced in [12] this behavior at infinity is related with the (modified) Ambrosetti-Rabinowitz condition $0 < \theta G(x, s) \leq g(x, s)s$ for some $\theta > 4$, any $x \in \mathbb{R}^N$, $s \neq 0$. This type of condition provides the boundedness of the Palais-Smale sequences of the associated functional. More generally, under suitable extra assumptions, it is possible to deal with the condition $\lim_{|s|\to+\infty} G(x, s)/s^4 = +\infty$ (see [15, 18]).

Here we do not consider superlinear nonlinearities. Instead, we suppose that $g(x, s) \sim K_0(x)s$ near the origin and $g(x, s) \sim K_\infty(x)|s|^{p-1}$, $2 < p \leq 4$, at infinity. It appears that there are few papers which deal with this type of nonlinearity at infinity. The first one is the paper [12] which states, among other results, the existence of positive solution for the autonomous nonlinearity $g(x, s) = s^3$ under different kind of hypothesis on the potential V. We have recently learned that the authors in [17] have obtained some existence results under the condition $\lim_{|s|\to+\infty} G(x, s)/s^4 > 0$ and other mild assumptions on g. We emphasize that we allow that K_∞ changes sign or even that $K_\infty \equiv 0$. We finally mention a recent paper of Fang and Szulkin [6] where they consider $g(x, s) = q(x)s^3$ and obtained the existence of infinitely many solutions under some symmetry conditions on the potential V.

In the proofs of the main theorems we apply minimization techniques. Although this kind of idea has already appeared in [13], we follow here a different approach. We use the change of variables introduced in [12] to define an Orlicz space E and an associated functional $J : E \to \mathbb{R}$ whose critical points are weak solutions of (P). Under the setting of Theorem 1.2 we are able to prove that J is coercive. Since we do not know if E is reflexive, we cannot assume that a minimizing sequence weakly converges in E. So, after proving that J satisfies the Palais-Smale condition, we can use the condition at the origin for proving that the infimum is negative and it is attained. The proof of Theorem 1.1 follows the same lines. The additional condition (g_2) is used to prove that the functional is bounded from below. This condition is in some sense related with a nonquadraticity condition introduced in [3] (see also [7]). We also emphasize that the compactness properties proved here use only the conditions (g_1) and (g_2). Hence, many other kinds of linking situations can be considered for the equation (P).

The paper is organized as follows: in section 2 we introduce the notation, some preliminaries results and useful tools. In Section 3 we prove Theorem. In Section 4 we lead with the resonant case by presenting the proof of Theorem 1.1. In the final Section 5 we obtain the compactness condition required in all the the former sections.

2 Variational framework

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. Moreover, for any $p \geq 1$, we denote by $\|u\|_p$ the $L^p(\mathbb{R}^N)$-norm of a function.

From the variational point of view, the problem (P) is formally the Euler-Lagrange equation associated to the functional

$$I(u) = \frac{1}{2}\int (1 + 2u^2)|\nabla u|^2 + \int V(x)u^2 - \int G(x, u). \tag{1.5}$$

Nevertheless, as quoted in [2], the term $\int u^2|\nabla u|^2$ is not well defined in $H^1(\mathbb{R}^N)$. Hence, following the idea introduced in [12] and the variational approach of [2], we reformulate the problem (P) by using the change of variable $f : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{cases} f'(t) = \dfrac{1}{\sqrt{1 + 2f(t)^2}}, & t \geq 0, \\ f(t) = -f(-t), & t \leq 0. \end{cases} \tag{1.6}$$

We present below the main properties of the function f.

Lemma 2.1 *The function f satisfies the following properties:*

(f_1) *f is uniquely determined, C^∞ and invertible;*

(f_2) $0 < f'(t) \leq 1$ *for all* $t \in \mathbb{R}$*;*

(f_3) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*

(f_4) $\lim_{t\to 0} \frac{f(t)}{t} = 1$*;*

(f_5) $\lim_{t\to +\infty} \frac{f(t)}{\sqrt{t}} = 2^{1/4}$*;*

(f_6) $\frac{f(t)}{2} \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$*;*

(f_7) $|f(t)| \leq 2^{1/4}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$*;*

(f_8) *there exists $\kappa > 0$ such that*

$$|f(t)| \geq \begin{cases} \kappa|t|, & |t| \leq 1, \\ \kappa|t|^{1/2}, & |t| \geq 1\,; \end{cases}$$

(f_9) $|f(t)f'(t)| \leq 2^{-1/2}$ *for all* $t \in \mathbb{R}$*;*

(f_{10}) *the function f^2 is strictly convex. In particular,* $f^2(st) \leq sf^2(t)$ *for all* $t \in \mathbb{R},\ s \in [0, 1]$*;*

(f_{11}) $f^2(st) \leq s^2 f^2(t)$ *for all* $t \in \mathbb{R},\ s \geq 1$*;*

(f_{12}) $f^2(s - t) \leq 4(f^2(s) + f^2(t))$ *for all* $s,\ t \in \mathbb{R}$.

Proof. We only prove (f_{11}) and (f_{12}). The other properties can be proved by using the ODE in (1.6) and arguing as in the papers [2, 12]. For proving (f_{11}) we notice that, since $f''(t) \leq 0$ in $[0, +\infty)$, we have that f' is non-increasing in this interval. For any $t \geq 0$ fixed we consider the function $h(s) := f(st) - sf(t)$ defined for $s \geq 1$. We have that $h'(s) = tf'(st) - f(t) \leq tf'(t) - f(t) \leq 0$, by (f_6). Since $h(1) = 0$ we conclude that $h(s) \leq 0$ for any $s \geq 1$, that is $f(st) \leq sf(t)$ for any $t \geq 0$ and $s \geq 1$. Thus

$$f^2(st) \leq s^2 f^2(t)$$

if $t \geq 0$ and $s \geq 1$. Since f^2 is even the item follows.

In order to establish item (f_{12}), we use the fact that f^2 is even and increasing in $(0, +\infty)$ together with (f_{10}) and (f_{11}) to get

$$\begin{aligned} f^2(s-t) &= f^2(|s-t|) \leq f^2(|s| + |t|) \\ &\leq f^2(2\max\{|s|, |t|\}) \leq 4(f^2(s) + f^2(t)), \end{aligned}$$

which concludes the proof. □

By using the solution f of (1.6) we can define the following Orlicz-Sobolev space

$$E := \left\{ v \in H^1(\mathbb{R}^N) : \int V(x) f^2(v) < \infty \right\}.$$

As we will see later, E is a Banach space when endowed with the norm

$$\|v\| := \|\nabla v\|_2 + |v|_f, \quad \text{for any } v \in E, \tag{1.7}$$

where

$$|v|_f := \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int V(x) f^2(\xi v) \right\}.$$

We summarize in the next proposition the main properties of the space E. Hereafter, we shall denote

$$Q(v) := \int |\nabla v|^2 + V(x) f^2(v), \quad v \in E.$$

Proposition 2.1 *Suppose that V satisfies $(V_1) - (V_2)$. Then E possesses the following properties:*

1. *If $v_n(x) \to v(x)$ a. e. in $\mathbb{R}^N$ and*

$$\lim_{n \to +\infty} \int V(x) f^2(v_n) = \int V(x) f^2(v),$$

then

$$\lim_{n \to +\infty} |v_n - v|_f = 0;$$

2. *The following embeddings are continuous: $X \hookrightarrow E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$;*

3. *The map $v \to f(v)$ from E to $L^q(\mathbb{R}^N)$ is continuous for any $q \in [2, 2 \cdot 2^*]$, and it is compact for any $q \in [2, 2 \cdot 2^*)$;*

4. *For any $v \in E$ there holds*

$$\left\| \frac{f(v)}{f'(v)} \right\| \leq 4\|v\|;$$

5. *If* $v_n \rightharpoonup 0$ *weakly in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $\left(\int V(x)f^2(v_n)\right)$ *is bounded then, up to a subsequence,* $f(v_n) \to 0$ *strongly in* $L^q(\mathbb{R}^N)$ *for any* $2 \le q < 2 \cdot 2^*$*;*

6. *For any* $v \in E$ *there holds*

$$|v|_f \le 2\max\left\{\int V(x)f^2(v), \left(\int V(x)f^2(v)\right)^{1/2}\right\};$$

7. *For any* $v \in E$ *there holds*

$$|v|_f \ge \min\left\{\int V(x)f^2(v), \left(\int V(x)f^2(v)\right)^{1/2}\right\}.$$

Proof. The three first items are proved in [4]. In order to prove the fourth we take $v \in E$ and notice that, by using (1.6) and a straightforward calculation, we get

$$\nabla\left(\frac{f(v)}{f'(v)}\right) = \left(1 + \frac{2f^2(v)}{1+2f^2(v)}\right)\nabla v \tag{1.8}$$

and therefore

$$\left\|\nabla\left(\frac{f(v)}{f'(v)}\right)\right\|_2 \le 2\|\nabla v\|_2. \tag{1.9}$$

By (f_6), we have that $1 \le \frac{f(t)}{tf'(t)} \le 2$ for any $t \ne 0$, and therefore we can use (f_{10}) to get

$$f^2\left(\xi\frac{f(t)}{f'(t)}\right) = f^2\left(\frac{f(t)}{tf'(t)}\xi t\right) \le \left(\frac{f(t)}{tf'(t)}\right)^2 f^2(\xi t) \le 4f^2(\xi t),$$

for any $t \in \mathbb{R}$, $\xi > 0$. Thus,

$$\left|\frac{f(v)}{f'(v)}\right|_f = \inf_{\xi>0}\left\{\frac{1}{\xi}\left(1 + \int V(x)f^2\left(\xi\frac{f(v)}{f'(v)}\right)\right)\right\} \le 4|v|_f.$$

Statement 4 follows from the above inequality and (1.9).

We now prove item 5. We may suppose that $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$. Since

$$\|f(v_n)\|_X^2 = \int\left(\frac{|\nabla v_n|^2}{1+2f^2(v_n)} + V(x)f^2(v_n)\right) \le \int\left(|\nabla v_n|^2 + V(x)f^2(v_n)\right),$$

the sequence $(f(v_n))$ is bounded in X. Hence, up to a subsequence, it weakly converges in X. The compactness of embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, for $2 \le q < 2^*$, and the pointwise convergence $f(v_n(x)) \to f(0) = 0$ a.e. in $\mathbb{R}^N$ imply that the weak limit is zero. So, $f(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$, whenever $2 \le q < 2^*$.

The Sobolev inequality and a straightforward calculation provide

$$\begin{aligned}\|f(v_n)\|_{2\cdot 2^*} &= \|f^2(v_n)\|_{2^*}^{1/2} \le c_1\|\nabla(f^2(v_n))\|_2^{1/2} \\ &= c_1\left(\int \frac{4f^2}{1+2f^2}|\nabla v_n|^2\right)^{1/4} \le 2c_1\|\nabla v_n\|_2^{1/2}.\end{aligned} \tag{1.10}$$

It follows from the interpolation inequality that $f(v_n) \to 0$ in $L^q(\mathbb{R}^N)$ for $2 \le q < 2 \cdot 2^*$.

For the proof of item 6 we argue as in [8]. By supposing that $v \not\equiv 0$ we may consider two cases. If $\int V(x)f^2(v) > 1$ then we set $\xi_0 = \left(\int V(x)f^2(v)\right)^{-1} < 1$ and use the definition of $|v|_f$ and (f_{10}) to get

$$\begin{aligned} |v|_f &\le \frac{1}{\xi_0}\left(1 + \int V(x)f^2(\xi_0 v)\right) \\ &\le \frac{1}{\xi_0}\left(1 + \xi_0 \int V(x)f^2(v)\right) = 2\int V(x)f^2(v). \end{aligned}$$

If $0 < \int V(x)f^2(v) \le 1$ we set $\xi_0 = \left(\int V(x)f^2(v)\right)^{-1/2}$, use (f_{11}) and argue as above to conclude that $|u|_f \le 2(\int V(x)f^2(v))^{1/2}$. This and the above expression finish the proof of item 6. The proof of item 7 is similar and we omit it. □

By a weak solution of (P) we mean a function $u \in H^1(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ such that

$$\int [(1 + 2u^2)\nabla u \nabla \varphi + 2u|\nabla u|^2\varphi + V(x)u\varphi] = \int g(x,u)\varphi,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. After the change of variables $u = f(v)$ in the map given in (1.5), we obtain the functional

$$J(v) := \frac{1}{2}\int \left(|\nabla v|^2 + \int V(x)f^2(v)\right) - \int G(x, f(v)), \tag{1.11}$$

for any $v \in E$. Under the growth condition (g_1) (or $(\widetilde{g_1})$) the functional J belongs to $C^1(E,\mathbb{R})$ and its critical points are weak solutions of the problem

$$-\Delta v + V(x)f'(v)f(v) = g(x, f(v))f'(v), \quad v \in E, \tag{1.12}$$

Moreover, if $v \in E \cap C^2(\mathbb{R}^N)$ is a critical point of J then the function $u = f(v)$ is a classical solution of (P) (see [2] for details). Thus, we deal in the sequel with the modified problem described above.

3 The coercive case

In this section we shall prove Theorem 1.2. The main point is that, in this setting, we are able to prove that J is coercive, as you can see in the next result.

Lemma 3.1 *Suppose that g satisfies (G_∞) and $(\widetilde{g_1})$. Then J is coercive on E provided one of the conditions $(C_1) - (C_3)$ stated in Theorem 1.2 holds.*

Proof. Suppose first that (C_1) holds, in such way that $r \in (2,4)$. For any $(x,s) \in \mathbb{R}^N \times \mathbb{R}$ we set

$$G_\infty(x,s) := \int_0^s g_\infty(x,t)\mathrm{d}t.$$

Given $v \in E$, it follows from $(\widetilde{g_1})$ and Hölder's inequality that

$$\begin{aligned} \int G_\infty(x, f(v)) &\le \frac{1}{2}\int \widetilde{a}(x)f^2(v) + \frac{1}{r}\int \widetilde{b}(x)|f(v)|^r \\ &\le \frac{1}{2}\|a\|_{2N/(N+2)}\|f(v)\|^2_{2\cdot 2^*} + \frac{1}{4}\|b\|_{2\cdot 2^*/(2\cdot 2^*-r)}\|f(v)\|^r_{2\cdot 2^*}. \end{aligned}$$

Hence, we can use the estimate in (1.10) to get

$$\int G_\infty(x, f(v)) \le c_1\|\nabla v\|_2 + c_2\|\nabla v\|_2^{r/2} \le c_3\left(Q(v)^{1/2} + Q(v)^{r/4}\right), \tag{1.13}$$

for some $c_3 > 0$. On the other hand, since $f^2(v) \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we can use the variational inequality for $\mu_1(K_\infty)$ and (f_9) to obtain

$$\int K_\infty(x)\left(f^2(v)\right)^2 \le \frac{4}{\mu_1(K_\infty)}\int (f(v)f'(v))^2|\nabla v|^2 \le \frac{2}{\mu_1(K_\infty)}Q(v). \tag{1.14}$$

This inequality and (1.13) provides, for any $v \in E$,

$$\begin{aligned} J(v) &= \frac{1}{2}Q(v) - \frac{1}{4}\int K_\infty(x)f^4(v) - \int G_\infty(x, f(v)) \\ &\ge \frac{1}{2}\left(1 - \frac{1}{\mu_1(K_\infty)}\right)Q(v) - c_3(Q(v)^{1/2} + Q(v)^{r/4}). \end{aligned} \tag{1.15}$$

Now suppose that $(v_n) \subset E$ is such that $\|v_n\| \to +\infty$ as $n \to \infty$. By using item 6 of Proposition 2.1 we get

$$\|v_n\| \le \|\nabla v_n\|_2 + 2\max\left\{\int V(x)f^2(v_n), \left(\int V(x)f^2(v_n)\right)^{1/2}\right\}.$$

Hence, along a subsequence, we have that either

$$\lim_{n\to+\infty}\|\nabla v_n\|_2 = +\infty \quad \text{or} \quad \lim_{n\to+\infty}\int V(x)f^2(v_n) = +\infty.$$

Anyway, we conclude that $Q(v_n) \to \infty$. Since $r < 4$ it follows from (1.15) that none subsequence of $(J(v_n))$ can be bounded, and therefore J is coercive.

If (C_2) holds then $r = 4$ and the equation (1.13) becomes

$$\int G_\infty(x, f(v)) \le c_1 Q^{1/2}(v) + \frac{1}{4}\|\widetilde{b}\|_{(2^*/2)'}Q(v).$$

Hence we can argue as in (1.14) – (1.15) to get

$$J(v) \ge \frac{1}{2}\left(1 - \frac{1}{\mu_1(K_\infty)} - \frac{1}{2}\|\widetilde{b}\|_{(2^*/2)'}\right)Q(v) - c_3 Q^{1/2}(v).$$

Since $\mu_1(K_\infty) > 1$, the term into the parenthesis above is positive if $\|\widetilde{b}\|_{(2^*/2)'}$ is small, and the proof follows as before.

Finnally, if (C_3) holds, then $g_\infty(x, s) = g(x, s)$. So (1.15) becomes

$$J(v) \ge \frac{1}{2}Q(v) - c_3(Q^{1/2}(v) + Q^{r/4}(v))$$

and the result follows as in case (C_1). □

In the next result we use the behavior of g near the origin to prove that J attains negative values.

Lemma 3.2 *Suppose that g satisfies (g_1) or $(\widetilde{g_1})$, and (G_0) with $\lambda_1(K_0) < 1$. Then there exists $v_0 \in E$ such that $J(v_0) < 0$.*

Proof. Let $\varphi_1 > 0$ be such that

$$-\Delta\varphi_1 + V(x)\varphi_1 = \lambda_1(K_0)K_0(x)\varphi_1 \ \text{ in } \mathbb{R}^N. \tag{1.16}$$

It follows from (f_3) that

$$J(s\varphi_1) \leq \frac{s^2}{2}\int(|\nabla\varphi_1|^2 + V(x)\varphi_1^2) - \int G(x, f(s\varphi_1)),$$

for any $s \in \mathbb{R}$. By using (1.16) we get

$$\frac{2J(s\varphi_1)}{s^2} \leq \lambda_1(K_0)\int K_0(x)\varphi_1^2 - \int \frac{2G(x, f(s\varphi_1))}{s^2}. \tag{1.17}$$

The growth condition $(\widetilde{g_1})$ (or even (g_1)) together with (f_3) and (f_4) show that the ratio $G(x, f(s\varphi_1))/s^2$ is uniformly bounded by an integrable function. Hence, we can use Fatou's lemma to obtain

$$\begin{aligned}\liminf_{s\to 0^+}\int \frac{2G(x, f(s\varphi_1))}{s^2} &\geq \int \liminf_{s\to 0^+}\left[\frac{2G(x, f(s\varphi_1))}{f^2(s\varphi_1)}\left(\frac{f(s\varphi_1)}{s\varphi_1}\right)^2\varphi_1^2\right] \\ &= \int K_0(x)\varphi_1^2,\end{aligned}$$

where we have used the continuity of f, $f(0) = 0$, (f_4) and (G_0). Coming back to (1.17) we conclude that

$$\limsup_{s\to 0}\frac{2J(s\varphi_1)}{s^2} \leq (\lambda_1(K_0) - 1)\int K_0(x)\varphi_1^2 < 0$$

and it suffices to take $v_0 := t\varphi_1$, with t sufficiently small, to get $J(v_0) < 0$. □

The proof of the Palais-Smale condition for J is quite long and technical. So, we prefer to present it only in the final section of the paper (see Proposition 5.2). Assuming that (PS) holds, we can obtain a nonzero weak solutions for (P) in the coercive cases as follows:

Proof of Theorem 1.2. Since J is coercive and maps bounded sets into bounded sets, we have that

$$c_0 := \inf_{v\in E} J(v) > -\infty.$$

The Ekeland Variational Principle provides a sequence $(v_n) \subset E$ such that $J(v_n) \to c_0$ and $J'(v_n) \to 0$. By coercivity of J it follows that (v_n) is a bounded sequence. The Palais-Smale condition guarantees that, along a subsequence, $v_n \to v$ strongly in E. Thus $J'(v) = 0$ and, by the last lemma, $J(v) = c_0 < 0$, that is, $v \not\equiv 0$ is the desired solution of the problem (1.12). □

Remark 3.1 We notice that the above proof could be considerable shortened if you could prove that the space E is reflexive. Actually, it follows from item (3) of Proposition 2.1 that $\int G(x, f(v_n)) \to \int G(x, f(v))$ whenever $v_n \rightharpoonup v$ weakly in E, and therefore J is lower semicontinuous on E. The point is that, since we do not know if E is reflexive, we cannot guarantee that a bounded sequence in E has a weakly convergent subsequence. Hence, the proof of the Palais-Smale condition seems to be necessary to get the result.

4 Resonance at the first eigenvalue

In this section we consider the (more delicate) situation when the resonance phenomenon occurs at the first positive eigenvalue. In this setting, we do not know if J is coercive. However, we are able to apply minimization procedure, as we can see from the next result.

Lemma 4.1 *Suppose g satisfies (g_2) and (G_∞) with $\mu_1(K_\infty) \geq 1$. Then the functional J is bounded from below.*

Proof. In the proof of this lemma we argue along the same lines of [3]. We first claim that

$$L(x,s) := G(x,s) - \frac{K_\infty(x)s^4}{4} \leq \frac{\Gamma(x)}{4},$$

for any $x \in \mathbb{R}^N$, $s \in \mathbb{R}$. Assuming the claim we can prove the lemma in the following way: given $v \in E$, we can use (1.14) and the above claim to get

$$\begin{aligned} J(v) &= \frac{1}{2}Q(v) - \frac{1}{4}\int K_\infty(x)f^4(v) - \int\left(G(x,f(v)) - \frac{1}{4}K_\infty(x)f^4(v)\right) \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\mu_1(K_\infty)}\right)Q(v) - \frac{1}{4}\|\Gamma\|_1 \geq -\frac{1}{4}\|\Gamma\|_1, \end{aligned}$$

where we have used $\mu_1(K_\infty) \geq 1$ in the last inequality.

It remains to prove the claim. Notice that, by (g_2), there holds

$$\frac{d}{d\tau}\left(\frac{L(x,\tau)}{\tau^4}\right) = \frac{g(x,\tau)\tau - 4G(x,\tau)}{\tau^5} \geq -\frac{\Gamma(x)}{\tau^5}, \qquad \tau > 0,$$

Integrating the above expression over $[s,t] \subset (0,+\infty)$ we get

$$\frac{L(x,t)}{t^4} \geq \frac{L(x,s)}{s^4} + \frac{\Gamma(x)}{4}\left(\frac{1}{t^4} - \frac{1}{s^4}\right). \tag{1.18}$$

In view of (G_∞) we have that

$$\limsup_{t\to+\infty} \frac{L(x,t)}{t^4} = \limsup_{t\to\infty} \frac{1}{4}\left(\frac{4G(x,t)}{t^4} - K_\infty(x)\right) = 0.$$

Thus, taking the limsup as $t \to +\infty$ in (1.18), we conclude that $L(x,s) \leq \Gamma(x)/4$, for any $x \in \mathbb{R}^N$, $s \geq 0$. The proof for $s < 0$ is analogous. □

As in the previous section, we have that J satisfies the Palais-Smale condition (see Proposition 5.1 in the next section). We are now ready to prove our result for the resonant case.

Proof of Theorem 1.1. It follows from Lemmas 3.2 and 4.1 that

$$-\infty < c_0 := \inf_{v\in E} J(v) < 0.$$

We can now apply the Ekeland Variational Principle and use the same argument of the proof of Theorem 1.2 to obtain a nonzero solution. □

5 The Palais-Smale condition

We devote the rest of the paper to proving that, under the hypotheses of any of our main theorems, the functional $J : E \to \mathbb{R}$ satisfies the Palais-Smale condition. This is crucial in our arguments due to the compactness required in variational methods. We start proving the boundedness of the (PS) sequences in E.

Lemma 5.1 *Suppose that g satisfies (g_1) and (g_2). Then any $(PS)_c$ sequence of J is bounded.*

Proof. Let $(v_n) \subset E$ be such that

$$\lim_{n\to+\infty} J(v_n) = c, \qquad \lim_{n\to+\infty} J'(v_n) = 0.$$

In view of item 4 of Proposition 2.1 we have that $f(v_n)/f'(v_n) \in E$. Hence, we can use (1.8) to compute

$$J'(v_n) \cdot \frac{f(v_n)}{f'(v_n)} \le 2\int |\nabla v_n|^2 + \int V(x)f^2(v_n) - \int g(x, f(v_n))f(v_n).$$

By using item 4 of Proposition 2.1 again we get

$$\begin{aligned} c + o_n(1)\|v_n\| \quad &\ge \quad J(v_n) - \frac{1}{4}J'(v_n) \cdot \frac{f(v_n)}{f'(v_n)} \\ &\ge \quad \frac{1}{4}\int V(x)f^2(v_n) + \frac{1}{4}\int \left(g(x, f(v_n))f(v_n) - 4G(x, f(v_n))\right), \end{aligned}$$

and therefore it follows from (g_2) that

$$\int V(x)f^2(v_n) \le 4c + \|\Gamma\|_1 + o_n(1)\|v_n\|. \tag{1.19}$$

Arguing by contradiction we suppose that, up to a subsequence, $\|v_n\| \to +\infty$ as $n \to +\infty$. We define $w_n := v_n/\|v_n\|$ and notice that, since we may suppose that $\|v_n\| \ge 1$, the above inequality and (f_{10}) provide

$$\int V(x)f^2(w_n) = \int V(x)f^2\left(\frac{v_n}{\|v_n\|}\right) \le \frac{1}{\|v_n\|}\int V(x)f^2(v_n) \to 0.$$

Since (w_n) is bounded in $\mathcal{D}^{1,2}$, up to a subsequence we have that $w_n \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. By Fatou's lemma $\int V(x)f^2(w) \le \liminf_{n\to\infty}\int V(x)f^2(w_n) = 0$, and therefore $w = 0$. It follows from item 1 of Proposition 2.1 that

$$|w_n|_f \to 0. \tag{1.20}$$

We now claim that

$$\lim_{n\to+\infty} \frac{1}{\|v_n\|^2}\int G(x, f(v_n)) = 0. \tag{1.21}$$

If this is true we can finish the proof by noticing that

$$\int |\nabla w_n|^2 = -\frac{J(v_n)}{\|v_n\|^2} - \frac{1}{\|v_n\|^2}\int V(x)f^2(v_n) + \frac{1}{\|v_n\|^2}\int G(x, f(v_n)) \to 0,$$

where we have used $J(v_n) \to c$, (1.19) and (1.21). This convergence and (1.20) implies that $1 = \|w_n\| = \|\nabla w_n\|_2^2 + |w_n|_f \to 0$, which does not make sense. This contradiction shows that (v_n) is bounded.

In the sequel we prove (1.21). We first notice that, since $v_n = \|v_n\|w_n$, we can use $(g_1), (f_3)$ and (f_7) to get

$$\begin{aligned}\frac{|G(x, f(v_n))|}{\|v_n\|^2} &\leq \frac{a(x)}{2}\frac{f^2(\|v_n\|w_n)}{\|v_n\|^2} + \frac{b(x)}{4}\frac{f^4(\|v_n\|w_n)}{\|v_n\|^2} \\ &\leq \frac{1}{2}\left(a(x) + b(x)\right) w_n^2.\end{aligned}$$

It follows from (f_8) that

$$|t| \leq \frac{1}{\kappa}|f(t)| + \frac{1}{\kappa^2} f^2(t), \quad \text{for any } t \in \mathbb{R}. \tag{1.22}$$

Hence

$$\int \frac{|G(x, f(v_n))|}{\|v_n\|^2} \leq c_1 \int (a(x) + b(x))(f^2(w_n) + f^4(w_n)). \tag{1.23}$$

On the other hand, item 5 of Proposition 2.1 implies that, up to a subsequence,

$$f(w_n) \to 0 \text{ strongly in } L^q(\mathbb{R}^N) \text{ for any } 2 \leq q < 2 \cdot 2^*. \tag{1.24}$$

Recalling that $b \in L^\alpha(\mathbb{R}^N)$ with $\alpha > N/2$, we can use Hölder's inequality to get

$$\int b(x) f^4(w_n) \leq \|b\|_\alpha \|f(w_n)\|^4_{4\alpha/(\alpha-1)} \to 0,$$

where we have used (1.24) and the fact that $4 < 4\alpha/(\alpha - 1) < 2 \cdot 2^*$. The same argument shows that

$$\max\left\{\int a(x) f^4(w_n), \int a(x) f^2(w_n), \int b(x) f^2(w_n)\right\} \to 0.$$

The lemma follows from the above convergences and (1.23). □

We are ready to prove that J satisfies the well known Palais-Smale condition.

Proposition 5.1 *Suppose that g satisfies (g_1) and (g_2). Then the functional J satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let $(v_n) \subset E$ be a (PS)-sequence. It follows from the last lemma that (v_n) is bounded in E. Hence, for some $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we have that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since we also have pointwise convergence we can use (1.19) and Fatou's lemma to get

$$\int V(x) f^2(v) \leq \liminf_{n\to+\infty} \int V(x) f^2(v_n) < \infty. \tag{1.25}$$

This and item 7 of Proposition 2.1 imply that $v \in E$. In the sequel we shall prove that $\|v_n - v\| \to 0$.

We start by noticing that, since f^2 is convex, the function $Q : E \to \mathbb{R}$ given by

$$Q(v) := \int |\nabla v|^2 + \int V(x) f^2(v),$$

is also convex. Hence,

$$\begin{aligned}Q(v) - Q(v_n) &\geq Q'(v_n) \cdot (v - v_n) \\ &= 2J'(v_n) \cdot (v - v_n) + 2\int g(x, f(v_n)) f'(v_n)(v - v_n)\end{aligned} \tag{1.26}$$

We claim that

$$\lim_{n\to+\infty}\int g(x,f(v_n))f'(v_n)(v-v_n)=0. \tag{1.27}$$

Assuming the claim, recalling that $J'(v_n)\to 0$ and taking the limit in (1.26) we get

$$\limsup_{n\to+\infty} Q(v_n)\le Q(v).$$

On the other hand, the weak converge of (v_n) in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ provides

$$\int|\nabla v|^2\le\liminf_{n\to+\infty}\int|\nabla v_n|^2. \tag{1.28}$$

Hence, we infer from (1.25) that $Q(v)\le\liminf_{n\to+\infty}Q(v_n)$, and therefore

$$\lim_{n\to+\infty}Q(v_n)=Q(v). \tag{1.29}$$

Before continuing the proof we justify the convergence in (1.27). From (f_{12}) and (1.19) we conclude that $(\int V(x)f^2(v_n-v))$ is a bounded sequence. Hence, the weak convergence $(v_n-v)\rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and item 5 of Proposition 2.1 imply that

$$f(v_n-v)\to 0 \text{ strongly in } L^q(\mathbb{R}^N) \text{ for all } 2\le q<2\cdot 2^*. \tag{1.30}$$

On the other hand, from (g_1), (f_2), (f_9), (f_3), (f_7) and (1.22), we get

$$\begin{aligned}|g(x,f(v_n))f'(v_n)| &\le a(x)|f(v_n)|+2^{-1/2}b(x)f^2(v_n)\\ &\le c_1(a(x)+b(x))(|f(v_n)|+f^2(v_n)),\end{aligned}$$

where $c_1:=\max\{\kappa^{-1},\kappa^{-2}\}$. The above expression and inequality (1.22) again provide $c_2>0$ such that

$$|g(x,f(v_n))f'(v_n)||v_n-v|\le\psi(x)h_n(x)(|f(v_n-v)|+f^2(v_n-v)), \tag{1.31}$$

with $\psi(x):=c_1(a(x)+b(x))\in L^\alpha(\mathbb{R}^N)$ and $h_n(x):=|f(v_n(x))|+f^2(v_n(x))$. If we set $q=2\alpha/(\alpha-1)$ we can use $\alpha>N/2$ to conclude that $2<q<2^*$. Hence, the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$, (f_3) and (f_7) imply that the sequence h_n is bounded in $L^q(\mathbb{R}^N)$. It follows from Hölder's inequality that

$$\int\psi(x)h_n(x)f^2(v_n-v)\le\|\psi\|_\alpha\|h_n\|_q\|f(v_n-v)\|_{2q}^2\to 0,$$

where we have used $4<2q<2\cdot 2^*$ and (1.30). Analogously,

$$\int\psi(x)h_n(x)|f(v_n-v)|\to 0.$$

The statement (1.27) is a consequence of inequality (1.31) and the two convergences above.

By using (1.29) we obtain

$$\begin{aligned}Q(v) &= \liminf_{n\to+\infty}Q(v_n)\\ &\ge \liminf_{n\to+\infty}\int|\nabla v_n|^2+\liminf_{n\to+\infty}\int V(x)f^2(v_n)\\ &\ge \int|\nabla u|^2+\int V(x)f^2(v)=Q(v).\end{aligned}$$

We infer from the above inequality, (1.25) and (1.28) that

$$\liminf_{n\to+\infty}\int|\nabla v_n|^2=\int|\nabla v|^2,\quad \liminf_{n\to+\infty}\int V(x)f^2(v_n)=\int V(x)f^2(v). \tag{1.32}$$

Hence

$$\begin{aligned} Q(v) &= \limsup_{n\to+\infty}\left(\int|\nabla v_n|^2+\int V(x)f^2(v_n)\right)\\ &\geq \limsup_{n\to+\infty}\int|\nabla v_n|^2+\liminf_{n\to+\infty}\int V(x)f^2(v_n)\\ &\geq \liminf_{n\to+\infty}\left(\int|\nabla v_n|^2+\int V(x)f^2(v_n)\right)=Q(v), \end{aligned}$$

and therefore we conclude that

$$\limsup_{n\to+\infty}\int|\nabla v_n|^2=\int|\nabla v|^2.$$

This and (1.32) imply that $\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\to\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$. So, the weak convergence of (v_n) imply that $v_n\to v$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, that is

$$\lim_{n\to+\infty}\|\nabla(v_n-v)\|_2=0. \tag{1.33}$$

Arguing as above we can also conclude that

$$\limsup_{n\to+\infty}\int V(x)f^2(v_n)=\int V(x)f^2(v).$$

and therefore we have that $\sqrt{V(x)f^2(v_n)}\to\sqrt{V(x)f^2(v)}$ strongly in $L^2(\mathbb{R}^N)$. Thus, up to a subsequence, we have that $\sqrt{V(x)f^2(v_n)}\leq\varphi(x)$ a.e. in $\mathbb{R}^n$ for some $\varphi\in L^2(\mathbb{R}^N)$. Thus, we can use (f_{12}) to obtain

$$V(x)f^2(v_n-v)\leq 4(V(x)f^2(v_n)+V(x)f^2(v))\leq 4(\varphi(x)^2+V(x)f^2(v)).$$

Since the right-hand side above belongs to $L^1(\mathbb{R}^N)$ it follows from the Lebesgue Theorem that $\int V(x)f^2(v_n-v)\to 0$. Thus, item 1 of Proposition 2.1 implies that

$$\lim_{n\to+\infty}|v_n-v|_f=0.$$

By using this equality and (1.33) we conclude that

$$\lim_{n\to+\infty}\|v_n-v\|=\lim_{n\to+\infty}(\|\nabla(v_n-v)\|_2+|v_n-v|_f)=0$$

and the proposition is now proved. □

Now we shall consider the Palais-Smale condition under $(\widetilde{g_1})$ instead of $(g_1)-(g_2)$.

Proposition 5.2 *Under the hypotheses of Theorem 1.2 the functional J satisfies the $(PS)_c$ condition for any $c\in\mathbb{R}$.*

Proof. As proved in Lemma 3.1 the functional J is coercive on E. Hence, any Palais-Smale sequence is bounded. It now suffices to notice that, in the proof of Proposition 5.1, we have used the condition (g_2) only to prove the boundedness of the PS-sequence. Since the condition $(\tilde{g}_1)$ implies (g_1), we can argue along the same lines of the proof of Proposition 5.1 to get the desired result. We omit the details. □

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